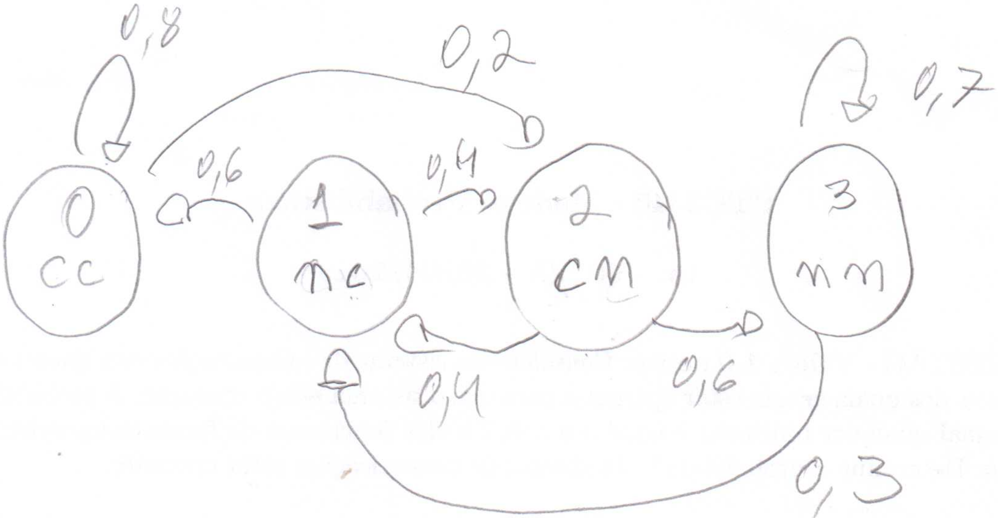
1º Q)



$$P = \begin{bmatrix} 0,8 & 0 & 0,2 & 0 \\ 0,6 & 0 & 0,4 & 0 \\ 0 & 0,4 & 0 & 0,6 \\ 0 & 0,3 & 0 & 0,7 \end{bmatrix}$$

2º 3º → 3º 4º 1
3º 4º → 4º 5º 2
4º 5º → 5º 6º 3
5º 6º → 6º Sab 4

a) $X_0 = \underset{1}{\overset{2^\circ\ 3^\circ}{nc}}$

$$P^4 = \begin{array}{c} \begin{array}{cccc} cc & nc & cn & nn \end{array} \\ \begin{bmatrix} 0,508 & 0,118 & 0,152 & 0,222 \\ 0,456 & 0,136 & 0,144 & 0,264 \\ 0,354 & 0,165 & 0,136 & 0,345 \\ 0,333 & 0,1725 & 0,132 & 0,3625 \end{bmatrix} \end{array}$$

$P(\text{chover Sabado} \mid \text{não chover } 2^\circ, \text{chover } 3^\circ) =$

$0,456 + 0,136 = 0,592$

b) $\pi_0 = 0,8\pi_0 + 0,6\pi_1$

$\pi_1 = 0,4\pi_2 + 0,3\pi_3$

$\pi_2 = 0,2\pi_0 + 0,4\pi_1$

$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$

$$\Rightarrow \begin{bmatrix} 0,2 & -0,6 & 0 & 0 \\ 0 & 1 & -0,4 & -0,3 \\ -0,2 & -0,4 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\pi_0 = 0,4286,\ \pi_1 = 0,1429,\ \pi_2 = 0,1429,\ \pi_3 = 0,2857$

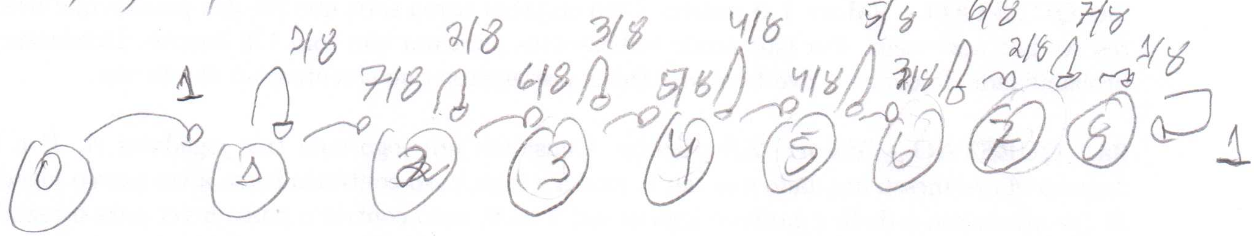
$\pi_0 + \pi_2 =$ % chuva $= \pi_0 + \pi_1 = 57,14\%$

2ªQ) X_n = número de urnas não vazias depois de de n bolas distribuídas

$X_n \in \{0, 1, \dots, 8\}$

$P_{ii} = \frac{i}{8} = 1 - P_{i\,i+1}$

$i = 0, 1, \dots, 8$



$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{8} & \frac{7}{8} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{8} & \frac{6}{8} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{8} & \frac{5}{8} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{8} & \frac{4}{8} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5/8 & 3/8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6/8 & 2/8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7/8 & 1/8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Probabilidade desejada

$$p_{03}^{(9)} = p_{13}^{(8)} \quad (\text{já que } p_{01} = 1)$$

Calculando $p_{13}^{(8)} = 0{,}00756 = 0{,}756\%$ //

3ºQ)
$$\begin{cases} \pi_1 = 0{,}45\pi_1 + 0{,}05\pi_2 + 0{,}01\pi_3 \\ \pi_2 = 0{,}48\pi_1 + 0{,}7\pi_2 + 0{,}5\pi_3 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

$$\begin{pmatrix} 0{,}55 & -0{,}05 & -0{,}01 \\ -0{,}48 & 0{,}3 & -0{,}5 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\pi_1 = 0{,}0624 = 6{,}24\% \rightarrow$ alta

$\pi_2 = 0{,}6234 = 62{,}34\% \rightarrow$ média

$\pi_3 = 0{,}3142 = 31{,}42\% \rightarrow$ baixa

4ºQ) $1 \rightarrow$ ganha $\quad p_{11} = 0{,}8$

$2 \rightarrow$ perde $\quad p_{21} = 0{,}3$

$$P = \begin{bmatrix} 0{,}8 & 0{,}2 \\ 0{,}3 & 0{,}7 \end{bmatrix}$$

$$P(\partial) = \underbrace{P(\partial|G)}_{0{,}7}\pi_1 + \underbrace{P(\partial|P)}_{0{,}2}\pi_2$$

$$\begin{cases}\pi_1 = 0{,}8\,\pi_1 + 0{,}3\,\pi_2 \\ \pi_1 + \pi_2 = 1\end{cases} \Rightarrow \begin{bmatrix}0{,}2 & -0{,}3\\ 1 & 1\end{bmatrix}\begin{bmatrix}\pi_1\\ \pi_2\end{bmatrix} = \begin{bmatrix}0\\ 1\end{bmatrix}$$

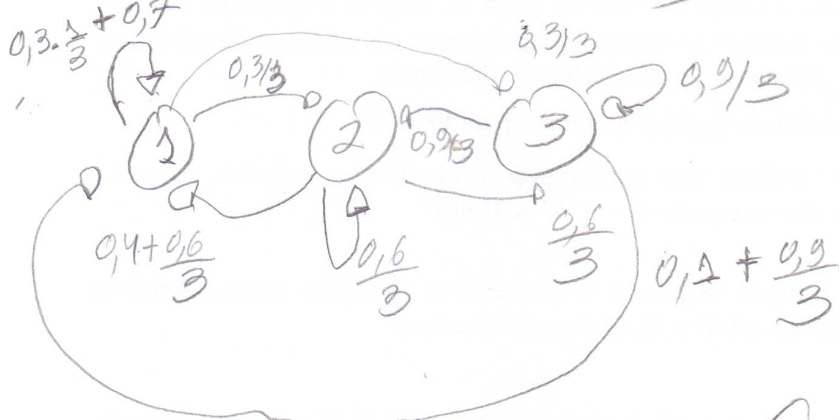
$$0{,}2\,\pi_1 = 0{,}3\,\pi_2 \Rightarrow \pi_1 = \frac{3}{2}\pi_2$$

$$\frac{3}{2}\pi_2 + \pi_2 = \frac{5}{2}\pi_2 = 1 \Rightarrow \pi_2 = \frac{2}{5},\ \pi_1 = \frac{3}{5}$$

$$\%\ \text{jantar} = P(J) = 0{,}7\cdot\frac{3}{5} + 0{,}2\cdot\frac{2}{5} = \frac{1}{2}\ //$$

