Exact Electromagnetic Duality I

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ABSTRACT

Exact results about the strong coupling regime of some supersymmetric gauge theories have been obtained in recent years. That was made possible mainly by the existence of a duality symmetry between the weak and strong coupling sectors of the theory, where the role of fundamental (gauge) particles and solitons (monopoles) are exchanged. The ideas involved are in fact a subtle and sophisticated generalization of the old duality of Mawxell's equations exchanging the roles of the electric and magnetic fields. These lectures are an introduction on the ideas and concepts involved in our understanding of the dual roles played by the electric and magnetic sectors of gauge theories, as well as of those played by fundamental particles and solitons in many non linear field theories. Starting with the duality in Maxwell's theory, we discuss how the concept of magnetic monopoles evolved from the beautiful Dirac's theory of 1931 to that of a soliton in a gauge theory with the symmetry spontaneously broken. These lectures are followed by another series of five lectures, by Prof. David Olive, on the same topic, and in fact they are meant to be an introduction of the students to the concepts treated there.

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1 Introduction

These notes constitute an introduction to the ideas and concepts involved in our understanding of the dual roles played by the electric and magnetic sectors of gauge theories, as well as of those played by fundamental particles and solitons in many non linear field theories. The subject has a long history and we make a review of the basic concepts necessary for the understanding of the most recent developments in the subject. This series of five lectures was designed taking into account the needs of the students attending the School. It is followed by another series of five lectures by Prof. David Olive, on the same topic [1], and in fact it is meant to be an introduction of the students to the concepts treated there.

It is well known that Maxwell's theory of electromagnetism, in the absence of sources, presents a striking symmetry between electricity and magnetism. However, when the fields are taken to be derived from a potential, the tip of a huge iceberg becomes dimly visible on the horizon. The set of Maxwell's equations breaks into two subsets. One which can be derived by a variational principle from a Lagrangean, and so has the usual meaning of imposing a dynamics to the potential. However, the second subset is made just of identities, if the electric and magnetic fields can indeed be derived from a potential. They have, as some people prefer to phrase it, a topological nature. At the quantum level, such differences become stronger since the potential gets the status of a quantity that best describes the way electric charges interact with the electromagnetic field. If at the classical level, magnetic sources could be consistently added by arguing that the physics is independent of the potential, now the presence of such sources introduce several difficulties. Dirac [2], in 1931, trying to get an understanding of the value of the fine-structure constant, was able to turn such drawbacks into one of the most beautiful and deep results not confirmed by experience so far. He argued that, according to the principles of quantum mechanics, the very existence of one single magnetic pole could explain why the electric charges of the electron and the proton are equal to such a high degree of accuracy, and why all the other electric charges observed in Nature are multiples of that unit. According to Dirac's result, any electric charge q and magnetic charge g should satisfy $q\,g = 2\pi n\hbar c$, with n an integer. The strength of the minimum allowed magnetic pole g, producing a magnetic field $B = \frac{g}{4\pi r^2}$, is therefore $g = \frac{2\pi\hbar c}{e}$, where e is the electron's charge. Since $\frac{e^2}{4\pi\hbar c} \sim \frac{1}{137}$, one observes that the attraction between two basic magnetic poles of opposite signs is something like 5×10^3 times the attraction between an electron and a positron. Dirac argued that such strong interaction between magnetic poles could explain why they have not been observed experimentally, since it would take a lot of energy to produce a separated pair of them. Dirac's argument, however, could not provide an estimation of the masses of magnetic monopoles.

The theory of magnetic monopoles undertook important developments in the 70's when 't Hooft and Polyakov [3, 4] showed that they can appear as classical soliton solutions of non abelian gauge theories with the symmetry spontaneously broken by the Higgs mechanism [5]. The magnetic charge of the solution appears as a topological charge, in the sense that its conservation is not a consequence of the dynamics, but of the configuration of the fields at spatial infinity. The charges are in fact determined by the classes of homotopy of the maps of the sphere S^2 at spatial infinity into the Higgs vacua. Such developments changed drastically our views about magnetic monopoles. Their magnetic charges are quantized due to the topological structures of the theory. Even though such quantization occurs at the classical theory, it is compatible with Dirac's quantization condition based purely on the principles of quantum mechanics. Additionally, the monopoles are not the type of particles we are used to. They are solitons, and therefore extended objects. Even though some aspects of their classical dynamics can be approximately inferred from multi-monopole solutions, their quantum theory is just a mystery. Several properties of the monopoles, like masses and couplings, depend on the inverse of the (gauge) coupling constants of the theory, and therefore can not in general be studied through perturbative methods.

From the phenomenological point of view, the theories where such monopoles are likely to appear are the unified theories (GUT, superstring, supergravity, etc) of strong and electroweak (and possibly gravity) interactions. The patterns of symmetry breaking in these theories are such that the monopoles would have a mass of about $10^{16} GeV$. Therefore, they are too heavy to be produced in present accelarators, and their production if it has happened, it was in the very early universe.

There is a class of magnetic monopoles that presents some very special properties. They are the so-called self-dual or BPS (Bogomolny-Prasad-Sommerfield) monopoles [6, 7]. They appear when the Higgs field is in the adjoint representation of the gauge group, and when the Higgs potential is taken to vanish, but keeping the vacuum expectation value of the Higgs field different from zero, so that the symmetry is still broken. They have the minimum value of the mass allowed by the theory (Bogomolny bound) and their classical solutions can be obtained exactly. The masses of the fundamental particles (gauge bosons) and solitons of such theories obey a universal mass formula, which is $m = a\sqrt{q^2 + g^2}$, where q and g are the electric and magnetic charges respectively of the particle, and a is the vacuum expectation value of the Higgs field. Several properties of these theories have led Montonen and Olive [8, 9] to conjecture that the magnetic monopoles are gauge particles of a theory dual to that where they appear as solitons. The gauge coupling of the dual theory is inversely related to the coupling constant of the starting theory. In addition, the original gauge bosons should appear as soliton in the dual theory. In such scheme, the monopoles should have spin 1. In order, for it to work at the quantum level, the mass formula should not suffer quantum corrections. For those reasons, it was soon established that the most promising theory for the conjecture was the $N = 4$ super Yang-Mills theory [10].

The calculations to verify or even to test such conjecture are very hard to perform, and no stimulating results were obtained for decades. However, in the early 90's some important observations were made concerning the spectrum of dyons in super Yang-Mills theories [11, 12, 13]. They indicated a $SL(2, \mathbb{Z})$ symmetry of modular transformations of the complex parameter $\tau \equiv \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$, where e is the gauge coupling and θ is the coefficient of the instanton term $F_{\mu\nu}\tilde{F}^{\mu\nu}$, in the Lagrangean, responsible for the violation of CP. Notice that such symmetry includes as a particular case, the tranformation $e \to \frac{1}{e}$ proposed by Montonen and Olive to related the gauge couplings of the dual theories. Inspired by these results, Vafa and Witten [14] have shown, using techniques of topological field theories, that the partition function of the $SU(2)$ $N = 4$ super Yang-Mills theory (with a twist to make it topological) is invariant under that $SL(2, \mathbb{Z})$. By the same time Seiberg and Witten [15] have shown that the $SU(2)$ $N = 2$ super Yang-Mills theory also possesses a similar type of duality symmetry. In fact, Seiberg and Witten were able to set up an scheme to obtain exact results about the several sectors of theory including the strong coupling regime. They even have shown that there is a point in the moduli where there exists massless magnetic monopoles. The condensation of such monopoles could produce a state which is like a dual superconductor, where colour confinement could occur through a dual Meissner effect. That would then realize (not in QCD unfortunately) one of the most promising mechanisms for confinement proposed in the 70's [16]. Those recent developments are the main motivations for this series of introductory lectures.

2 Duality in Maxwell's theory of electromagnetism

The classical theory of electromagnetism is perhaps one of the most beautiful physical theories known. It presents a high level of symmetries and seems to be "designed" to live in four dimensional space-time. It is described by Maxwell's equations, which in the absence of charges and currents, are given in terms of the antisymmetric field tensor $F^{\mu\nu}$ as[1]

$$\partial_\mu F^{\mu\nu} = 0 \, , \qquad \partial_\mu \tilde{F}^{\mu\nu} = 0 \tag{2.1}$$

where $\tilde{F}_{\mu\nu}$ is the dual of $F_{\mu\nu}$, i.e.

$$\tilde{F}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \tag{2.2}$$

The first set of equations are the Euler-Lagrange equations obtained from Mawxell's Lagrangean

$$\mathcal{L} \equiv -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{2.3}$$

where the variational calculus is with respect to a vector potential A_μ, from which the field tensor is derived

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.4}$$

As a consequence of (2.4) and (2.2), the second set of Maxwell's equations are trivially satisfied and are the so-called Bianchi identities.

The eletric and magnetic fields are related to the field tensor $F^{\mu\nu}$ by

$$E^i \equiv F^{0i} \, , \qquad B^i \equiv \frac{1}{2} \epsilon^{ijk} F_{jk} \tag{2.5}$$

or equivalently

$$B^i \equiv \tilde{F}^{0i} \, , \qquad E^i \equiv -\frac{1}{2} \epsilon^{ijk} \tilde{F}_{jk} \tag{2.6}$$

The Maxwell's equations (2.1) are conformally invariant, and their versions in dimensions other than four are just Poincaré invariant. However, one of the most impressive symmetries is the duality transformation. It is valid only in four dimensions because it interchanges the field tensor and its dual, and only in that dimension they have the same rank. Writing (2.1) in complex notation

$$\partial_\mu \left(F^{\mu\nu} + i \, \tilde{F}^{\mu\nu} \right) = 0 \tag{2.7}$$

one sees it is clearly invariant under the duality transformation [17]

$$\left(F^{\mu\nu} + i \, \tilde{F}^{\mu\nu} \right) \rightarrow e^{i\theta} \left(F^{\mu\nu} + i \, \tilde{F}^{\mu\nu} \right) \tag{2.8}$$

with θ being a real constant. In terms of the eletric and magnetic fields it becomes

$$\left(E^i + i \, B^i \right) \rightarrow e^{i\theta} \left(E^i + i \, B^i \right) \tag{2.9}$$

[1] We take the signature of the Minkowski metric as $(+,-,-,-)$, and use greek letters $\mu, \nu, \ldots = 0,1,2,3$ to label space-time indices, and english letters $i, j, k \ldots = 1,2,3$ to label space indices.

The density of energy

$$\frac{1}{2} \mid F^{0i} + i\tilde{F}^{0i} \mid^2 = \frac{1}{4} \mid F^{ij} + i\tilde{F}^{ij} \mid^2 = \frac{1}{2}\left(E^2 + B^2\right) \tag{2.10}$$

and the density of momentum

$$-\frac{1}{2}\left(F^{0i} + i\tilde{F}^{0i}\right)\left(F_{ij} + i\tilde{F}_{ij}\right)^* = E \wedge B \tag{2.11}$$

are invariant under (2.8).