5ºQ) X_n = exame aplicado na n-ésima vez

$$X_n \in \{1, 2, 3\}$$



$$P = \begin{bmatrix}0{,}8 & 0{,}1 & 0{,}1\\ 0{,}6 & 0{,}2 & 0{,}2\\ 0{,}4 & 0{,}3 & 0{,}3\end{bmatrix}$$

$$\begin{cases}\pi_1 = 0{,}8\,\pi_1 + 0{,}6\,\pi_2 + 0{,}4\,\pi_3\\ \pi_2 = 0{,}1\,\pi_1 + 0{,}2\,\pi_2 + 0{,}3\,\pi_3\\ 1 = \pi_1 + \pi_2 + \pi_3\end{cases}$$

$$\pi_1 = 5/7,\ \pi_2 = \pi_3 = 1/2$$

Exame 1 → 71,5%
" 2 → 14,25%
" 3 → 14,25%

(6) $X \to$ Poisson com média λ (8)

$\lambda \to$ exponencial com média 1.

$$P(X=n) = E\left(P(X=n \mid \lambda)\right)$$

$$P(X=n \mid \lambda=z) = e^{-z}\frac{z^n}{n!}$$

$$E\left(P(X=n \mid \lambda=z)\right) = \int_0^\infty e^{-z}\frac{z^n}{n!}e^{-z}\,dz =$$

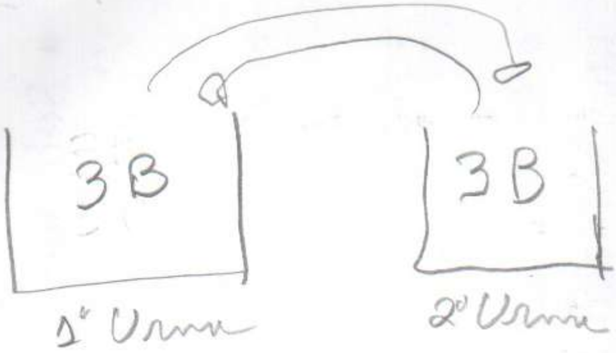
$$\int_0^\infty \frac{z^n}{n!}e^{-2z}\,dz = \frac{1}{2^{n+1}}\underbrace{\left(\int_0^\infty 2(2z)^n e^{-2z}\,dz\right)\frac{1}{n!}}_{=1 \text{ distribuição gama com parâmetros } n+1 \text{ e } 2} = \frac{1}{2^{n+1}}$$

Logo,

$$P(X=n) = \frac{1}{2^{n+1}}, \quad n = 0, 1, 2, \ldots$$

7

a)



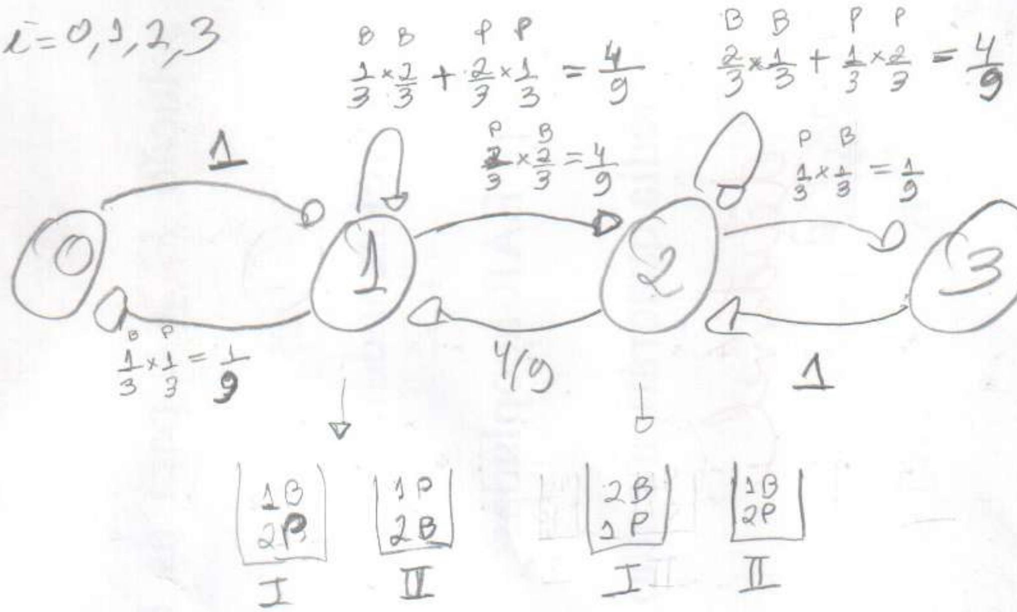
3 Brancos
3 Pretos

X_n = nº de bolas brancas na 1ª urna na $n^{\text{ésima}}$ etapa

$X_n \in \{0,1,2,3\}$

estados: i → i bolas brancas na 1ª urna

$i = 0,1,2,3$



$$P = \begin{array}{c c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/9 & 4/9 & 4/9 & 0 \\ 0 & 4/9 & 4/9 & 1/9 \\ 0 & 0 & 1 & 0 \end{bmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \end{array}$$

10

(b)

$$P^6 = \begin{array}{c|cccc|c} & 0 & 1 & 2 & 3 & \\ \hline & 0,0507 & 0,45 & 0,45 & 0,0493 & 0 \\ & 0,05 & 0,4507 & 0,4493 & 0,05 & 1 \\ & 0,05 & 0,4493 & 0,4507 & 0,05 & 2 \\ & 0,0493 & 0,45 & 0,45 & 0,0507 & 3 \end{array}$$

P(pelo menos 2 bolas brancas na 6ª rodada) =

$0,45 + 0,0507 = 0,5007$

c) Probabilidades limites $\pi' = \pi' P$

$$\pi_0 = 1/9\, \pi_1$$

$$\pi_1 = \pi_0 + \frac{4}{9}\pi_1 + \frac{4}{9}\pi_2 \qquad = 5/9\, \pi_1 + \frac{4}{9}\pi_2$$

$$\pi_2 = \frac{4}{9}\pi_1 + \frac{4}{9}\pi_2 + \pi_3$$

$$-\frac{4}{9}\pi_1 + \frac{4}{9}\pi_2 = 0 \Rightarrow \pi_1 = \pi_2$$

$$\frac{4}{9}\pi_1 - \frac{5}{9}\pi_2 + \pi_3 = 0 \Rightarrow \pi_3 = \frac{\pi_1}{9}$$

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \Rightarrow$$

$$1/9\, \pi_1 + \pi_1 + \pi_1 + 1/9\, \pi_1 = (2+9+9)\frac{\pi_1}{9} = 1 \Rightarrow$$

$$\frac{20}{9}\pi_1 = 1 \Rightarrow \pi_1 = \frac{9}{20} = \pi_2, \quad \pi_0 = \frac{1}{20} = \pi_3$$

$$\pi_0 = \pi_3 = 0{,}05$$

$$\pi_1 = \pi_2 = 0{,}45$$

$$\pi_0 = p\,\pi_0 + (1-p)\,\pi_1 \Rightarrow \pi_0 = \pi_1$$
$$\pi_0 + \pi_1 = 1 \Rightarrow \pi_0 = \pi_1 = \frac{1}{2}$$

8) $$P = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

$$P^{(n)} = \begin{bmatrix} 1/2 + 1/2(2p-1)^n & 1/2 - 1/2(2p-1)^n \\ 1/2 - 1/2(2p-1)^n & 1/2 + 1/2(2p-1)^n \end{bmatrix}$$

$n = 1$: $(1,1) \rightarrow \cancel{1/2} + \cancel{1/2}(\cancel{2}p \cancel{-1}) = p$ $\quad (1,2) \rightarrow 1/2 - 1/2(2p-1) = 1-p$

OK

$(2,1) \rightarrow 1/2 - 1/2(2p-1) = 1-p$ $\quad (2,2) \rightarrow 1/2 + 1/2(2p-1) = p$

$n \Rightarrow n+1$:

$$P^{(n+1)} = P^{(n)}P = \begin{bmatrix} 1/2 + 1/2(2p-1)^n & 1/2 - 1/2(2p-1)^n \\ 1/2 - 1/2(2p-1)^n & 1/2 + 1/2(2p-1)^n \end{bmatrix} \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix} =$$

$$\begin{bmatrix} 1/2\left[\left(1+(2p-1)^n\right)p + \left(1-(2p-1)^n\right)(1-p)\right] & 1/2\left[\left(1+(2p-1)^n\right)(1-p) + \left(1-(2p-1)^n\right)p\right] \\ 1/2\left[\left(1-(2p-1)^n\right)p + \left(1+(2p-1)^n\right)(1-p)\right] & 1/2\left[\left(1-(2p-1)^n\right)(1-p) + \left(1+(2p-1)^n\right)p\right] \end{bmatrix}$$

13

$$\begin{bmatrix} 1/2\left[1+(2p-1)(2p-1)^n\right] & 1/2\left[1-(2p-1)(2p-1)^n\right] \\ 1/2\left[1-(2p-1)(2p-1)^n\right] & 1/2\left[1+(2p-1)(2p-1)^n\right] \end{bmatrix} =$$

$$\begin{bmatrix} 1/2 + 1/2(2p-1)^{n+1} & 1/2 - 1/2(2p-1)^{n+1} \\ 1/2 - 1/2(2p-1)^{n+1} & 1/2 + 1/2(2p-1)^{n+1} \end{bmatrix}$$

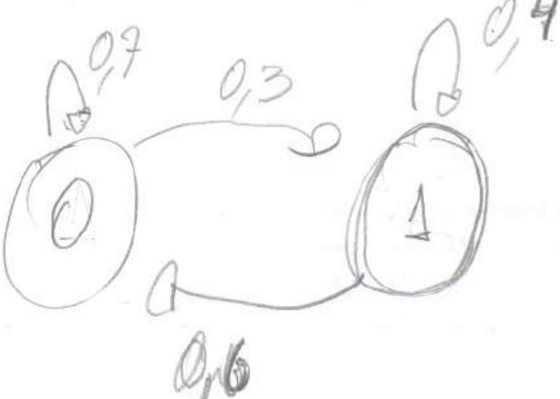
10 $\sum_i p_{ij} = 1$, todo j

$$\pi_j = \sum_i \pi_i p_{ij} \quad , \quad \sum \pi_i = 1$$

$$\pi_j = \frac{1}{M+1} \Rightarrow \begin{cases} \sum_{j=0}^{M} \pi_j = 1 \quad \text{OK} \\ \pi_j = \frac{1}{M+1} = \sum_{i=0}^{M} \pi_i p_{ij} = \frac{1}{M+1} \sum_{i=0}^{M} p_{ij} = 1 \end{cases}$$

(9) $p_1 = 0{,}7$ $\quad X_n = \begin{cases} 0 & \text{se deu cara no } n\text{ésimo dia} \\ 1 & \text{" " coroa " " "} \end{cases}$

$p_2 = 0{,}6$



Y_n = moeda escolhida no $n$ésimo dia

$$P(X_0=0)=P(X_0=1)=\frac{1}{2}$$

$$P(Y_3=1) = P(Y_3=1, X_2=0) + P(Y_3=1, X_2=1)$$

$$= P(Y_3=1 / X_2=0)\, P(X_2=0) + P(Y_3=1 | X_2=1) . P(X_2=1)$$

$$= 1 . P(X_2=0) + 0 . \cancel{P(X_2=1)} \to 0$$

$$P = \begin{bmatrix} 0{,}7 & 0{,}3 \\ 0{,}6 & 0{,}4 \end{bmatrix}, \quad P^2 = \begin{array}{c|c} \begin{array}{cc} 0 & 1 \end{array} & \\ \begin{bmatrix} 0{,}67 & 0{,}33 \\ 0{,}66 & 0{,}34 \end{bmatrix} & \begin{array}{c} 0 \\ 1 \end{array} \end{array}$$

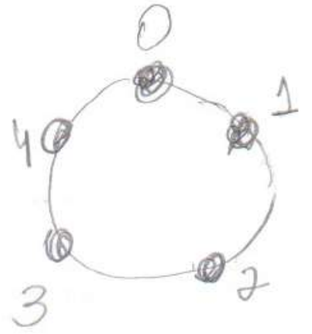
$$P(X_2=0) = P(X_2=0|X_0=0) . \overset{1/2}{P(X_0=0)} + P(X_2=0|X_0=1) . \overset{1/2}{P(X_0=1)}$$

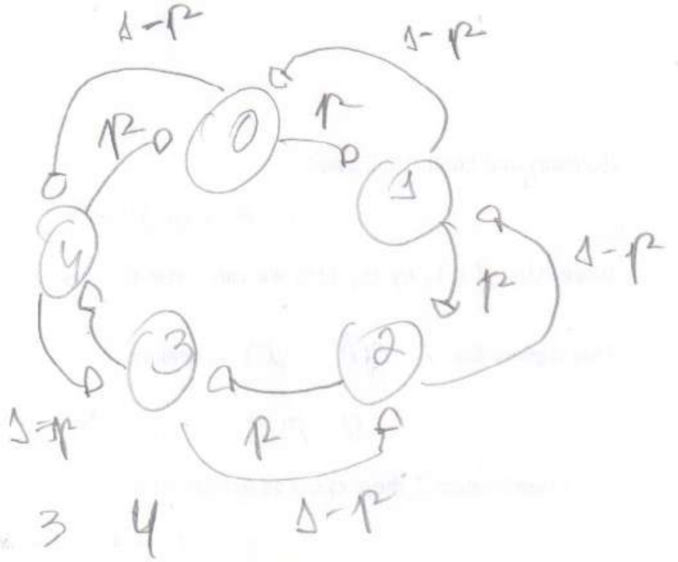
$$= \frac{1}{2}(0{,}67 + 0{,}66) = 0{,}665$$

$$P(Y_3=1) = 0{,}665$$

11

15





a)

$$P = \begin{array}{c} \\ \\ \end{array}\begin{array}{c|ccccc|c} & 0 & 1 & 2 & 3 & 4 & \\ \hline & 0 & p & 0 & 0 & 1-p & 0 \\ & 1-p & 0 & p & 0 & 0 & 1 \\ & 0 & 1-p & 0 & p & 0 & 2 \\ & 0 & 0 & 1-p & 0 & p & 3 \\ & p & 0 & 0 & 1-p & 0 & 4 \end{array}$$

b) $p = 1/2 \quad P_0^8 = \begin{array}{ccccc} 0 & 1 & 2 & & \\ \left[0{,}2734\right. & 0{,}1406 & 0{,}2227 & 0{,}2227 & \left.0{,}1406\right] \end{array}$ a linha 0

$P_{02}^8 = 0{,}2227$

c) $\pi_i = \dfrac{1}{5} = 0{,}2 \,, \quad i = 0,1,2,3,4$ pois P é duplamente estocástica

12) Y_n = soma de n dados independentes

$$Y_n = I_n \times 13 + X_n \qquad I_n = \left[\frac{Y_n}{13}\right]$$ → parte inteira da divisão

$$X_n \in \{0, 1, 2, \dots, 12\}$$

$$P(X_{n+1} = i + j \mid X_n = i) = \frac{1}{6}, \quad i = 0, 1, 2, \dots, 11$$
$$j = 1, \dots, \min\{6, 12 - i\}$$

$$P(X_{n+1} = j - (13 - i) \mid X_n = i) = \frac{1}{6}, \quad i = 7, \dots, 12$$
$$j = 13 - i, \dots, 6$$

$$P = \begin{bmatrix}
0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\
1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 \\
1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 \\
1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 \\
1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 \\
1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

~~Soma~~ dos linhas é 1

$$\sum_i p_{ij} = 1 \Rightarrow \lim_{n\to\infty} P(X_n = 0) = \frac{1}{13}$$

13 $M_i = E(Z \mid X_0 = i)$

Z = número de jogadas até acabar o jogo

$M_0 = 0, \; M_N = 0$

$M_i = E\left(E(Z \mid X_1) \mid X_0 = i\right)$

$E(Z \mid X_1 = i+1) = M_{i+1} + 1 \rightarrow$ prob p

$E(Z \mid Z_1 = i-1) = M_{i-1} + 1 \rightarrow$ prob $1-p$

$$M_i = (1 + M_{i+1})\, p + (1 + M_{i-1})(1-p) =$$

$$= 1 + p M_{i+1} + q M_{i-1}$$

(18)

$N = 4, \quad p = 1/4, \quad q = 3/4$

$$M_1 = 1 + \frac{1}{4} M_2 + \cancel{\frac{3}{4} M_0}^{\,0} = 1 + \frac{1}{4} M_2$$

$$M_2 = 1 + \frac{1}{4} M_3 + \frac{3}{4} M_1 = \frac{7}{4} + \frac{1}{4} M_3 + \frac{3}{16} M_2$$

$$M_3 = 1 + \cancel{\frac{1}{4} M_4}^{\,0} + \frac{3}{4} M_2$$

$$\begin{cases} \frac{13}{16} M_2 - \frac{1}{4} M_3 = \frac{7}{4} \quad \left(\frac{12}{13}\right) \Rightarrow \\ -\frac{3}{4} M_2 + M_3 = 1 \end{cases}$$

$$\cancel{\frac{3}{4} M_2} - \frac{3}{13} M_3 = \frac{21}{13}$$

$$\cancel{-\frac{3}{4} M_2} + M_3 = 1$$

$$\frac{10}{13} M_3 = \frac{34}{13}$$

$M_3 = 3.4, \quad M_2 = 3.2, \quad M_1 = 1.8$

Fórmula (exercício 18):

$$M_1 = \frac{1}{1/2} - \frac{4}{1/2} \frac{1-3}{1-3^4} = 2 - 8 \times \frac{2}{80} = 2 - 0.2 = 1.8$$

$$M_2 = \frac{2}{1/2} - \frac{4}{1/2} \frac{1-9}{1-3^4} = 4 - 8 \times \frac{8}{80} = 4 - 0.8 = 3.2$$

$$M_3 = \frac{3}{1/2} - \frac{4}{1/2} \frac{1-27}{1-3^4} = 6 - 8 \times \frac{26}{80} = 6 - 2.6 = 3.4$$