The Lagrangean (2.3) on the other hand, is invariant under (2.8) only for $\theta = \pi$. However, it is worth noticing that the Lagrangean is the real part of the complex quantity[2]

$$\frac{1}{2}\left(F^{\mu\nu} + i\tilde{F}^{\mu\nu}\right)^2 = F_{\mu\nu}^2 + iF^{\mu\nu}\tilde{F}_{\mu\nu} \tag{2.12}$$

and its imaginary part is a total derivative, i.e.

$$F_{\mu\nu}\,\tilde{F}^{\mu\nu} = 2\partial_\mu\,W^\mu = 12E^i B_i \tag{2.13}$$

where

$$W^\mu = \epsilon^{\mu\nu\rho\sigma}\,A_\nu\,\partial_\rho\,A_\sigma \tag{2.14}$$

Therefore, it is right to say that the Lagrangean and the topological term (2.13) transform as a doublet of (2.8).

Notice that we could have a scale parameter in (2.8), i.e. $\lambda e^{i\theta}$ instead of just $e^{i\theta}$, that (2.7) would still be invariant. However, that would break the invariance of the energy and momentum. Notice that $\mid F^{\mu\nu} + i\,\tilde{F}^{\mu\nu} \mid^2 = 0$, and so, that is perhaps the only quadratic quantity invariant under (2.8) with the scale parameter included.

If we want the symmetry (2.8) to hold true in the presence of matter we encounter serious difficulties. We could naively add two types of currents to (2.7) as

$$\partial_\mu\,\left(F^{\mu\nu} + i\,\tilde{F}^{\mu\nu}\right) = j^\nu + i\tilde{j}^\nu \tag{2.15}$$

and impose that they transform under (2.8) as

$$\left(j^\nu + i\tilde{j}^\nu\right) \rightarrow e^{i\theta}\,\left(j^\nu + i\tilde{j}^\nu\right) \tag{2.16}$$

Since $\tilde{j}^\nu$ becomes a source for $\tilde{F}^{\mu\nu}$ we can not derive the field tensor from a vector potential as in (2.4). That is fine at the classical level, since all physical quantities are described in terms of the field tensor. However, at the quantum level the vector potential acquires physical importance and the implementations of such ideas, as we describe in the next section, becomes very subtle. But the great restrictions come from the experimental side. The charges associated to the currents $\tilde{j}^\nu$ are static sources for the magnetic field, and so far no experimental support has appeared for their existence.

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[2]Remember that $F_{\mu\nu}^2 = -\tilde{F}_{\mu\nu}^2$, since $\epsilon^{\mu\nu\lambda\delta}\,\epsilon_{\rho\sigma\lambda\delta} = -2\left(\delta_\rho^\mu\delta_\sigma^\nu - \delta_\sigma^\mu\delta_\rho^\nu\right)$

3 Dirac's argument for the quantization of charges

The quantum mechanical description of electromagnetic interaction of a charged particle can be expressed in a very elegant way. The effect of the electromagnetic field is to change the phase of the wave function at each point of space-time. However, the change of the phase at each point has not a definite value. Only the difference of the phase between any two neighbouring points is uniquely determined by the electromagnetic interaction. For two distant points there is a definite phase difference only relative to some curve joining them. In addition, it will be different for different curves. Therefore, the effect of the electromagnetic interaction is to make the phase of the wave function non-integrable. One requirement is that the change of phase round a closed curve should be the same for all wave functions of a given system. By fixing the phase of the wave function at some reference point x_0, the wave function $\psi_{\text{int.}}(x)$ of a particle of charge q, in the presence of an electromagnetic field described by the potential A_μ, relative to the wave function $\psi_{\text{free}}(x)$ of this same particle in the absence of the electromagnetic interaction, is given by

$$\psi_{\text{int.}}(x) = \psi_{\text{free}}(x)\, e^{\frac{iq}{\hbar c}\int_C dx^\mu A_\mu} \tag{3.1}$$

where C is some contour starting at the reference point x_0 and ending at x.

That is a very important concept in Physics. Not only the electromagnetic interaction is governed by it, but also the weak and strong interactions. It is called the gauge principle [18], and one can easily check that the so-called minimum coupling follows from the above statements. Indeed, if $\psi_{\text{free}}(x)$ satisfies a wave equation, involving the momentum and energy operators, i.e. $-i\hbar\partial_\mu$, then $\psi_{\text{int.}}(x)$ satisfies the same wave equation with those operators replaced by the covariant derivatives, $-i\hbar\partial_\mu - \frac{q}{c}A_\mu$.

The phase of $\psi_{\text{free}}(x)$ is integrable, and so its change round a closed curve is zero. Therefore, the change of phase of $\psi_{\text{int.}}(x)$ round a closed contour Γ is, by the use of Stokes' theorem,

$$\frac{q}{\hbar c}\int_\Gamma dx^\mu A_\mu = \frac{q}{\hbar c}\int_S ds^{\mu\nu} F_{\mu\nu} \tag{3.2}$$

where S is a surface with boundary Γ, and $ds^{\mu\nu}$ is an element of the surface. Consequently the change of phase round a closed curve is determined solely by the electromagnetic field and it is the same for all wave functions of the system.

All the above is compatible with the known facts about the quantum description of interaction of particles with a classical electromagnetic field. However, in 1931 Dirac [2] has shown that new features could be added without running into inconsistencies. He noticed that the change in phase round a closed curve could be different for different wave functions of the system by multiples of 2π, and that difference did not have to be interpreted in terms of the electromagnetic field. We reproduce here his arguments.

If one considers a small loop, one expects that the change in phase round it to be small, since the wave function has to be continuous. Therefore, for small loops the change in phase can not be different by multiples of 2π for different wave functions. However, if the ψ vanishes at a given point, its phase there has no meaning. Since ψ is complex, its vanishing involves two conditions. In three space dimensions the points where ψ vanishes should lie along a line, which Dirac calls the nodal line. Then, if we consider a wave function with nodal line passing through the small loop, its change in phase round the loop does not have to be small. The change could be some value near $2\pi n$, and the integer n is charecteristic of the nodal line. The difference between the change in phase and the nearest $2\pi n$ is what should be interpreted as the eletromagnetic flux (3.2). So, the change in phase round such small space loop is

$$2\pi n + \frac{q}{\hbar c} \int_{\rm loop} ds^{\mu\nu} F_{\mu\nu} \tag{3.3}$$

The case of large closed contour Γ can be treated by considering a surface S whose border is Γ, and dividing S into small loops. The total change in phase round Γ will be the sum of the changes in phases round the small loops. If Γ has no components in time direction, we observe from (3.2) and (2.5) that only the magnetic flux will contribute to the change in phase. Therefore, the total change in phase round Γ is

$$2\pi \sum n + \frac{q}{\hbar c} \int_S ds_i B^i \tag{3.4}$$

where $ds_i \equiv \epsilon_{ijk} ds^{jk}$, and the sum is over all the small loops scanning S.

By shrinking Γ to a point we get that the change in phase (3.4) should vanish. But now S is a closed surface, and it follows that the nodal lines crossing it should satisfy

$$2\pi \sum n = -\frac{q}{\hbar c} \int_S ds_i B^i \tag{3.5}$$

But, the right hand side of this equation should be the same for all wave functions since it only depends on the electromagnetic field. Therefore, $2\pi \sum n$ should be the same for all wave functions. If that vanishes there is no problem because if we take a wave function with no nodal lines then $2\pi \sum n$ should vanish indeed, and if we take one with nodal lines it means that such lines are crossing S twice and giving contributions of opposite sign at each crossing. However, if $2\pi \sum n$ does not vanishes it means some nodal lines must end inside the surface. Since this sum must be the same for all wave functions, and since the result applies to any surface enclosing the end points, we conclude that the end points of nodal lines should be the same for all wave functions, irrespective of these functions having zeros or not. Then such end points should be points of singularities of the electromagnetic field. Taking a small closed surface surrounding an end point one gets, from (3.5), that the total magnetic flux through it should equal $2\pi n \frac{\hbar c}{q}$. Then, there should be at such point a magnetic charge g, with a magnetic field $B = \frac{g}{4\pi r^2}$, and so a magnetic flux g satisfying

$$qg = 2\pi n \hbar c \tag{3.6}$$

Therefore, the existence of magnetic charges is consistent with the laws of quantum mechanics, as long as their strengths are multiples of $\frac{2\pi\hbar c}{q}$. On the other hand, the existence of just one single magnetic charge g_0 in the universe, leads to the quantization of the electric charge in units of $\frac{2\pi\hbar c}{g_0}$. Such quantization of the electric charge is strongly supported by all known exeperimental results. However, magnetic monopoles have not been observed in Nature so far. Since, the above considerations do not put any constraint in their masses, one could argue that perhaps they are too heavy for a pair of them to be produced in present accelerators.

4 Dirac-Schwinger-Zwanziger-Saha quantization condition

The quantization condition we derived in the previous section applies to particles that carry either electric or magnetic charges, but not both. The generalization of Dirac's quantization condition to dyons, i.e. particles carrying both electric and magnetic charges, was performed by Schwinger [19] and Zwanziger [20]. Here we give a derivation of it based on the ideas of Saha [21] which uses the quantization of the angular momentum. Although not very rigorous, the argument is quite suggestive.

Consider the non relavistic motion of a particle of mass m, electric charge e_1 and magnetic charge g_1 around a stationary body of electric charge e_2 and magnetic charge g_2. The equations of motion are

$$m\frac{d\vec{v}}{dt} = e_1\left(\vec{E} + \frac{1}{c}\vec{v}\wedge\vec{B}\right) + g_1\left(\vec{B} - \frac{1}{c}\vec{v}\wedge\vec{E}\right) \tag{4.1}$$

The electric and magnetic fields of the stationary body are

$$\vec{E} = \frac{e_2}{4\pi r^2}\hat{r} \qquad \vec{B} = \frac{g_2}{4\pi r^2}\hat{r} \tag{4.2}$$

Substituting into (4.1) one gets

$$m\frac{d\vec{v}}{dt} = \frac{(e_1e_2 + g_1g_2)}{4\pi}\frac{\hat{r}}{r^2} + \frac{(e_1g_2 - g_1e_2)}{4\pi}\frac{1}{c}\vec{v}\wedge\frac{\hat{r}}{r^2} \tag{4.3}$$

Notice that the angular momentum of the particle is not conserved

$$\frac{d}{dt}\left(m\vec{r}\wedge\vec{v}\right) = m\vec{r}\wedge\frac{d\vec{v}}{dt} = \frac{(e_1g_2 - g_1e_2)}{4\pi}\frac{1}{c}\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) \tag{4.4}$$

The quantity

$$\vec{J} \equiv m\vec{r}\wedge\vec{v} - \frac{(e_1g_2 - g_1e_2)}{4\pi c}\hat{r} \tag{4.5}$$

is conserved

$$\frac{d\vec{J}}{dt} = 0 \tag{4.6}$$

We can interpret the second term on the r.h.s. of (4.5) as the angular momentum of the electromagnetic field.