14

A

B

C

$P(T_A > T_B + T_C) = ?$

a) $T_A = T_B = T_C = 10 \Rightarrow P(T_A > T_B + T_C) = 0$

b) $P(T = i) = 1/3, \; i = 1,2,3$

$P(T_A > T_B + T_C) = P(T_A = 3, T_B = 1, T_C = 1) =$

$P(T_A = 3) \cdot P(T_B = 1) \cdot P(T_C = 1) = 1/27$

c) $T \rightarrow$ exponenciais com parâmetro $1/\mu$

$\lambda e^{-\lambda t} \dfrac{(\lambda t)^{n-1}}{(n-1)!}$

$Y = T_B + T_C \rightarrow$ gama com parâmetros $(2, \mu)$

$f_Y(y) = (\lambda y)\, \lambda e^{-\lambda y}$ $\qquad X = T_A \rightarrow$ exponencial

$$P(X > Y) = \int_0^\infty \int_y^\infty \lambda e^{-\lambda x} dx \left(\lambda y \lambda e^{-\lambda y}\right) dy$$

$$= \int_0^\infty e^{-\lambda y} \lambda y e^{-\lambda y} dy = \int_0^\infty y \lambda^2 e^{-2\lambda y} dy$$

$$= \frac{\cancel{\lambda}}{2} \underbrace{\int_0^\infty y (2\lambda) e^{-2\lambda y} dy}_{1/2\lambda} = \frac{\cancel{\lambda}}{2} \times \frac{1}{2\cancel{\lambda}} = \frac{1}{4}$$

15 20

$$P\big(N(s)=k \mid N(t)=n\big) = \frac{P\big(N(s)=k, N(t)=n\big)}{P\big(N(t)=n\big)} =$$

$$\frac{P\big(N(s)=k, N(t)-N(s)=n-k\big)}{P\big(N(t)=n\big)} =$$

$$\frac{P\big(N(s)=k\big) \cdot P\big(N(t)-N(s)=n-k\big)}{P\big(N(t)=n\big)} =$$

$$\frac{\cancel{e^{-\lambda s}}\dfrac{(\lambda s)^k}{k!} \times \cancel{e^{-\lambda(t-s)}}\dfrac{\big(\lambda(t-s)\big)^{n-k}}{(n-k)!}}{\cancel{e^{-\lambda t}}\dfrac{(\lambda t)^n}{n!}} =$$

$$\frac{n!}{k!(n-k)!}\left(\frac{s}{t}\right)^k\left(1-\frac{s}{t}\right)^{n-k} = \binom{n}{k}\left(\frac{s}{t}\right)^k\left(1-\frac{s}{t}\right)^{n-k}$$

$k=0,\dots,n, \quad s<t$

17) a) $P(N(1/3)=2 \mid N(1)=2) =$

$$\frac{P(N(1/3)=2,\ N(1)=2)}{P(N(1)=2)} = \frac{P(N(1/3)=2,\ N(1)-N(1/3)=0)}{P(N(1)=2)}$$

$$\frac{P(N(1/3)=2)\cdot P(N(2/3)=0)}{P(N(1)=2)} = \frac{\cancel{e^{-\lambda 1/3}}\frac{(\lambda 1/3)^2}{\cancel{2!}} \times \cancel{e^{-\lambda 2/3}}}{\cancel{e^{-\lambda}}\frac{(\lambda)^2}{\cancel{2!}}}$$

$= 1/9$

b) $P(N(1/3) \geq 1 \mid N(1)=2) = P(N(1/3)=1 \mid N(1)=2) +$

$P(N(1/3)=2 \mid N(1)=2)$

$$P(N(1/3)=1 \mid N(1)=2) = \frac{P(N(1/3)=1,\ N(1)-N(1/3)=1)}{P(N(1)=2)} =$$

$$\frac{P(N(1/3)=1)\cdot P(N(2/3)=1)}{P(N(1)=2)} = \frac{\cancel{e^{-\lambda 1/3}}(\lambda 1/3) \times \cancel{e^{-\lambda 2/3}}(\lambda 2/3)}{\cancel{e^{-\lambda}}\frac{\lambda^2}{2!}} = [illegible]$$

$$P(N(1/3) \geq 1 \mid N(1)=2) = \frac{5}{9}$$

18 $V(4) = \sum_{H=1}^{N(4)} Y_H$ $\quad$ $Y_H \to$ quanto paga ao $H^{\text{ésimo}}$ cliente $\quad$ 22

$V(4) =$ pagamento feito em 4 semanas $\quad$ $N(4) \to$ número de pedidos em 4 semanas

$$P(N(4)=n) = \frac{(5.4)^n}{n!} e^{-5.4} \quad , \quad E(Y_H) = 2000{,}00$$

$$E(V(4)) = E\big(E(V(4) \mid N(4))\big)$$

$$E(V(4) \mid N(4)=n) = \sum_{H=1}^{n} E(Y_H) = 2000 \times n$$

$$E(V(4)) = 2000 \cdot E(N(4)) = 2000 \cdot 5.4 = 40.000{,}00 \quad //$$

$$E(V(4)^2) = E\big(E(V(4)^2 \mid N(4))\big)$$

$$E(V(4)^2 \mid N(4)=n) = E\left(\left(\sum_{H=1}^{n} Y_H\right)^2\right) =$$

$$E\left(\left(\sum_{H=1}^{n} (Y_H - 2000) + 2000 \times n\right)^2\right) = \overbrace{E\left(\left(\sum_{H=1}^{n} (Y_H - 2000)\right)^2\right)}^{Var(Y_1 + \dots + Y_n)} + (2000n)^2$$

$$= \sum_{H=1}^{n} \overbrace{Var(Y_H)}^{(2000)^2} + (2000 \cdot n)^2 = n(2000)^2 + (2000 \cdot n)^2$$

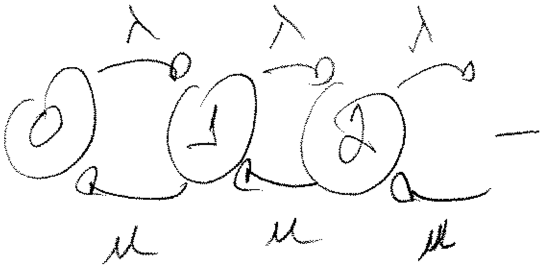
3ª Lista – Gabarito

1

$\lambda = 6/h$ $10/h por máquina

$\mu = 8/h$

$X(t)$ = número de máquinas aguardando reparo / em reparo

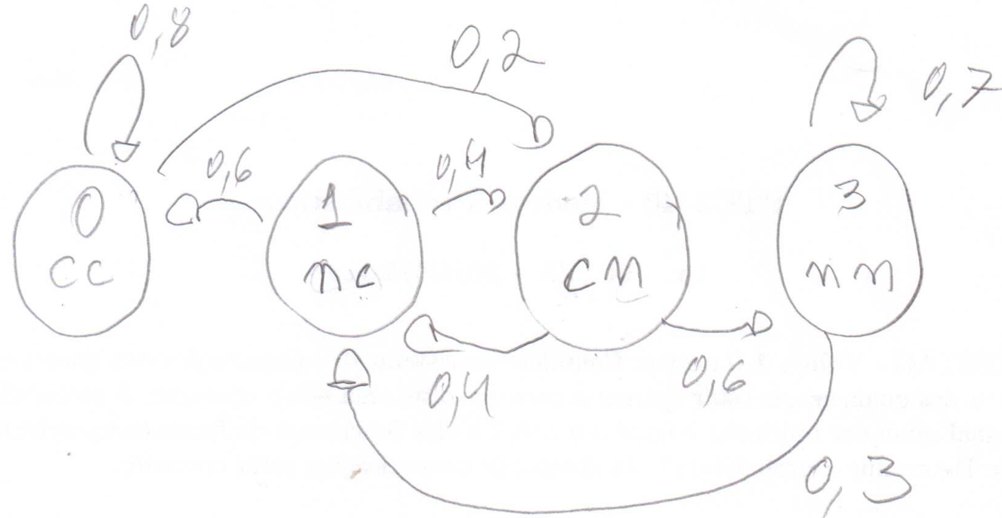


$$W = \frac{1}{\mu - \lambda} = \frac{1}{8-6} = \frac{1}{2}$$

$L = 6 \cdot W = 3$ → o número médio de máquinas aguardando reparo / em reparo

$C = 10 \times 3 = \$30/\text{hora}$

1º Q)



$$P = \begin{bmatrix} 0,8 & 0 & 0,2 & 0 \\ 0,6 & 0 & 0,4 & 0 \\ 0 & 0,4 & 0 & 0,6 \\ 0 & 0,3 & 0 & 0,7 \end{bmatrix}$$

2º 3º → 3º 4º 1
3º 4º → 4º 5º 2
4º 5º → 5º 6º 3
5º 6 → 6 Sab 4

a) $X_0 = \overset{2^o\,3^o}{\underset{}{nc}}\;1$

$$P^4 = \begin{array}{cccc} cc & nc & cm & nn \end{array}$$
$$\begin{bmatrix} 0,508 & 0,118 & 0,152 & 0,222 \\ 0,456 & 0,136 & 0,144 & 0,264 \\ 0,354 & 0,165 & 0,136 & 0,345 \\ 0,333 & 0,1725 & 0,132 & 0,3625 \end{bmatrix}$$

P(chover Sabado | n chover 2º, chover 3º) =

$0,456 + 0,136 = 0,592$

b) $\pi_0 = 0,8\pi_0 + 0,6\pi_1$

$\pi_1 = 0,4\pi_2 + 0,3\pi_3$

$\pi_2 = 0,2\pi_0 + 0,4\pi_1$

$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$

$$\Rightarrow \begin{bmatrix} 0,2 & -0,6 & 0 & 0 \\ 0 & 1 & -0,4 & -0,3 \\ -0,2 & -0,4 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\pi_0 = 0,4286,\ \pi_1 = 0,1429,\ \pi_2 = 0,1429,\ \pi_3 = 0,2857$

$\pi_0 + \pi_2 =$ % chuva $= \pi_0 + \pi_1 = 57,14\%$

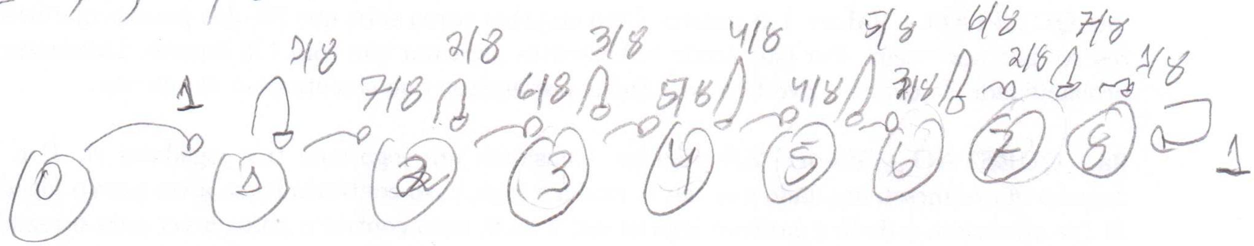
②

2ª Q) X_n = número de urnas não vazias depois de de n bolas distribuídas

$$X_n \in \{0, 1, \dots, 8\}$$

$$p_{ii} = \frac{i}{8} = 1 - p_{i\,i+1}$$

$i = 0, 1, \dots, 8$



$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{8} & \frac{7}{8} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{8} & \frac{6}{8} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{8} & \frac{5}{8} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{8} & \frac{4}{8} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5/8 & 3/8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6/8 & 2/8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7/8 & 1/8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Probabilidade desejada

$$p_{03}^{(9)} = p_{13}^{(8)} \quad (\text{já que } p_{01} = 1)$$

Calculando $p_{13}^{(18)} = 0{,}00756 = 0{,}756\%$ //

3º Q)
$$\begin{cases} \pi_1 = 0{,}45\pi_1 + 0{,}05\pi_2 + 0{,}01\pi_3 \\ \pi_2 = 0{,}48\pi_1 + 0{,}7\pi_2 + 0{,}5\pi_3 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

$$\begin{pmatrix} 0{,}55 & -0{,}05 & -0{,}01 \\ -0{,}48 & 0{,}3 & -0{,}5 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\pi_1 = 0{,}0624 = 6{,}24\%$ → alta

$\pi_2 = 0{,}6234 = 62{,}34\%$ → média

$\pi_3 = 0{,}3142 = 31{,}42\%$ → baixa

4º Q) 1 → ganha $\quad p_{11} = 0{,}8$
2 → perde $\quad p_{21} = 0{,}3$

$$P = \begin{bmatrix} 0{,}8 & 0{,}2 \\ 0{,}3 & 0{,}7 \end{bmatrix}$$

$$P(2) = \underbrace{P(2|G)}_{0{,}7}\pi_1 + \underbrace{P(2|P)}_{0{,}2}\pi_2$$

$$\begin{cases}\pi_1 = 0,8\,\pi_1 + 0,3\,\pi_2 \\ \pi_1 + \pi_2 = 1\end{cases} \Rightarrow \begin{bmatrix}0,2 & -0,3\\ 1 & 1\end{bmatrix}\begin{bmatrix}\pi_1\\ \pi_2\end{bmatrix} = \begin{bmatrix}0\\ 1\end{bmatrix}$$

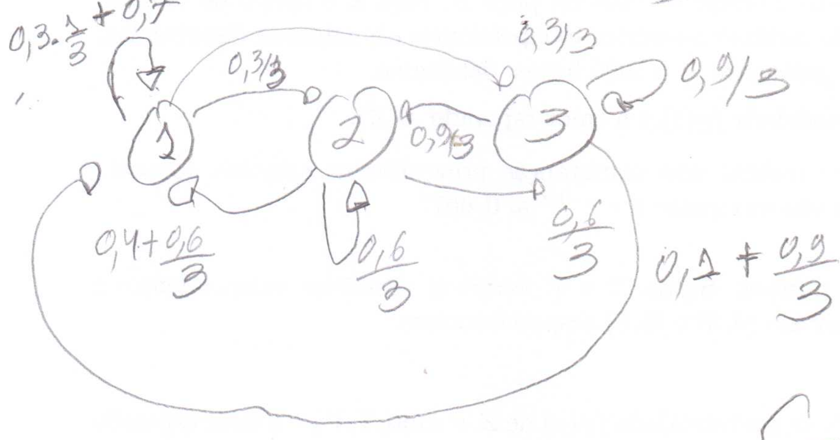
$$0,2\,\pi_1 = 0,3\,\pi_2 \Rightarrow \pi_1 = \frac{3}{2}\pi_2$$

$$\frac{3}{2}\pi_2 + \pi_2 = \frac{5}{2}\pi_2 = 1 \Rightarrow \pi_2 = \frac{2}{5},\ \pi_1 = \frac{3}{5}$$

$$\%\ \text{jantar} = P(J) = 0,7\cdot\frac{3}{5} + 0,2\cdot\frac{2}{5} = \frac{1}{2}\ //$$

5ºQ) X_n = exame aplicado na n-ésima vez

$$X_n \in \{1,2,3\}$$



$$P = \begin{bmatrix}0,8 & 0,1 & 0,1\\ 0,6 & 0,2 & 0,2\\ 0,4 & 0,3 & 0,3\end{bmatrix}$$

$$\begin{cases}\pi_1 = 0,8\,\pi_1 + 0,6\,\pi_2 + 0,4\,\pi_3\\ \pi_2 = 0,1\,\pi_1 + 0,2\,\pi_2 + 0,3\,\pi_3\\ 1 = \pi_1 + \pi_2 + \pi_3\end{cases}$$

$$\pi_1 = 5/7,\ \pi_2 = \pi_3 = 1/2$$

Exame 1 → 71,5%

" 2 → 14,25%

" 3 → 14,25%

(6) $X \to$ Poisson com média λ (8)

$\lambda \to$ exponencial com média 1.