When we quantize the component of $\vec{J}$ along $\vec{r}$, i.e. $\hat{r}\cdot\vec{J}$, we get that it should be a multiple of $\frac{\hbar}{2}$. Therefore we get

$$e_1g_2 - g_1e_2 = 2\pi n\hbar c \tag{4.7}$$

5 Yang's relationship between the quantization of the electric charge and compactness of the gauge group

In 1970 C.N. Yang [22] called attention to the fact that the quantization of the electric charge and the compactness of the gauge group are intimately related. Consider a global gauge transformation of charged fields ψ_j of charge e_j

$$\psi_j \to \psi'_j = e^{ie_j \alpha} \psi_j \tag{5.1}$$

Notice that if the charges e_j's are not commensurate with each other, then there is not two real values of the parameter α, let us say α_1 and α_2, such that the corresponding transformations are the same for all fields. Indeed, for that to happen for the charges e_j and e_k for instance, one needs

$$ie_j \alpha_1 = ie_j \alpha_2 + 2\pi i n \qquad ie_k \alpha_1 = ie_k \alpha_2 + 2\pi i m \tag{5.2}$$

and so

$$\frac{e_j}{e_k} = \frac{n}{m} \tag{5.3}$$

Therefore, the group with elements $e^{i\alpha Q}$ is not compact, since its elements are different for all real values of α.

On the other hand, if the charges e_j's are all integral multiples of a universal unit charge e, then for any two values of α differing by a multiple of $\frac{2\pi}{e}$, the transformation (5.1) is the same for all fields. Consequently, the gauge group is compact since α should now be taken in the interval between 0 and $\frac{2\pi}{e}$.

If one finds a good physical reason for the compactness of the electromagnetic gauge group then the quantization of the electric charge would be implied, without assuming the existence of magnetic monopoles. In the case of non abelian gauge theories (not in electromagnetism) the non compactness of the gauge group implies that the classical energy is not bounded below. In the Weinberg-Salam model the gauge group is non abelian, namely $SU(2) \otimes U(1)_Y$. The electromagnetic gauge group $U(1)_{\rm e.m.}$ is generated by a linear combination of the hypercharge Y and the generator T_3 of $SU(2)$. So, the compactness of $U(1)_{\rm e.m.}$ relies on the compactness of $U(1)_Y$, and so it is not established. The compactness of $U(1)_{\rm e.m.}$ could be achieved in models of grand unification with semisimple gauge groups (without $U(1)$ factors). However, the theories of that type of physical interest all have a pattern of symmetry breaking that implies the existence of 't Hooft-Polyakov type monopoles. So, the quantization of the electric charge and monopoles are still going together.

6 Sine-Gordon Model: an example of two types of particles

As we will discuss below, the monopole (if it exists) is a different type of particle. It does not appear as the excitations of the fields present in the Lagrangean defining the theory. It appears as classical solutions with very special properties known as solitons. In addition, the magnetic charge is not a charge associated to a symmetry of the Lagrangean, like the electric charge, but it appears as a topological charge. Therefore, to illustrate such ideas let us discuss a toy model where similar things happens.

The sine-Gordon model [23] is a two dimensional field theory with a scalar field ϕ, defined by the Lagrangean

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{\mu^2}{\beta^2}\left(\cos\left(\beta\phi\right) - 1\right) \\ &= \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{2\mu^2}{\beta^2}\left(\sin\frac{1}{2}\beta\phi\right)^2 \equiv T - V \end{aligned} \tag{6.1}$$

The corresponding equation of motion is

$$\partial^2\phi + \frac{\mu^2}{\beta}\sin\beta\phi = 0 \tag{6.2}$$

Notice that the potential V is periodic, and the theory is invariant under the discrete symmetries

$$\phi \rightarrow \phi + \frac{2\pi n}{\beta}\,; \qquad\qquad \phi \rightarrow -\phi \tag{6.3}$$

The theory possesses therefore an infinite number of vacua. That will play an important role in its properties.

First type of particle

Take the linearization of the equation (6.2)

$$\partial^2\phi + \mu^2\phi = 0 \tag{6.4}$$

The quantum fluctuations around the vacuum solution $\phi = 0$, or any other vacuum $\phi = \frac{2\pi n}{\beta}$, give particles of mass

$$m = \mu\hbar \tag{6.5}$$

The are the particles associated to the filed ϕ, and we call them fundamental particles.

Second type of particle

Let us look for classical solutions of the equation (6.2) with finite energy. In order for that to happen we need that the field should approach a vacuum at space infinity ($\phi \rightarrow \frac{2\pi n}{\beta}$

as $x \to \pm\infty$). We shall denote $\sigma = \beta\phi$. The energy of the system (6.1) is then

$$
\begin{aligned}
E &= \frac{1}{2\beta^2}\int_{-\infty}^{\infty} dx\,\left(\dot\sigma^2 + \sigma'^2 + \left(2\mu\sin\frac{\sigma}{2}\right)^2\right)\\
&= \frac{1}{2\beta^2}\int_{-\infty}^{\infty} dx\,\left(\dot\sigma^2 + \left(\sigma' \pm 2\mu\sin\frac{\sigma}{2}\right)^2 \mp 4\mu\sigma'\sin\frac{\sigma}{2}\right)\\
&\geq \mid \frac{2\mu}{\beta^2}\int_{\sigma(-\infty)}^{\sigma(\infty)} d\sigma\,\sin\frac{\sigma}{2} \mid = \mid \frac{4\mu}{\beta^2}\cos\frac{\sigma}{2} \mid_{\sigma(-\infty)}^{\sigma(\infty)}\\
&= \frac{8\mu}{\beta^2}\,; \qquad \text{if } \sigma(-\infty) = 0 \text{ and } \sigma(\infty) = 2\pi
\end{aligned}
\tag{6.6}
$$

That constitutes some sort of Bogomolny bound on the energy, namely $E \geq \frac{8\mu}{\beta^2}$, and the saturation of such bound occurs when

$$
\begin{aligned}
\dot\sigma &= 0\\
\sigma' &= \pm 2\mu\sin\frac{\sigma}{2}
\end{aligned}
\tag{6.7}
$$

These two first order differential equations imply the (second order) sine-Gordon equation of motion (6.2). In addition, notice that if $\sigma'(x_0) = 0$ for some finite x_0, then (6.7) imply that $\sigma^{(n)}(x_0) = 0$, i.e. all space derivatives at x_0 vanish. Therefore, non trivial solutions of (6.7) can not have vanishing derivative. The only way is for σ to interpolate between successive vacua as x varies from $-\infty$ to ∞. It can not interpolate between non successive vacua because $\sin\frac{\sigma}{2}$ and so σ' would vanish as it passes through intermediate vacua.

Consider the topological current

$$
j^\mu \equiv \frac{1}{2\pi}\varepsilon^{\mu\nu}\partial_\nu\sigma
\tag{6.8}
$$

with corresponding topological charge

$$
Q \equiv \int_{-\infty}^{\infty} dx\, j^0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\,\sigma' = \frac{1}{2\pi}\left(\sigma(\infty) - \sigma(-\infty)\right)
\tag{6.9}
$$

For the finite energy solutions one needs $\sigma(\pm\infty)$ to be a vacuum, and so such topological charge is quantized (classically), i.e. $Q \in \mathbb{Z}$. The equations (6.7) have two non trivial solutions with charges $Q = \pm 1$. For $Q = 1$ the solution interpolates between $\sigma(-\infty) = 0$ and $\sigma(\infty) = 2\pi$, whilst for $Q = -1$ it interpolates between $\sigma(-\infty) = 2\pi$ and $\sigma(\infty) = 0$. They are the soliton and anti-soliton solutions, respectively, of the sine-Gordon model. These solutions propagate without dispersion, and scatter with each other without losing their identities (the effect of scattering is a phase shift). Therefore, they behave like particles. It is legitimate to interpret the energy of such classical configurations in their rest frame (static solutions) as their classical masses. The soliton (and anti-soliton) mass is then

$$
M = \frac{8\mu}{\beta^2} = \frac{8\mu}{\hbar\beta^2}m
\tag{6.10}
$$

where m is the mass (6.5) of the fundamental particle associated to the field ϕ. Notice, that the weaker the coupling is (small β) the larger is the soliton mass. That already indicates that the soliton properties can not be studied perturbatively.

At the quantum level we have two types of particles. The quantum fluctuations of the field ϕ, with mass $\mu\hbar$ and vanishing topological charge, and solitons and anti-solitons of mass $\frac{8\mu}{\beta^2}$, with topological charges $Q = \pm 1$. The question now is what dynamics the solitons obey. The sine-Gordon model is one of the few theories where that question has been answered in a more detailed way [24, 25, 26]. The field associated to the solitons and anti-solitons obey the dynamics of the massive Thirring model, defined by the Lagrangean

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m_F\bar{\psi}\psi - \frac{g}{2}\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi \tag{6.11}$$

The massless Thirring model is exactly solvable [27] and one can then make a perturbation expansion in powers of m_F. Such expansion involves the n-point function of the operator $\bar{\psi}\psi$. On the other hand, one can make a perturbation expansion of the sine-Gordon model in powers of μ^2, which involves therefore the n-point functions of the operator $\cos\beta\phi$. Coleman [25] has shown that such n-point functions are the same for any n, if the following identifications are made

$$\begin{aligned}
\frac{\mu^2}{\beta^2}\cos\beta\phi &= -m_F\bar{\psi}\psi \\
-\frac{\beta}{2\pi}\varepsilon^{\mu\nu}\partial_\nu\phi &= \frac{1}{\hbar}\bar{\psi}\gamma^\mu\psi \\
\frac{\beta^2\hbar}{4\pi} &= \frac{1}{1+g\hbar/\pi}
\end{aligned} \tag{6.12}$$

In addition, such equivalence is only valid for $\beta^2 < 8\pi$, since for $\beta^2 > 8\pi$ the sine-Gordon Hamiltonian is unbounded below (for a more detailed discussion of such equivalence, see chap. 7 of [28]). The relations (6.12) are called the bosonization rules.