$$P(X=n) = E(P(X=n \mid \lambda))$$

$$P(X=n \mid \lambda=z) = e^{-z}\frac{z^n}{n!},$$

$$E(P(X=n \mid \lambda=z)) = \int_0^\infty e^{-z}\frac{z^n}{n!}e^{-z}\,dz =$$

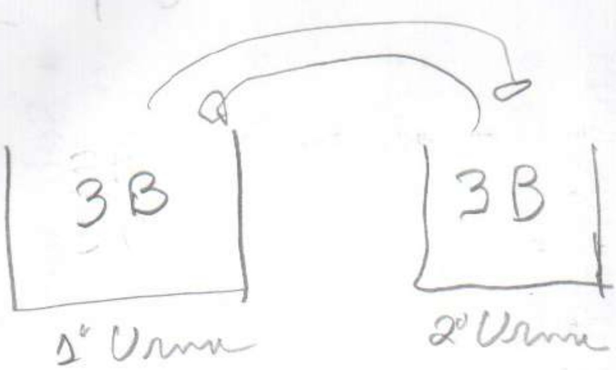
$$\int_0^\infty \frac{z^n}{n!}e^{-2z}\,dz = \frac{1}{2^{n+1}}\underbrace{\left(\int_0^\infty 2(2z)^n e^{-2z}\,dz\right)\frac{1}{n!}}_{\text{= 1 distribuição gama com parâmetros } n+1 \text{ e } 2} = \frac{1}{2^{n+1}}$$

Logo,

$$P(X=n) = \frac{1}{2^{n+1}},\ n=0,1,2,\dots$$

7

a)



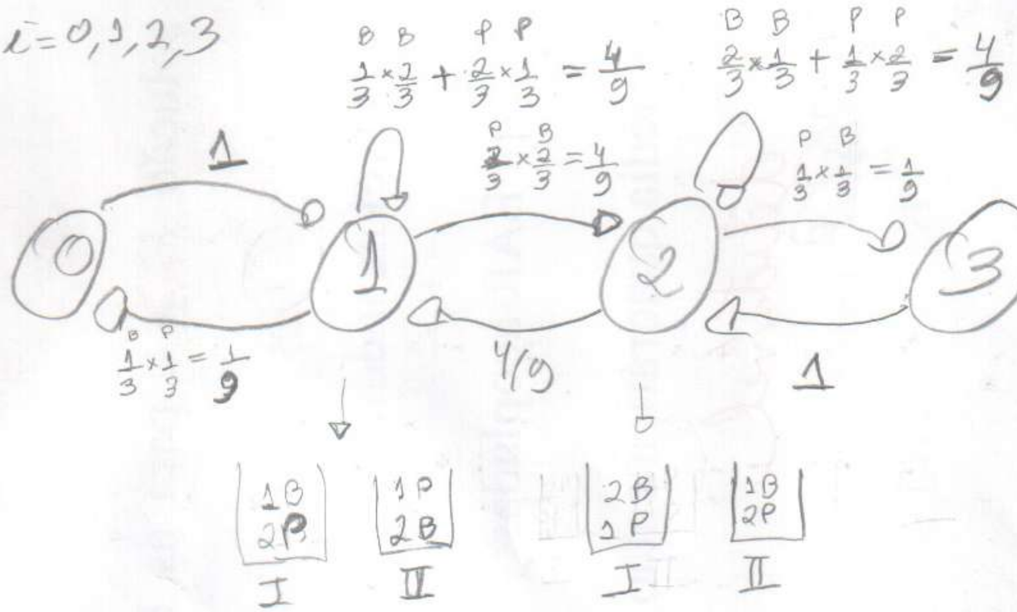
3 Brancos
3 Pretos

X_n = nº de bolas brancas na 1ª urna na $n^{\text{ésima}}$ etapa

$X_n \in \{0, 1, 2, 3\}$

estados: $i \to$ i bolas brancas na 1ª urna

$i = 0, 1, 2, 3$



$$
P = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}
\begin{array}{cccc|c}
0 & 1 & 2 & 3 & \\
\end{array}
$$

$$
P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/9 & 4/9 & 4/9 & 0 \\ 0 & 4/9 & 4/9 & 1/9 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array}
$$

10

(b)

$$
P^6 = \begin{array}{c|cccc|c}
 & 0 & 1 & 2 & 3 & \\
\hline
 & 0{,}0507 & 0{,}45 & 0{,}45 & 0{,}0493 & 0 \\
 & 0{,}05 & 0{,}4507 & 0{,}4493 & 0{,}05 & 1 \\
 & 0{,}05 & 0{,}4493 & 0{,}4507 & 0{,}05 & 2 \\
 & 0{,}0493 & 0{,}45 & 0{,}45 & 0{,}0507 & 3
\end{array}
$$

P(pelo menos 2 bolas brancas na 6ª rodada) =

$0{,}45 + 0{,}0507 = 0{,}5007$

c) Probabilidades limites $\pi' = \pi' P$

$\pi_0 = 1/9\, \pi_1$

$\pi_1 = \pi_0 + \frac{4}{9}\pi_1 + \frac{4}{9}\pi_2 \quad = 5/9\, \pi_1 + \frac{4}{9}\pi_2$

$\pi_2 = \quad \frac{4}{9}\pi_1 + \frac{4}{9}\pi_2 + \pi_3$

$-\frac{4}{9}\pi_1 + \frac{4}{9}\pi_2 = 0 \Rightarrow \pi_1 = \pi_2$

$\frac{4}{9}\pi_1 - \frac{5}{9}\pi_2 + \pi_3 = 0 \Rightarrow \pi_3 = \frac{\pi_1}{9}$

$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \Rightarrow$

$1/9\, \pi_1 + \pi_1 + \pi_1 + 1/9\, \pi_1 = (2+9+9)\frac{\pi_1}{9} = 1 \Rightarrow$

$\frac{20}{9}\pi_1 = 1 \Rightarrow \pi_1 = \frac{9}{20} = \pi_2, \; \pi_0 = \frac{1}{20} = \pi_3$

$\pi_0 = \pi_3 = 0{,}05$

$\pi_1 = \pi_2 = 0{,}45$

(12)

$\pi_0 = p\pi_0 + (1-p)\pi_1 \Rightarrow \pi_0 = \pi_1$

$\pi_0 + \pi_1 = 1 \Rightarrow \pi_0 = \pi_1 = \frac{1}{2}$

(8) $P = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$

$$P^{(n)} = \begin{bmatrix} 1/2 + 1/2(2p-1)^n & 1/2 - 1/2(2p-1)^n \\ 1/2 - 1/2(2p-1)^n & 1/2 + 1/2(2p-1)^n \end{bmatrix}$$

$n=1$: $(1,1) \Rightarrow \cancel{1/2} + \cancel{1/2}(\cancel{2}p \cancel{-1}) = p$ $\quad (1,2) \Rightarrow 1/2 - 1/2(2p-1) = 1-p$

OK

$(2,1) \Rightarrow 1/2 - 1/2(2p-1) = 1-p$ $\quad (2,2) \Rightarrow 1/2 + 1/2(2p-1) = p$

$n \Rightarrow n+1$:

$$P^{(n+1)} = P^{(n)}P = \begin{bmatrix} 1/2 + 1/2(2p-1)^n & 1/2 - 1/2(2p-1)^n \\ 1/2 - 1/2(2p-1)^n & 1/2 + 1/2(2p-1)^n \end{bmatrix} \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix} =$$

$$\begin{bmatrix} 1/2\left[\left(1+(2p-1)^n\right)p + \left(1-(2p-1)^n\right)(1-p)\right] & 1/2\left[\left(1+(2p-1)^n\right)(1-p) + \left(1-(2p-1)^n\right)p\right] \\ 1/2\left[\left(1-(2p-1)^n\right)p + \left(1+(2p-1)^n\right)(1-p)\right] & 1/2\left[\left(1-(2p-1)^n\right)(1-p) + \left(1+(2p-1)^n\right)p\right] \end{bmatrix}$$

13

$$\begin{bmatrix} \frac{1}{2}\left[1+(2p-1)(2p-1)^n\right] & \frac{1}{2}\left[1-(2p-1)(2p-1)^n\right] \\ \frac{1}{2}\left[1-(2p-1)(2p-1)^n\right] & \frac{1}{2}\left[1+(2p-1)(2p-1)^n\right] \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^{n+1} & \frac{1}{2}-\frac{1}{2}(2p-1)^{n+1} \\ \frac{1}{2}-\frac{1}{2}(2p-1)^{n+1} & \frac{1}{2}+\frac{1}{2}(2p-1)^{n+1} \end{bmatrix}$$

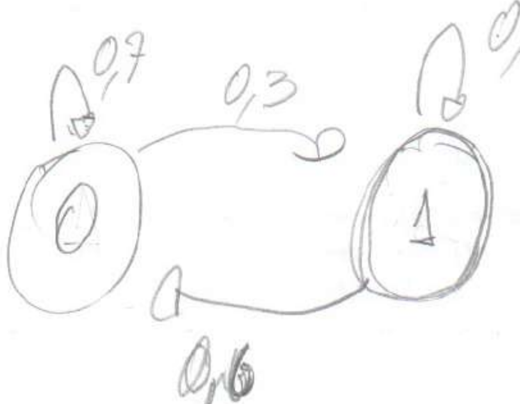
10) $\sum_i p_{ij} = 1$, todo j

$$\pi_j = \sum_i \pi_i p_{ij} \quad , \quad \sum \pi_i = 1$$

$$\pi_j = \frac{1}{M+1} \Rightarrow \begin{cases} \sum_{j=0}^{M} \pi_j = 1 \quad \text{OK} \\ \pi_j = \frac{1}{M+1} = \sum_{i=0}^{M} \pi_i p_{ij} = \frac{1}{M+1} \sum_{i=0}^{M} p_{ij} = 1 \end{cases}$$

(9) $p_1 = 0{,}7$ $\quad X_n = \begin{cases} 0 & \text{se deu cara no } n\text{ésimo dia} \\ 1 & \text{" " coroa " " "} \end{cases}$

$p_2 = 0{,}6$



Y_n = moeda escolhida no $n$ésimo dia

$$P(X_0 = 0) = P(X_0 = 1) = \frac{1}{2}$$

$$P(Y_3 = 1) = P(Y_3 = 1, X_2 = 0) + P(Y_3 = 1, X_2 = 1)$$

$$= P(Y_3 = 1 / X_2 = 0) \cdot P(X_2 = 0) + P(Y_3 = 1 / X_2 = 1) \cdot P(X_2 = 1)$$

$$= 1 \cdot P(X_2 = 0) + \cancel{0 \cdot P(X_2 = 1)} \to 0$$

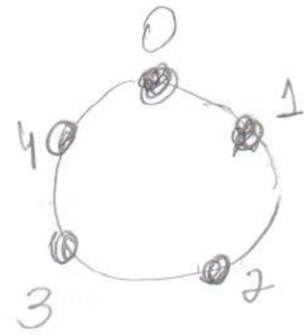
$$P = \begin{bmatrix} 0{,}7 & 0{,}3 \\ 0{,}6 & 0{,}4 \end{bmatrix}, \quad P^2 = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ & \begin{bmatrix} 0{,}67 & 0{,}33 \\ 0{,}66 & 0{,}34 \end{bmatrix} \begin{array}{c} 0 \\ 1 \end{array} \end{array}$$

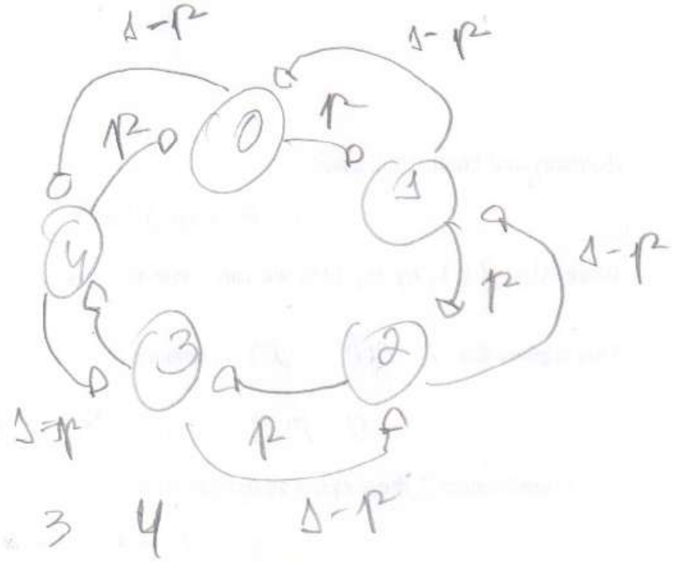
$$P(X_2 = 0) = P(X_2 = 0 | X_0 = 0) \cdot \overset{1/2}{P(X_0 = 0)} + P(X_2 = 0 | X_0 = 1) \cdot \overset{1/2}{P(X_0 = 1)}$$

$$= \frac{1}{2}(0{,}67 + 0{,}66) = 0{,}665$$

$$P(Y_3 = 1) = 0{,}665$$

11

a) 



$$P = \begin{array}{c} \\ \end{array}\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array}$$

$$P = \begin{array}{cc} \left[\begin{array}{ccccc} 0 & p & 0 & 0 & 1-p \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & p \\ p & 0 & 0 & 1-p & 0 \end{array}\right] & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

b) $p = 1/2$ $\quad P_0^8 = \begin{array}{ccccc} 0 & 1 & 2 & & \\ \left[0{,}2734 \right. & 0{,}1406 & 0{,}2227 & 0{,}2227 & \left. 0{,}1406 \right] \end{array}$ a linha 0

$P_{02}^{8} = 0{,}2227$

c) $\pi_i = \dfrac{1}{5} = 0{,}2$, $i = 0,1,2,3,4$ pois P é duplamente estocástica

12) Y_n = soma de n dados independentes

$$Y_n = I_n \times 13 + X_n \qquad I_n = \left[\frac{Y_n}{13}\right]$$ → parte inteira da divisão

$$X_n \in \{0, 1, 2, \ldots, 12\}$$

$$P(X_{n+1} = i + j \mid X_n = i) = \frac{1}{6}, \quad i = 0, 1, 2, \ldots, 11, \quad j = 1, \ldots, \min\{6, 12 - i\}$$

$$P(X_{n+1} = j - (13 - i) \mid X_n = i) = \frac{1}{6}, \quad i = 7, \ldots, 12, \quad j = 13 - i, \ldots, 6$$

$$P = \begin{bmatrix}
0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\
1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\
1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 \\
1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 \\
1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 \\
1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 \\
1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Soma dos linhas é 1 (17)

$$\sum_i p_{ij} = 1 \Rightarrow \lim_{n \to \infty} P(X_n = 0) = \frac{1}{13}$$

13) $M_i = E(\cancel{Z} \mid X_0 = i)$

$\cancel{Z}$ = número de jogadas até acabar o jogo

$M_0 = 0, \; M_N = 0$

$M_i = E(E(Z \mid X_1) \mid X_0 = i)$

$E(Z \mid X_1 = i+1) = M_{i+1} + 1 \to$ prob p

$E(Z \mid Z_1 = i-1) = M_{i-1} + 1 \to$ prob $1-p$

$$M_i = (1 + M_{i+1})p + (1 + M_{i-1})(1-p) =$$

$$= 1 + pM_{i+1} + qM_{i-1}$$

(18)

$N = 4, \ p = 1/4, \ q = 3/4$

$$M_1 = 1 + \frac{1}{4}M_2 + \cancel{\frac{3}{4}M_0}^{0} = 1 + \frac{1}{4}M_2$$

$$M_2 = 1 + \frac{1}{4}M_3 + \frac{3}{4}M_1 = \frac{7}{4} + \frac{1}{4}M_3 + \frac{3}{16}M_2$$

$$M_3 = 1 + \cancel{\frac{1}{4}M_4}^{0} + \frac{3}{4}M_2$$

$$\begin{cases} \frac{13}{16}M_2 - \frac{1}{4}M_3 = \frac{7}{4} \quad \left(\frac{12}{13}\right) \Rightarrow \\ -\frac{3}{4}M_2 + M_3 = 1 \end{cases}$$

$$\cancel{\frac{3}{4}M_2} - \frac{3}{13}M_3 = \frac{21}{13}$$

$$\cancel{-\frac{3}{4}M_2} + M_3 = 1$$

$$\frac{10}{13}M_3 = \frac{34}{13}$$

$M_3 = 3.4, \ M_2 = 3.2, \ M_1 = 1.8$

Fórmula (exercício 18):

$$M_1 = \frac{1}{1/2} - \frac{4}{1/2}\frac{1-3}{1-3^4} = 2 - 8 \times \frac{2}{80} = 2 - 0.2 = 1.8$$

$$M_2 = \frac{2}{1/2} - \frac{4}{1/2}\frac{1-9}{1-3^4} = 4 - 8 \times \frac{8}{80} = 4 - 0.8 = 3.2$$

$$M_3 = \frac{3}{1/2} - \frac{4}{1/2}\frac{1-27}{1-3^4} = 6 - 8 \times \frac{26}{80} = 6 - 2.6 = 3.4$$