The Thirring model has a $U(1)$ global symmetry

$$\psi \to e^{i\theta}\psi \tag{6.13}$$

and the corresponding conserved Noether current and charge are

$$j^\mu_{\rm Noet.} = \frac{1}{\hbar}\bar{\psi}\gamma^\mu\psi\,; \qquad Q_{\rm Noet.} = \frac{1}{\hbar}\int_{-\infty}^{\infty} dx\,\psi^\dagger\psi \tag{6.14}$$

So, the fields $\psi^\dagger$ and ψ have $Q_{\rm Noet.}$ charges 1 and -1 respectively. Therefore, following Skyrme [24] one can identify the quantum solitons of the sine-Gordon theory as the particles associated to the Fermi fields of the Thirring model. The topological charge of sine-Gordon is then identified as the Noether charge of the Thirring model. Such equivalence between solitons and topological charges of one theory with fundamental particles and Noether charges of other theory, as we will see in the next sections, is perhaps what happens with magnetic monopoles and gauge particles in a non abelian gauge theory with symmetry spontaneously broken by a Higgs field. Such equivalence will constitute in fact, a generalization of the Maxwell-Dirac electromagnetic duality.

7 Gauge Theories with Symmetry Spontaneously Broken

Several features of the sine-Gordon model in two dimensional space-time can be present in theories in four dimensions. Non-abelian gauge theories present classical solutions with solitonic properties, and in many cases such solutions carry non trivial topological charges like the sine-Gordon solitons. One example are the 't Hooft-Polyakov [3, 4] magnetic monopoles which appear in gauge theories when the symmetry is spontaneously broken in a suitable way. The structures of these theories is the subject of this section.

We consider the model

$$\mathcal{L} = -\frac{1}{4}\,\mathrm{Tr}\,(F_{\mu\nu})^2 + (\mathcal{D}_\mu\phi)^\dagger\,\mathcal{D}^\mu\phi - V(\phi) \tag{7.1}$$

which is a non abelian gauge theory with a gauge group G coupled to a multiplet of scalar fields ϕ, the Higgs fields, transforming under a representation D of the gauge group G

$$\begin{aligned} \phi &\rightarrow D(g)\phi \\ \mathcal{D}_\mu\phi &\rightarrow D(g)\mathcal{D}_\mu\phi \end{aligned} \tag{7.2}$$

and where the covariant derivative is given by

$$\mathcal{D}_\mu\phi \equiv \partial_\mu\phi + ieD(A_\mu)\phi \tag{7.3}$$

The gauge fields A_μ, live in the Lie algebra $\mathcal{G}$ of G, $A_\mu = A_\mu^a T_a$, where T_a, $a = 1, 2, \ldots \dim G$ are the generators of G. The field tensor $F_{\mu\nu}$ also lives in $\mathcal{G}$ and is given by $(F_{\mu\nu} = F_{\mu\nu}^a T_a)$

$$D(F_{\mu\nu}) = \frac{1}{ie}\left[\mathcal{D}_\mu\,,\,\mathcal{D}_\nu\right] = \partial_\mu D(A_\nu) - \partial_\nu D(A_\mu) + ie[D(A_\mu)\,,\,D(A_\nu)] \tag{7.4}$$

Under the gauge transformations they behave as

$$\begin{aligned} D(A_\mu) &\rightarrow D(g)D(A_\mu)D(g^{-1}) + \frac{i}{e}\partial_\mu D(g)D(g^{-1}) \\ D(F_{\mu\nu}) &\rightarrow D(g)D(F_{\mu\nu})D(g^{-1}) \end{aligned} \tag{7.5}$$

We require the Higgs potential V to be gauge invariant, $V(D(g)\phi) = V(\phi)$, and renormalizable. The usual form is

$$V(\phi) = \frac{\lambda}{4}\left(\mid\phi\mid^2 - a^2\right)^2 \tag{7.6}$$

where λ is a coupling constant and a a given constant, the vacuum expectation value of the Higgs field.

The gauge invariance of the theory requires the gauge fields to be massless, since a term like $A_\mu A^\mu$ would break the symmetry (7.5). However, the only known massless vector particle is the photon. So, the theory (7.1) can only be useful if we can generate mass for for the gauge particles. That is done through the so called Higgs mechanism [5], as in the Weinberg-Salam model of electro-weak interactions. In Quantum Chromodynamics (QCD), the theory of strong interactions, the gauge particles (gluons) are massless, but it is believed that they are confined inside the hadrons, through a mechanism dual to the Meissner effect, where charges are confined inside a magnetic superconductor.

The idea of the Higgs mechanism is very simple, even though it took some years to realize its importance in High Energy Physics. Consider a system of spins S_i with nearest neighbor interaction $H \sim -\sum_{i,j} S_i \cdot S_j$. The system is invariant under rotation. However, the state of minimum energy correspond to a configuration where all spins are aligned. Such state is not invariant under rotation. In fact, under a rotation it is transformed into another state of that type. So, for very low temperatures the configuration of the system is dominated by a vacuum state and therefore the rotation is not a symmetry. We then say that the symmetry of the system is spontaneously broken. Notice that, such thing can happen if there exist more than one vacuum state. If the vacuum state is unique, it is bound to be a singlet of the symmetry group and so invariant.

The Hamiltonian of the theory (7.1) is the sum of three positive terms. In order to minimize the energy all three terms must vanish. The terms involving the Higgs field should then satisfy [29]

$$V(\phi) = 0 \, ; \qquad\qquad \mathcal{D}_\mu \phi = 0 \tag{7.7}$$

The other term implies $F_{\mu\nu} = 0$ and so $A_\mu = \partial_\mu g g^{-1}$.

Notice that, in order to satisfy (7.7) the field configurations do not have to be constant in space time. If there is a continuum of Higgs vacua, $V(\phi) = 0$, the fields can be moving in that space of vacua. Such movement of the Higgs field in the vacua is associated with the Goldstone bosons degrees of freedom [30]. However, if one wants a constant vacuum configuration , then necessarily $A_\mu = 0$, since there is no way of having $\mathcal{D}_\mu \phi = \partial_\mu \phi + ieD(A_\mu)\phi = 0$, with $\partial_\mu \phi = 0$, and $A_\mu \neq 0$.

The Higgs potential $V(\phi)$ is gauge invariant, and therefore if ϕ is a vacuum, so is $D(g)\phi$. We then define the Higgs vacuum manifold as

$$M_0 \equiv \{\phi \mid V(\phi) = 0\} \tag{7.8}$$

Notice that, the other condition in (7.7), namely $\mathcal{D}_\mu \phi = 0$ coud be put more restriction on M_0. But remember we have the gauge field in $\mathcal{D}_\mu$ to absorb them.

An important concept is the little group. Given a vacuum ϕ, consider the subgroup H_ϕ of G

$$H_\phi \equiv \{g \in G \mid D(g)\phi = \phi\} \tag{7.9}$$

If the system gets frozen in the vacuum ϕ then the gauge symmetry left is that of H_ϕ. We then say that G was spontaneously broken to H_ϕ. From (7.4) and (7.7) one observes that

$$D(F_{\mu\nu})\phi = \frac{1}{ie}\left[\mathcal{D}_\mu \, , \, \mathcal{D}_\nu\right]\phi = 0 \, ; \qquad \rightarrow \qquad F^a_{\mu\nu} D(T_a)\phi = 0 \tag{7.10}$$

Writing an infinitesimal element $h \in H_\phi$ as $h = 1 + i\varepsilon T$, one gets from $D(h)\phi = \phi$ that $D(T)\phi = 0$. Therefore, the generators of H_ϕ annihilate the vacuum ϕ. Consequently, from (7.10) one observes that the only components of $F_{\mu\nu}$ that do not have to vanish in the Higgs vacuum are those associated to the subgroup H_ϕ. The components of $F_{\mu\nu}$ not associated to H_ϕ are expelled from the Higgs vacumm, and we then have something similar to what happens to magnetic fields in a superconductor. The analogy between Higgs vacuum and superconductor can in fact be pushed much further. The distance scale that the non H_ϕ components of $F_{\mu\nu}$, and so of A_μ, can penetrate into the Higgs vacuum can then be associated to a mass for such gauge particles.

Consider now the orbit of a point of M_0. These are constructed by acting with G on that point to obtain the set

$$\mathcal{O}_\phi = \{D(g)\phi \mid g \in G\} \tag{7.11}$$

Since the elements of H_ϕ leave ϕ invariant, it follows that all elements of the form gh, with $g \in G$ fixed and any $h \in H_\phi$, map ϕ into the same point. Therefore, $\mathcal{O}_\phi$ has the structure of the coset space

$$\mathcal{O}_\phi = G/H_\phi \tag{7.12}$$

We then have two possibilities: either M_0 is made of several orbits, or then it is just one orbit, i.e. $M_0 \equiv G/H_\phi$. In the second case it follows that the little groups of all vacua are isomorphic. The reason is that in such case any two vacua are linked by a gauge transformation $\phi' = D(g)\phi$. Therefore, if $D(h)$ leaves ϕ invariant, then $D(g)D(h)D(h^{-1})$ leaves ϕ' invariant. So, $H_{\phi'} = gH_\phi g^{-1}$. Consequently, if all Higgs vacua constitute just one orbit of the gauge group then all little groups are isomorphic. If there are several orbits the little groups may vary from orbit to orbit. Such considerations will play an important role when we discuss the topological (magnetic) charges of some classical solutions of (7.1).

7.1 The masses of the gauge particles

For energies much lower than the energy scale determined by the Higgs potential parameters, the system will be in states close to the Higgs vacuum. Therefore, the vacuum expectation value of the Higgs field will be $\langle\phi\rangle = a$ where a is the minimum of the Higgs potential. Therefore, the quantum fluctuations of the Higgs field will be around a vaccum $\phi_0 = \text{const.}$, with $\phi_0^2 = a^2$, which the system will choose. Then we can write the Higgs field as $\phi = \phi_0 + \eta$. Substituting that into the Lagrangian we will get a mass term for the gauge particles coming from the minimal coupling of the Higgs field to the gauge particles. Considering the η independent term in the kinetic term, one gets

$$\begin{aligned} (\mathcal{D}_\mu\phi_0)^\dagger\, \mathcal{D}^\mu\phi_0 &= e^2\phi_0^\dagger D(A_\mu)D(A^\mu)\phi_0 \\ &= \frac{1}{2\hbar^2}M_{ab}^2 A_\mu^a A^{\mu b} \end{aligned} \tag{7.13}$$

where

$$M_{ab}^2 = e^2\hbar^2\phi_0^\dagger D(\{T_a\,,\,T_b\})\phi_0 \tag{7.14}$$

For hermitian generators $T_a^\dagger = T_a$, the mass matrix M_{ab}^2 is real and symmetric, and so diagonalizable by an orthogonal transformation. Notice, that for the generators of the little group of ϕ_0 we get from (7.9) that $D(T)\phi_0 = 0$, and so the mass of the corresponding gauge particles vanishes. So, we still have a gauge theory for H_{ϕ_0} with massless gauge particles. The remaining gauge bosons acquire masses of the order of $e\hbar a$.

7.2 The case of the adjoint representation

In the discussions about duality that will follow, it will become clear that theories with Higgs field in the adjoint representation have special properties. The adjoint representation is defined by the action of the group on its Lie algebra by

$$gT_ag^{-1} = T_b d^b_a(g) \qquad d(g)d(g') = d(gg') \tag{7.15}$$

In addition for elements infinitesimally close to the identity $g \sim 1 + i\varepsilon^a T_a$ on gets from (7.15) that

$$d^c_b(T_a) = if^c_{ab}\, ; \qquad [T_a\, , T_b] = if^c_{ab}T_c \tag{7.16}$$

For compact semisimple Lie algebras one can always choose a special basis where the structure constants are totally antisymmetric, i.e. $f^c_{ab} = f_{abc}$, with $\mathrm{Tr}(T_aT_b) = \delta_{ab}$.

Since in this case the Higgs has the same number of components as the dimension of the algebra, one can define Lie algebra valued quantities

$$\phi \equiv \phi^a T_a \tag{7.17}$$

The Higgs potential (7.6) can then be written as

$$V(\phi) = \frac{\lambda}{4}\left(\mathrm{Tr}\,\phi^2 - a^2\right) \tag{7.18}$$

So, choosing a vacuum ϕ_0, with $\mathrm{Tr}\,\phi_0^2 = a^2$, one is choosing a direction in the algebra. Therefore, the little group of ϕ_0 is generated by the generator defined by ϕ_0 and all the others which commute with it. So, the little group has necessarily a $U(1)$ factor and we write it as

$$H_{\phi_0} \equiv U(1) \otimes K \tag{7.19}$$

where K is the subgroup of G commuting with ϕ_0. We choose to normalize the generator of the $U(1)$ as

$$Q = \frac{e\hbar}{a}\phi^a_0 T_a \equiv \frac{e\hbar}{a}\phi_0 \tag{7.20}$$

The mass formula (7.14) in this case becomes (the factor $\frac{1}{2}$ accounts for the fact that the adjoint Higgs is real)

$$\begin{aligned} M^2_{ab} &= \frac{1}{2}e^2\hbar^2\phi_0^\dagger d(\{T_a\, , T_b\})\phi_0 \\ &= \frac{1}{2}e^2\hbar^2\left(\left(d^\dagger(T_a)\phi_0\right)^\dagger (d(T_b)\phi_0) + \left(d^\dagger(T_b)\phi_0\right)^\dagger (d(T_a)\phi_0)\right) \\ &= e^2\hbar^2 d_{ac}(\phi_0)d_{cb}(\phi_0) \\ &\equiv a^2 Q^2_{ab} \end{aligned} \tag{7.21}$$

where we have used the fact that, as a consequence of (7.16) and (7.17) one has $(d(T_a)\phi_0)_b = d_{ab}(\phi_0)$, and where we have introduced, following (7.20),

$$Q_{ab} \equiv d_{ab}(\frac{e\hbar}{a}\phi_0) \tag{7.22}$$

We then conclude that the masses of the gauge bosons are proportional to the eigenvalues (charges) of the $U(1)$ generator. In fact, we have the elegant mass formula for gauge bosons

$$M = a \mid q \mid \tag{7.23}$$

where q stand for their $U(1)$ charges. Such formula will have an important role in the duality conjectures which we will discuss.

8 Finite energy classical solutions

The energy density, or the Θ_{00} component of the energy momentum tensor, for the theory (7.1) is

$$\Theta_{00} = \frac{1}{2}\left((E_i^a)^2 + (B_i^a)^2 + (\Pi^a)^2 + ((\mathcal{D}_i\phi)^a)^2\right) + V(\phi) \tag{8.1}$$

where

$$E_i^a = -F_{0i}^a \; ; \qquad B_i^a = -\frac{1}{2}\varepsilon_{ijk}F_a^{jk} \; ; \qquad \Pi_a = ((\mathcal{D}_0\phi)_a \tag{8.2}$$

We are interested in classical solutions that have finite energy. Therefore, in order for $\int d^3x\Theta_{00}$ to converge, we need Θ_{00} to vanish fast enough as $\mid x \mid\rightarrow \infty$. So, the field should approach some vacuum configuration as $\mid x \mid\rightarrow \infty$. In the example of the monopole we discuss how fast the fields should reach the vacuum. So, in a finite energy solution the Higgs field should belong to M_0 defined in (7.7). Taking some sphere S^2 of radius sufficiently large (at spatial infinity) one observes a given solution defines a map from S^2 to M_0

$$\phi \; : \quad S^2 \rightarrow M_0 \tag{8.3}$$

However, a given fixed sphere S^2 can not be given a physical meaning. Therefore, if we are to get some physical interpretation out of these maps, we should consider equivalent those maps that can be continously deformed one into the other, when the sphere S^2 is changed but still kept at spatial infinity. So, we are interested in the homotopy classes of the maps (8.3). Since, we are mapping two dimensional spheres, those constitute the second homotopy classes of M_0, $\tilde{\Pi}_2(M_0)$. The finite energy solutions of the theory (7.1) can then be classified according to $\tilde{\Pi}_2(M_0)$. In addition, we will see that these homotopy classes characterize quantities which are conserved in time independently of the equations of motion, i.e. they constitute topological charges. Let us discuss an explicit example.

Consider the theory (7.1) with gauge group $G = SO(3)$ and the Higgs field in the triplet (adjoint) representation. We then have, $[T_a\, , T_b] = i\varepsilon_{abc}T_c$, and so from (7.16) $d_{ac}(T_b) = i\varepsilon_{abc}$. Therefore, the covariant derivative (7.3) becomes

$$(\mathcal{D}_\mu\phi)_a = \partial_\mu\phi_a - e\varepsilon_{abc}A_\mu^b\phi^c \tag{8.4}$$

Since the Higgs and gauge fields are three vectors w.r.t. group indices, we use a vectorial notation and write $\varepsilon_{abc}v^bu^c \equiv (\vec{v}\wedge\vec{u})_a$. One can easily check that the configurations [29]

$$\vec{A}_\mu = \frac{1}{a^2e}\vec{\phi}\wedge\partial_\mu\vec{\phi} + \frac{1}{a}\vec{\phi}B_\mu \; ; \qquad \qquad \vec{\phi}^2 = a^2 \tag{8.5}$$

where B_μ is an arbitrary four vector, satisfy the vacuum conditions (7.7).

Notice that the choice of a fixed Higgs vacuum $\vec{\phi}_0$, with $\vec{\phi}_0^2 = a^2$, breaks the gauge symmetry from $SO(3)$ down to $SO(2)$ (or $U(1)$), i.e. the theory is now invariant under rotations only around the axis defined by $\vec{\phi}_0$. In addition, the Higgs vacuum M_0, defined in (7.8), is the sphere S^2, defined by $\vec{\phi}^2 = a^2$, Since any two points of S^2 can be joined by an $SO(3)$ rotation, we see M_0 is just one orbit of the gauge group. So, from (7.8), (7.9) and (7.12) we have

$$G = SO(3) \;\rightarrow\; H_{\phi_0} = SO(2) \equiv U(1) \;; \qquad\qquad M_0 = S^2 \equiv SO(3)/SO(2) \tag{8.6}$$

One then observes that the term $\frac{1}{a}\vec{\phi}B_\mu$ in (8.5) lies in the algebra of the little group $U(1)$. The field tensor in the vacuum becomes

$$\vec{F}_{\mu\nu} = \partial_\mu \vec{A}_\nu - \partial_\nu - e\vec{A}_\mu \wedge \vec{A}_\nu = \frac{1}{a}\vec{\phi}G_{\mu\nu} \tag{8.7}$$

where $\vec{A_\mu}$ is given by (8.5), and

$$G_{\mu\nu} = \frac{1}{a^3 e}\vec{\phi}\cdot\left(\partial_\mu\vec{\phi}\wedge\partial_\nu\vec{\phi}\right) + \partial_\mu B_\nu - \partial_\nu B_\mu \tag{8.8}$$

As a consequence of the equations of motion one has

$$\partial_\mu G^{\mu\nu} = 0 \;; \qquad\qquad \partial_\mu \tilde{G}^{\mu\nu} = 0 \tag{8.9}$$

where

$$\tilde{G}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}G^{\rho\sigma} \tag{8.10}$$

Therefore, we have Maxwell's equations holding true in the Higgs vacuum. Such Maxwell theory is associated to the $U(1)$ subgroup, the little group of the vacuum.

Consider now the magnetic flux through a closed surface Σ at the spatial infinity, and so, in the Higgs vacuum. We have

$$g_\Sigma = \int_\Sigma \vec{B}\cdot d\vec{s} = -\frac{1}{2}\varepsilon_{ijk}\int_\Sigma G^{jk}ds^i = -\frac{1}{2a^3 e}\int_\Sigma \varepsilon_{ijk}\vec{\phi}\cdot\left(\partial^j\vec{\phi}\wedge\partial^k\vec{\phi}\right)ds^i \tag{8.11}$$

since the contribution from B_μ vanishes by Stoke's theorem. Now, one can show that g_Σ is invariant under continuous variations of $\vec{\phi}$ as long as it is kept in the Higgs vacuum, i.e.

$$\vec{\phi}' = \vec{\phi} + \delta\vec{\phi}\;; \qquad \text{with} \qquad \vec{\phi}\cdot\delta\vec{\phi} = 0 \tag{8.12}$$

In evaluating the variation of the integrand in (8.11), one should use the fact that, since $\vec{\phi}$ is a unit vector in a three dimensional space, $\partial^j\vec{\phi}\wedge\partial^k\vec{\phi}$ and $\delta\vec{\phi}\wedge\partial^k\vec{\phi}$ are parallel to $\vec{\phi}$ and so orthogonal to $\delta\vec{\phi}$ or $\partial^j\vec{\phi}$. Therefore, after integrating by parts one gets

$$\delta\left(\vec{\phi}\cdot\left(\partial^j\vec{\phi}\wedge\partial^k\vec{\phi}\right)\right) = \partial^j\left(\vec{\phi}\cdot\left(\delta\vec{\phi}\wedge\partial^k\vec{\phi}\right)\right) - \partial^k\left(\vec{\phi}\cdot\left(\delta\vec{\phi}\wedge\partial^j\vec{\phi}\right)\right) \tag{8.13}$$

But that vanishes due to Stokes' theorem.

Therefore, any small variation of $\vec{\phi}$ inside the Higgs vacuum does not change g_Σ, and so g_Σ depends on the homotopy classes of the mapping $\Sigma \to M_0$. Such invariance of g_Σ can be extended to any change in $\vec{\phi}$ that can be built from small variations. The time evolution of the solution provokes smooth variations in $\vec{\phi}$. However, if Σ is kept far away the variations of $\vec{\phi}$ on Σ will not take it out of the Higgs vacuum. Consequently, if Σ is sufficiently large we observes that g_Σ is time independent. In addition, smooth gauge transformations also provokes smooth changes in $\vec{\phi}$, and so should not change g_Σ. Therefore, g_Σ is a gauge invariant and conserved charge.

Since g_Σ depends on the homotopy classes of the mapping $\Sigma \to M_0$, we conclude that it is quantized. The reason is that in the case under discussion the homotopy classes are given by the integers, since $\Sigma \sim M_0 \sim S^2$, and $\tilde{\Pi}_2(S^2) = \mathbb{Z}$. The integer labeling the homotopy classes gives the widing number, i.e. the number of times one sphere wraps around the other under the mapping $\Sigma \to M_0$. In fact, we have

$$g_\Sigma = -\frac{4\pi N}{e} \; ; \qquad N \equiv \frac{1}{8\pi a^3} \int_\Sigma \varepsilon_{ijk} \vec{\phi} \cdot \left(\partial^j \vec{\phi} \wedge \partial^k \vec{\phi} \right) ds^i \tag{8.14}$$

and N is an integer that counts the number of times Σ covers M_0. It is the widing number, the so called Brouwer degree or the Poincaré-Hopf index of the map.

The smallest unit of electric charge in the theory (7.1) is $q = \frac{1}{2} e\hbar$, and therefore

$$q \, g_\Sigma = -2\pi N \hbar \tag{8.15}$$

Therefore, we get the Dirac's quantization rule from the topology of classical solutions.

Notice that the above considerations lead us to the conclusion that a necessary condition for the existence of solutions carrying magnetic charges is that the second homotopy group of the Higgs vacuum should be non trivial, i.e.

$$\tilde{\Pi}_2(M_0) \text{ non trivial} \quad \to \quad \text{necessary condition for monopole solutions} \tag{8.16}$$

8.1 The 't Hooft-Polyakov monopole solution

Consider the theory (7.1) for the gauge group $SO(3)$ with Higgs field in the triplet (adjoint) representation. We want a static, finite energy solution which looks like a monopole seen from large distances. Since we want a static configuration we break the Lorentz symmetry down to the spatial rotations. The fact it is localized in space it breaks the translation symmetry. In addition, we can not have the solution invariant under rotations and gauge transformations at the same time. The reason is that invariance under the $SO(3)$ gauge group forces the Higgs field to vanish everywhere, and therefore we do not get finite energy since ϕ does not approach the Higgs vacuum at spatial infinity. The invariance under the $SO(3)$ rotation group forces ϕ to be constant asymptotically and therefore the mapping $\Sigma \to M_0$, discussed above, will be trivial, and so the magnetic charge vanishes. Consequently, we look for solutions invariant under the diagonal $SO(3)$, i.e.

$$J \equiv -i\vec{r} \wedge \vec{\nabla} + T \tag{8.17}$$

where $-i\vec{r}\wedge\vec{\nabla}$ generates the spatial rotations and T are the generators of the $SO(3)$ gauge group.

In 1974 't Hooft [3] and Polyakov [4] constructed a static solution with such properties, and the appropriate ansatz is (we use $a, b, c = 1, 2, 3$ as $SO(3)$ gauge indices, and $i, j, k = 1, 2, 3$ as space indices)

$$\begin{aligned} \phi_a &= \frac{r_a}{er^2}H(\xi) \\ A_i^a &= -\varepsilon_{aij}\frac{r^j}{er^2}(1-K(\xi)) \\ A_0^a &= 0 \end{aligned} \tag{8.18}$$

where $\xi \equiv \langle\phi\rangle er$, with $\langle\phi\rangle \equiv a$ being the vacuum expectation value of the Higgs field (see (7.6)).

The equations of motion for the theory (7.1) becomes

$$\begin{aligned} \xi^2\frac{d^2K}{d\xi^2} &= K\,H^2 - K(K^2-1) \\ \xi^2\frac{d^2H}{d\xi^2} &= 2K^2\,H + \frac{\lambda}{e^2}H(H^2-\xi^2) \end{aligned} \tag{8.19}$$

The boundary conditions for finite energy are

$$\begin{aligned} K-1 \leq O(\xi) \quad &; \quad H \leq O(\xi) \qquad \text{for } \xi \to 0 \\ K \to 0 \quad &; \quad H \sim \xi \qquad \text{for } \xi \to \infty \end{aligned} \tag{8.20}$$

The solution was worked out numerically. However, at large distances one has

$$F_{ij}^a \sim \frac{1}{er^4}\varepsilon_{ijk}r^a\,r^k \sim \frac{1}{er^3}\varepsilon_{ijk}r^k\,\frac{\phi^a}{\langle\phi\rangle} \tag{8.21}$$

So, the only component of the field tensor that survives is in the direction of the Higgs field, as we saw in (8.7). The $U(1)$ magnetic field at large distances is then

$$B^i = -\frac{1}{e}\frac{r^i}{r^3} \tag{8.22}$$

and so, the magnetic charge is

$$g = -\frac{4\pi}{e} \tag{8.23}$$

Since, the smallest possible charge of the theory is $\frac{1}{2}e\hbar$ one has $q\,g = -2\pi\hbar$. Therefore, according to Dirac's quantization condition, the 't Hooft-Polyakov solution carries a unit magnetic charge.

The mass of the monopole is given by

$$M = \frac{4\pi\langle\phi\rangle}{e}f(\frac{\lambda}{e^2}) \tag{8.24}$$

where $f(\frac{\lambda}{e^2})$ is a slowly varying function. In fact, $f(0) = 1$, $f(0.1) = 1.1$, $f(0.5) = 1.42$, $f(10) = 1.44$, and so on.

One can check that as $r \to \infty$

$$K = O\left(\exp\left(-\frac{mr}{\hbar}\right)\right) \qquad H - \xi = O\left(\exp\left(-\frac{\mu r}{\hbar}\right)\right) \tag{8.25}$$

where $m = \langle\phi\rangle e\hbar$ and $\mu = \sqrt{e\lambda}\langle\phi\rangle\hbar$. Therefore, the size of the monopole is determined by the Compton wavelength $\frac{\hbar}{m}$ and $\frac{\hbar}{\mu}$ of the heavy particles of the theory. On the other hand, that is determined by the vacuum expectation value of the Higgs field.

In 1975 Julia and Zee [31] constructed a solution like the 't Hooft-Polyakov monopole, but it also carried an electric charge besides the magnetic charge. Following Schwinger it was called dyon solution. It has a magnetic charge $\frac{4\pi}{e}$, but the electric charge is not fixed. In fact, it appears as a Noether charge and not as a topological charge.

It is worth mentioning that other types of solitonic solutions can appear in a theory like (7.1) with the gauge symmetry spontaneously broken. For instance if the $\tilde{\Pi}_1(M_0)$ is non trivial we can have solutions with the form of a string, which due to their importance in models of galaxy formation are called cosmic string. In addition, if $\tilde{\Pi}_0(M_0)$ is non trivial one can have the so called domain walls solutions [32]

9 The Bogomolny bound on the monopole mass

We now consider the theory (7.1) with the Higgs field in the adjoint representation of the gauge group G. As we have seen in (7.19) the little group of the Higgs vacuum has a $U(1)$ component. Therefore in a finite energy solution there is, at large distances, a $U(1)$ component of the field tensor which survives. It is obtained by projecting the field tensor on the $U(1)$ generator (see (7.20) and (8.7))

$$G_{\mu\nu} = \frac{1}{a}\phi_0 \cdot F_{\mu\nu} = \frac{1}{a}\,\mathrm{Tr}\left(\phi_0 F_{\mu\nu}\right) \tag{9.1}$$

where ϕ_0 is the value of the Higgs field in the vacuum.

The magnetic charge associated to such $U(1)$ gauge group is

$$g = \int_\Sigma \vec{B} \cdot d\vec{s} = \frac{1}{a}\int_\Sigma B^j_a \phi^a_0 ds^j = \frac{1}{a}\int_\Sigma \mathrm{Tr}\left(B^j \phi_0\right) ds^j \tag{9.2}$$

where we have denoted $B^j \equiv B^j_a T_a$ and $\phi_0 \equiv \phi^a_0 T_a$, and where Σ is a closed surface at spatial infinity.

The Yang-Mills equations associated to (7.1) are

$$\mathcal{D}_\nu F^{\mu\nu} = ie[\phi\,,\,\mathcal{D}^\mu\phi] \tag{9.3}$$

$$\mathcal{D}_\nu \tilde{F}^{\mu\nu} = 0 \tag{9.4}$$

where

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \tag{9.5}$$

Using (9.4) for $\nu = 0$

$$\mathcal{D}_j B^j = 0\,; \qquad\qquad B^i = -\frac{1}{2}\varepsilon^{0ijk}F_{jk} \tag{9.6}$$

Then, using the Gauss law one gets

$$\begin{aligned}
g &= \frac{1}{a}\int_V \partial_j \,\mathrm{Tr}\left(B^j\phi_0\right) dV \\
&= \frac{1}{a}\int_V \left(\partial_j \,\mathrm{Tr}\left(B^j\phi_0\right) + ie\,\mathrm{Tr}\left(\left[A_j\,,\,B^j\phi_0\right]\right)\right) dV \\
&= \frac{1}{a}\int_V \mathrm{Tr}\left(\left(\mathcal{D}_j B^j\right)\phi_0 + B^j\mathcal{D}_j\phi_0\right) dV \\
&= \frac{1}{a}\int_V \mathrm{Tr}\left(B^j\mathcal{D}_j\phi_0\right) dV
\end{aligned} \tag{9.7}$$

By similar arguments one obtains that the electric charge is ($E^j = -F^{0j}$)

$$\begin{aligned}
q &= \int_\Sigma \vec{E}\cdot d\vec{s} \\
&= \frac{1}{a}\int_V \mathrm{Tr}\left(E^j\mathcal{D}_j\phi_0\right) dV
\end{aligned} \tag{9.8}$$

Now, the classical energy for such theory is the space integral of (8.1), and so we have

$$\begin{aligned}
\mathcal{E} &= \int_V dV\Theta_{00} \\
&\geq \frac{1}{2}\int_V dV\left(\left(E^j_a\right)^2 + \left(B^j_a\right)^2 + \left(\left(\mathcal{D}^j\phi\right)_a\right)^2\right) \\
&= \frac{1}{2}\int_V dV\left(\left(E^j_a - \left(\mathcal{D}^j\phi\right)_a\sin\theta\right)^2 + \left(B^j_a - \left(\mathcal{D}^j\phi\right)_a\cos\theta\right)^2\right. \\
&+ \left. 2E^j_a\left(\mathcal{D}_j\phi\right)_a\sin\theta + 2B^j_a\left(\mathcal{D}_j\phi\right)_a\cos\theta\right) \\
&= \frac{1}{2}\int_V dV\left(\left(E^j_a - \left(\mathcal{D}^j\phi\right)_a\sin\theta\right)^2 + \left(B^j_a - \left(\mathcal{D}^j\phi\right)_a\cos\theta\right)^2\right) \\
&+ a\left(q\sin\theta + g\cos\theta\right)
\end{aligned} \tag{9.9}$$

Therefore, for a classical solution at its rest frame (static) we get that its mass must satisfies

$$M \geq a\left(q\sin\theta + g\cos\theta\right) \tag{9.10}$$

The most stringent limit occurs for[3]

$$M \geq a\sqrt{q^2+g^2} \tag{9.11}$$

This is called the Bogomolny bound [6].

[3]Notice that the maximum of $q\sin\theta + g\sin\theta$ occurs at an angle θ_M such that $\tan\theta_M = q/g$, and therefore we have a right triangle with sides of length q, g and $\sqrt{q^2+g^2} = q\sin\theta_M + g\cos\theta_M$.

In obtaining such bound we have dropped the terms $(\mathcal{D}_0\phi)^2$ and the Higgs potential $V(\phi) = \frac{\lambda}{4}\left(\phi^2 - a^2\right)^2$ from Θ_{00}. So, if we want to saturate the Bogomolny bound (9.11) we have to take a vanishing Higgs potential. However, in order to get non trivial magnetic charge we have to break the gauge symmetry spontaneously. Therefore, we take the limit $\lambda \to 0$ but still keep $\langle\phi\rangle \to a$ as $r \to \infty$. That is called the Prasad-Sommerfield limit [7].

If we consider solutions with no electric charge then we should take $E^j_a = 0$, and so $\theta = 0, \pi$. Therefore, to saturate the Bogomolny bound (9.11), i.e.

$$M = a \mid g \mid \tag{9.12}$$

we have to impose

$$\mathcal{D}_0\phi = 0 \; ; \qquad E^j_a = 0 \; ; \qquad V(\phi) = 0 \tag{9.13}$$

and also

$$B^j_a = \pm \left(\mathcal{D}^j\phi\right)_a \tag{9.14}$$

The relations (9.14) are called the Bogomolny equations [6].

One can check that the first order differential equations (9.13) and (9.14) imply the (second order) equations of motion for the theory (7.1) with the Higgs field in the adjoint representation and vanishing Higgs potential. That resembles the case of instantons which are solutions of the self-duality equations $F_{\mu\nu} = \pm\tilde{F}_{\mu\nu}$, which are of first order and imply the Yang-Mills equations which are of second order. For that reason, the Bogomolny equations (9.14) are refered as a self-duality equation. In fact, the analogy with self-dual Yang-Mills in Euclidean space can be pushed further. Since the Higgs is in the adjoint representaion, it has the same number of components as the dimension of the group. Therefore, introduce a fictitious fifth space dimension with cordinate x_4 and introduce a fifth component of the gauge potential as [33]

$$A^a_4 \equiv \phi^a \tag{9.15}$$

Then, since nothing depends upon x_4, we have

$$\mathcal{D}_\mu\phi = \partial_\mu A_4 + ie[A_\mu \, , \, A_4] = \partial_\mu A_4 - \partial_4 A_\mu + ie[A_\mu \, , \, A_4] = F_{\mu 4} \tag{9.16}$$

and $(\alpha = 0, 1, 2, 3, 4)$

$$\mathcal{D}_\mu\mathcal{D}^\mu\phi = -\mathcal{D}_\mu F^{\mu 4} = -\mathcal{D}_\alpha F^{\alpha 4} \tag{9.17}$$

In addition

$$\mathcal{D}_\mu F^{\mu\nu} + ie[A_4 \, , \, F^{4\nu}] = \mathcal{D}_\mu F^{\mu\nu} + \mathcal{D}_4 F^{4\nu} = \mathcal{D}_\alpha F^{\alpha\nu} \tag{9.18}$$

Therefore, the equations

$$\mathcal{D}_\alpha F^{\alpha\beta} = 0 \; ; \qquad \alpha, \beta = 0, 1, 2, 3, 4 \tag{9.19}$$

are the same as the the equations of motion of the theory (7.1) for vanishing Higgs potential

$$\begin{aligned} \mathcal{D}_\nu F^{\mu\nu} &= ie[\phi \, , \, \mathcal{D}^\mu\phi] \\ \mathcal{D}^\mu\mathcal{D}_\mu\phi &= 0 \end{aligned} \tag{9.20}$$

Introducing the dual

$$f^{\alpha\beta\gamma} \equiv \frac{1}{2}\varepsilon^{\alpha\beta\gamma\sigma}F_{\gamma\sigma} \tag{9.21}$$

we get the identity

$$\mathcal{D}_\alpha f^{\alpha\beta\gamma} = 0 \tag{9.22}$$

If we now impose

$$F^{\alpha\beta} = f^{\alpha\beta\gamma}v_\gamma \tag{9.23}$$

for some constant vector v_γ we get that (9.19) is automatically satisfied. Choosing

$$v = (1,0,0,0,0) \tag{9.24}$$

one gets that (9.23) becomes

$$F^{mn} = \tilde{F}^{mn} \; ; \qquad F^{0m} = 0 \; ; \qquad m,n = 1,2,3,4 \tag{9.25}$$

where $\tilde{F}^{mn}\frac{1}{2}\varepsilon^{mnpq}F_{pq}$. But (9.25), together with $V(\phi) = 0$ are equivalent to (9.13) and (9.14). So, the Bogomolny equations are mathematically very close to the self-dual Yang-Mills equations.

9.1 The BPS monopole

Using the ansatz (8.18) one can check that the equations (9.13) and (9.14) are equivalent to (for time independent configurations)

$$\xi\frac{dK}{d\xi} = -K\,H \; ; \qquad \xi\frac{dH}{d\xi} = H - (K^2 - 1) \tag{9.26}$$

The solution found by Prasad and Sommerfield in 1975 is quite simple

$$H = \xi\cosh\xi - 1 \; ; \qquad K = \frac{\xi}{\sinh\xi} \tag{9.27}$$

Such solution corresponds to a monopole with magnetic charge $g = \frac{4\pi}{e}$ and mass $M = a \mid g \mid$. Contrary to the 't Hooft-Polyakov monopole, this is an exact solution. These solutions are called the BPS (Bogomolny-Prasad-Sommerfield) or self-dual monopoles.

At large distances we have

$$H(\xi) - \xi = 1 + O(\exp(-\xi)) \;\; ; \qquad \xi \to \infty \tag{9.28}$$

and therefore

$$\phi^a \to \langle\phi\rangle\frac{r^a}{r} + \frac{r^a}{er^2} + \ldots \tag{9.29}$$

So, the Higgs field approaches the vacuum at large distance much slower than in the 't Hooft-Polyakov monopole solution. That has to do with the fact that, although we have spontaneous breakdown of the gauge symmetry, generating mass for some gauge fields, the Higgs field itself is massless. Therefore, the BPS monopoles, contrary to the 't Hooft-Polyakov monopole, are not localized in space.

The fact that the photon and the Higgs are massless has an interesting consequence. The Higgs field intermediate a Coulomb type interaction between the charged gauge bosons which is always attractive, irrespective of the relative sign of their charges (as it is always true for interactions intermediated by spin zero particles). Since the Higgs couples to the gauge bosons with the same strength as the photon, it follows that it exactly cancels the repulsion, intermediate by the photon, between gauge bosons of the same charge, and doubles the attraction between bosons of opposite charges. That can be verified by calculating the scattering of gauge bosons at tree level.

An interesting result about BPS monopole dynamics was obtained by Manton in 1977 [34] by studying their scattering. He showed that BPS monopoles of opposite charge interact via a Coulomb potential when well separated. In addition, monopoles of the same charge do not interact when at rest, and monopoles of opposite charge have an attraction which is the double of that expected from the strength of their charges. Such result, compared to the static interaction of gauge particles, has reinforced the arguments in favor of a duality conjucture about the quantum theory of BPS monopoles [9, 8]. Montonen and Olive [8] have conjectured that the theory describing the dynamics of such monopoles is a gauge theory where the massive gauge bosons are the BPS monopoles, with the gauge coupling e replaced by its inverse. The massive gauge bosons would appear in the dual theory as solitons, like the BPS monopoles have risen in the original theory. In addition, the photon and the Higgs would be the same in both theories. The ideas concerning such conjecture have evolved a lot since then, and are the subject of the series of lectures by David Olive in this School [1].

One can also construct BPS dyon solutions. Their masses saturate the Bogomolny bound (9.11), and so are given $M = a\sqrt{q^2+g^2}$. The electric charge is given by $q = g\tan\theta = -\frac{4\pi}{e}\tan\theta$. Therefore, like the Julia-Zee dyons the electric charge of the BPS dyons are not fixed but can vary in a continuous range.

9.2 The Bogomolny moduli

Manton's result [34] mentioned above, allows a nice interpretation of the number of parameters entering in a N-monopole solution. The solution for one monopole has 4 parameters, where 3 of them account for its position and 1 for the $U(1)$ gauge symmetry (one can always rotate the solution in the internal space, since $U(1)$ is still a gauge symmetry of the theory). According to Manton, BPS monopoles of the same charge do not interact when at rest. Therefore, one can find static N-monopole solutions for the Bogomolny equations. The number of parameters of such solution is $4N$, with $3N$ to account for their positions and N for their $U(1)$ phases. One can in fact eliminate one parameter which correspond to a global rotation of the solution. Only the relative $U(1)$ phases are relevant. So, the N-monopole solution has in fact, $4N-1$ parameters.

Such result can be obtained more rigorously by studying the Bogomolny equations directly [35]. The Bogomolny equations are (9.14)

$$B_i^a = (\mathcal{D}_i\phi)^a \quad ; \qquad\qquad B_i^a = \frac{1}{2}\varepsilon_{ijk}F_{jk}^a \tag{9.30}$$

where $i, j, k = 1, 2, 3$ and $a = 1, 2, 3$ are space and $SO(3)$ indices respectively.

We want to determine the number of physical zero modes about an arbitrary BPS solution with magnectic charge $4\pi N$. In other words, we want to find the deformations of the solutions which still mantain them as solutions of (9.30). We denote

$$\phi \equiv \phi^a T_a \qquad A_i \equiv A_i^a T_a \qquad B_i \equiv B_i^a T_a \tag{9.31}$$

where T_a are the generators of $SO(3)$.

The equation (9.30) then reads

$$\partial_i \phi + [A_i\, ,\, \phi] = \varepsilon_{ijk} \left(\partial_j A_k + \frac{1}{2} [A_j\, ,\, A_k] \right) \tag{9.32}$$

Given a solution (A, ϕ), if we want $(A + \delta A, \phi + \delta\phi)$ to be a solution too, we require

$$\mathcal{D}_i \delta\phi - [\phi\, ,\, \delta A_i] - \varepsilon_{ijk} \mathcal{D}_j \delta A_k = 0 \tag{9.33}$$

Among the solutions of (9.33) there are many which simply correspond to gauge tranformations. These are excluded by imposing the background gauge condition (Gauss law)

$$\mathcal{D}_i \delta A_i + [\phi\, ,\, \delta\phi] = 0 \tag{9.34}$$

We now introduce

$$\Psi = \mathbb{1} \delta\phi + i\sigma_j \delta A_j \tag{9.35}$$

and consider the relation

$$\triangle \Psi \equiv \left(-i\sigma_j \mathcal{D}_j + \mathbb{1}\phi \right) \Psi = 0 \tag{9.36}$$

One can easily check that

$$\triangle \Psi = i\sigma_j \left(-\mathcal{D}_j \delta\phi + [\phi\, ,\, \delta A_j] + \varepsilon_{ikj} \mathcal{D}_i \delta A_k \right) + \mathbb{1} \left([\phi\, ,\, \delta\phi] + \mathcal{D}_j \delta A_j \right) \tag{9.37}$$

and so, (9.36) is equivalent to (9.33) and (9.34) together.

Therefore, the problem of finding the number of parameters of a N-monopole solution of the Bogomolny equation has been formulated as that of finding the zero modes of the Dirac like operator $\triangle = -i\sigma_j \mathcal{D}_j + \phi$. Eric Weinberg [35] has solved that problem and has shown the number of parameters is indeed $4N - 1$. The formulation of the problem in terms of such Dirac like operator links to index theorems [36].

10 The Witten effect

We now discuss the relationship between the allowed values of the electric charge of dyons and CP non conserving theories. We follow the arguments of Witten in [37].

Consider the Schwinger-Zwanziger quantization condition for dyons (4.7). Since there are electrons in Nature of charge $(e, 0)$, then (4.7) applied to a dyon of charges (q, g) implies that $g = \frac{2\pi n\hbar c}{e}$, and so the magnetic charge of the dyon is quantized. However, nothing is said about its electric charge. Consider now two dyons of minimum magnetic charge $g_0 = \frac{2\pi\hbar c}{e}$, (q, g_0) and (q', g_0). Then (4.7) implies that $q - q' = ne$. Therefore the difference of the electric charges of the two dyons must be a multiple of the electron charge. Now, if CP is a symmetry of Nature, the existence of a dyon of charges (q, g) implies the existence of another dyon of charges $(-q, g)$, since under CP, $q \to -q$ and $g \to g$. Then, (4.7) for these two dyons implies $q = \frac{n}{2}e$. So, if CP is a symmetry we get $q - q' = ne$ and $q = \frac{n}{2}e$, and consequently $q = me$ or $q = \left(m + \frac{1}{2}\right)e$. Therefore, all dyons have either integer or half-integer electric charges, in units of the electron charge.

In order to study the effect of CP violation on the dyon electric charge, consider the theory (7.1) with gauge group $SO(3)$ and Higgs field in the adjoint (triplet) representation, and add the CP violating term to the Lagrangean

$$\theta \frac{e^2}{32\pi^2} \text{Tr}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right) \quad ; \qquad \tilde{F}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{10.1}$$

Such term is a total divergence (see (2.13)) and consequently does not affect the classical equations of motion. Therefore, the classical monopole and dyon solutions of such theory is the same as the ones we have been considering so far. However, the physics of the system is changed since (10.1) implies CP is not conserved and the parameters θ gives a measure of that violation. Ref. [38] contains a semi-classical analysis of the dyon charge without the term (10.1). Their result is that in a gauge in which the gauge field vanishes at infinity, the classical dyon solution is periodic in time. The semi-classical quantization condition is that $S + ET = 2\pi n$, where S is the action per period, E the energy and T the period. But since $S = TI$, where I is the action per unit of time one gets $T(I + E) = 2\pi n$. Their explict calculations gives $I + E = kq^2$ and $T = \frac{2\pi}{ek}\frac{1}{q}$, where q is the dyon electric charge and k a calculable constant, and therefore

$$q = ne \tag{10.2}$$

Now let us take into account the CP violating term (10.1). Since the classical equations of motion do not change T and E are the same as before, i.e. as in [38]. However, I gets a contribution from (10.1). It can be shown that

$$I + E = kq^2 + keq\frac{\theta}{2\pi} \tag{10.3}$$

with the same k as before. Then $T(I + E) = 2\pi n$ implies

$$q = ne + \frac{\theta}{2\pi}e \tag{10.4}$$

So, the dyon electric charge is not really integer but depends on the parameter θ. In fact, if $\theta \neq 0$ there is no neutral dyon.

11 Monopoles in $N = 2$ supersymmetric gauge theories

The $N = 2$ supersymmetric gauge theory has a gauge potential A^a_μ, a Dirac spinor χ^a and two real scalar fields P^a and S^a [39]. All fields transform under the adjoint representaion of gauge group G. We denote by T_a, $a = 1, 2, \ldots \dim G$, the generators of G, and $[T_a \, , \, T_b] = f_{abc} T_c$, $\mathrm{Tr}\,(T_a T_b) = \delta_{ab}$. Writing $A_\mu = A^a_\mu T_a$, $\chi = \chi^a T_a$, $P = P^a T_a$ and $S = S^a T_a$, we get that the Lagrangean is given by

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4}\,\mathrm{Tr}\,(F_{\mu\nu})^2 + \frac{1}{2}\,\mathrm{Tr}\,(\mathcal{D}_\mu P)^2 + \frac{1}{2}\,\mathrm{Tr}\,(\mathcal{D}_\mu S)^2 + i\,\mathrm{Tr}\,(\bar{\chi}\gamma^\mu \mathcal{D}_\mu \chi) \\
&- \frac{1}{2} e^2\,\mathrm{Tr}\,([S\, , \, P])^2 + e\,\mathrm{Tr}\,(\bar{\chi}\gamma_5 [\chi\, , \, P]) + ie\,\mathrm{Tr}\,(\bar{\chi}[\chi\, , \, S])
\end{aligned} \tag{11.1}
$$

The Lagrangean (11.1) is invariant (up to total derivatives) under the supersymmetric transformations

$$
\begin{aligned}
\delta A^a_\mu &= i\,(\bar{\epsilon}\gamma_\mu \chi - \bar{\chi}\gamma_\mu \epsilon) \\
\delta P^a &= \bar{\epsilon}\gamma_5 \chi - \bar{\chi}\gamma_5 \epsilon \\
\delta S^a &= i\,(\bar{\epsilon}\chi - \bar{\chi}\epsilon) \\
\delta \chi^a &= \left(\sigma^{\mu\nu} F^a_{\mu\nu} - \gamma_\mu \left(\mathcal{D}_\mu S\right)^a + i\gamma_\mu \left(\mathcal{D}_\mu P\right)^a \gamma_5 - ie f^{abc} P^b S^c \gamma_5\right) \epsilon
\end{aligned} \tag{11.2}
$$

where the parameter of the transformation ϵ is a Grassmann Dirac spinor. That is a $N = 2$ supersymmetry because the Dirac spinor can be seen as two Majorana spinor.

We now want to construct time independent classical solutions of the theory (11.1) which constitute supersymmetric generalizations of magnetic and dyon solutions of the theory (7.1) in the limit of Prasad-Sommerfield. The equation for the fermion χ is linear in χ and so we look for solutions with $\chi = 0$. Solutions with $\chi \neq 0$ can then be obtained by applying the supersymmetry transformation (11.2). Considering the case $G = SO(3)$, we take the following ansatz, $a = 1, 2, 3$, (compare with (8.18))

$$
\begin{aligned}
A^a_0 &= \frac{x^a J(r)}{e r^2} \; ; \qquad A^a_i = \varepsilon_{iab} \frac{x^b}{e r^2} \left(K(r) - 1\right) \\
S^a &= \frac{x^a}{e r^2} H(r) \; ; \qquad P^a = \frac{x^a}{e r^2} L(r) \; ; \qquad \chi = 0
\end{aligned} \tag{11.3}
$$

The equations of motion take then the form (compare with (8.19))

$$
\begin{aligned}
r^2 H''(r) &= 2HK^2 \\
r^2 L''(r) &= 2LK^2 \\
r^2 J''(r) &= 2JK^2 \\
r^2 K''(r) &= K\left(K^2 - 1\right) + K\left(H^2 + L^2 - J^2\right)
\end{aligned} \tag{11.4}
$$

The solution is given by

$$
\begin{aligned}
J(r) &= \alpha h(r) \; ; \qquad H(r) = \beta h(r) \; ; \qquad L(r) = \gamma h(r) \\
K(r) &= \frac{Cr}{\sinh Cr} \; ; \qquad h(r) = Cr \coth Cr - 1
\end{aligned} \tag{11.5}
$$

with $\beta^2 + \gamma^2 - \alpha^2 = 1$. Notice that it is an exact solution and it is very similar to the BPS monopole solution (9.27). However, (11.5) is a dyon solution since it also has an electric charge.

The gauge symmetry is broken by the potential $\text{Tr}\left([S\,,\,P]\right)^2$, from $SO(3)$ down to $SO(2)$ or $U(1)$, i.e. the little group of S and P, which are rotations around $n^a \equiv \frac{x^a}{r}$. Notice that S and P are parallel in group space in the solution (11.5), and so $\text{Tr}\left([S\,,\,P]\right)^2 = 0$. The difference here is that S and P, which play the role of Higgs fields do not have to have a fixed vacuum expectation value to minimize the potential energy.

The $U(1)$ tensor field is then (compare with (8.8))

$$F_{\mu\nu} = \partial_\mu \left(n^a A^a_\nu\right) - \partial_\nu \left(n^a A^a_\mu\right) - \frac{1}{e}\varepsilon^{abc} n^a \partial_\mu n^b \partial_\nu n^c \;; \qquad n^a = \frac{x^a}{r} \tag{11.6}$$

One can then check that the magnetic and electric charges of the solution are

$$g = -\frac{4\pi}{e} \;; \qquad q = -\frac{4\pi}{e}\alpha \tag{11.7}$$

The mass of such dyon is obtained by integrating the Θ_{00} component of the stress tensor, and it is

$$M = \frac{4\pi}{e^2} C \left(1 + \alpha^2\right) \tag{11.8}$$

using the asymptotic value at spatial infinity of S and P one gets

$$\sqrt{S^2 + P^2} \equiv v = \frac{C}{e}\sqrt{1+\alpha^2} \tag{11.9}$$

One then gets

$$M = v\sqrt{q^2 + g^2} \tag{11.10}$$

Therefore the solution (11.5) saturates the Bogomolny bound (9.11) for the mass.

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