Cap 5

14

A

B

C

$P(T_A > T_B + T_C) = ?$

a) $T_A = T_B = T_C = 10 \Rightarrow P(T_A > T_B + T_C) = 0$

b) $P(T = i) = 1/3, \; i = 1, 2, 3$

$P(T_A > T_B + T_C) = P(T_A = 3, T_B = 1, T_C = 1) =$

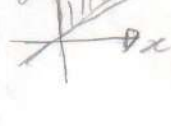
$P(T_A = 3) \cdot P(T_B = 1) \cdot P(T_C = 1) = 1/27$

c) $T \rightarrow$ exponenciais com parâmetro $1/\mu$

$\lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$

$Y = T_B + T_C \rightarrow$ gama com parâmetros $(2, \mu)$

$f_Y(y) = (\lambda y)\, \lambda e^{-\lambda y}$ $\qquad X = T_A \rightarrow$ exponencial



$$P(X > Y) = \int_0^\infty \int_y^\infty \lambda e^{-\lambda x} dx \left(\lambda y \lambda e^{-\lambda y}\right) dy$$

$$= \int_0^\infty e^{-\lambda y} \lambda y e^{-\lambda y} dy = \int_0^\infty y \lambda^2 e^{-2\lambda y} dy$$

$$= \frac{\lambda}{2} \overbrace{\int_0^\infty y (2\lambda) e^{-2\lambda y} dy}^{1/2\lambda} = \frac{\cancel{\lambda}}{2} \times \frac{1}{2\cancel{\lambda}} = \frac{1}{4}$$

15 $$P\big(N(s)=k \mid N(t)=n\big) = \frac{P\big(N(s)=k, N(t)=n\big)}{P\big(N(t)=n\big)} =$$

20

$$\frac{P\big(N(s)=k, N(t)-N(s)=n-k\big)}{P\big(N(t)=n\big)} =$$

$$\frac{P\big(N(s)=k\big) \cdot P\big(N(t)-N(s)=n-k\big)}{P\big(N(t)=n\big)} =$$

$$\frac{\cancel{e^{-\lambda s}}\dfrac{(\lambda s)^k}{k!} \times \cancel{e^{-\lambda(t-s)}}\dfrac{\big(\lambda(t-s)\big)^{n-k}}{(n-k)!}}{\cancel{e^{-\lambda t}}\dfrac{(\lambda t)^n}{n!}} =$$

$$\frac{n!}{k!(n-k)!}\left(\frac{s}{t}\right)^k\left(1-\frac{s}{t}\right)^{n-k} = \binom{n}{k}\left(\frac{s}{t}\right)^k\left(1-\frac{s}{t}\right)^{n-k}$$

$k=0,\dots,n, \quad s<t$

17 a) $P(N(1/3)=2 \mid N(1)=2) =$

$$\frac{P(N(1/3)=2,\ N(1)=2)}{P(N(1)=2)} = \frac{P(N(1/3)=2,\ N(1)-N(1/3)=0)}{P(N(1)=2)}$$

$$\frac{P(N(1/3)=2)\cdot P(N(2/3)=0)}{P(N(1)=2)} = \frac{\cancel{e^{-\lambda 1/3}}\dfrac{(\lambda 1/3)^2}{\cancel{2!}} \times \cancel{e^{-\lambda 2/3}}}{\cancel{e^{-\lambda}}\dfrac{(\lambda)^2}{\cancel{2!}}}$$

$$= 1/9$$

b) $P(N(1/3)\geq 1 \mid N(1)=2) = P(N(1/3)=1 \mid N(1)=2) +$

$P(N(1/3)=2 \mid N(1)=2)$

$$P(N(1/3)=1 \mid N(1)=2) = \frac{P(N(1/3)=1,\ N(1)-N(1/3)=1)}{P(N(1)=2)} =$$

$$\frac{P(N(1/3)=1)\cdot P(N(2/3)=1)}{P(N(1)=2)} = \frac{\cancel{e^{-\lambda 1/3}}(\lambda 1/3)\times \cancel{e^{-\lambda 2/3}}(\lambda 2/3)}{\cancel{e^{-\lambda}}\dfrac{\lambda^2}{2!}} = \frac{4}{9}$$

$$P(N(1/3)\geq 1 \mid N(1)=2) = \frac{5}{9}$$

(18) $V(4) = \sum_{i=1}^{N(4)} Y_i$ (22)

$Y_i \to$ quantia paga ao $i^{\text{ésimo}}$ cliente

$V(4)$ = pagamento feito em 4 semanas

$N(4) \to$ número de pedidos em 4 semanas

$$P(N(4)=n) = \frac{(5.4)^n}{n!} e^{-5.4}, \quad E(Y_i) = 2000{,}00$$

$$E(V(4)) = E\left(E(V(4) \mid N(4))\right)$$

$$E(V(4) \mid N(4)=n) = \sum_{i=1}^{n} E(Y_i) = 2000 \times n$$

$$E(V(4)) = 2000 \cdot E(N(4)) = 2000 \cdot 5.4 = 40.000{,}00$$

$$E(V(4)^2) = E\left(E(V(4)^2 \mid N(4))\right)$$

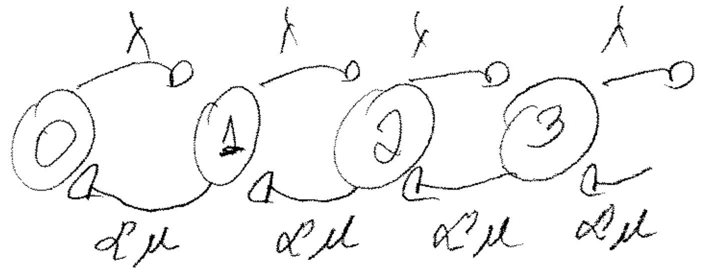
$$E(V(4)^2 \mid N(4)=n) = E\left(\left(\sum_{i=1}^{n} Y_i\right)^2\right) =$$

$$E\left(\left(\sum_{i=1}^{n}(Y_i-2000) + 2000\times n\right)^2\right) = \overbrace{E\left(\left(\sum_{i=1}^{n}(Y_i-2000)\right)^2\right)}^{Var(Y_1+\dots+Y_n)} + (2000n)^2$$

$$= \sum_{i=1}^{n} \overbrace{Var(Y_i)}^{(2000)^2} + (2000 \cdot n)^2 = n(2000)^2 + (2000 \cdot n)^2$$

(2)

a)



Condição de equilíbrio

$$\frac{\lambda}{\alpha\mu} < 1$$

$$P_n = \left(1 - \frac{\lambda}{\alpha\mu}\right)\left(\frac{\lambda}{\alpha\mu}\right)^n, \quad n = 0, 1, \dots$$

b) $$W_1 = \sum_{n=0}^{\infty} E(T/X=n) \cdot P_n = \sum_{n=0}^{\infty} \left(\frac{n}{\mu}\right)\left(\frac{\lambda}{\alpha\mu}\right)^n \left(1 - \frac{\lambda}{\alpha\mu}\right)$$

$$E(T/X=n) = \frac{n}{\mu}$$

$$W_1 = \alpha \sum_{n=0}^{\infty} \left(\frac{n}{\alpha\mu}\right)\left(\frac{\lambda}{\alpha\mu}\right)^n \left(1 - \frac{\lambda}{\alpha\mu}\right) = \cancel{\alpha} \frac{\lambda}{\cancel{\alpha\mu}(\alpha\mu - \lambda)}$$

$$= \frac{\lambda}{\mu(\alpha\mu - \lambda)}$$

c) $$P(N=n) = (1-\alpha)^{n-1}\alpha, \quad n = 1, 2, \dots$$

d) $$E(T_s) = E\left(E(T_s|N)\right) = \sum_{n=1}^{\infty} \alpha(1-\alpha)^{n-1} \frac{n}{\mu}$$

$$= \frac{1}{\mu} \times \frac{1}{\alpha} = \frac{1}{\alpha\mu}$$

(2')

e) Qual é o tempo médio gasto na fila?

$$W_Q = E(E(T_Q | N))$$

$$E(T_Q | N = n) = \frac{n\lambda}{\mu(\alpha\mu - \lambda)}$$

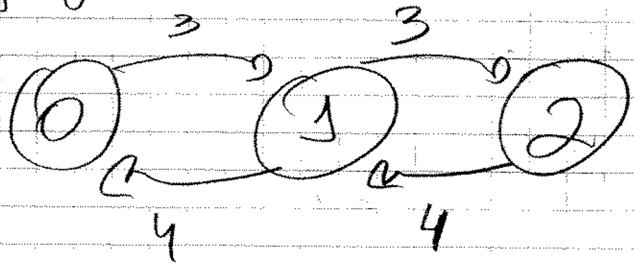
$$W_Q = \frac{\lambda}{\mu(\alpha\mu - \lambda)} \sum_{n=1}^{\infty} n\,\alpha(1-\alpha)^{n-1} \quad 1/\alpha = \frac{\lambda}{\alpha\mu(\alpha\mu - \lambda)}$$

f) $L = \sum_{n=0}^{\infty} n \cdot P_n = \frac{\lambda}{\alpha\mu - \lambda}$

$$W = \frac{1}{\alpha\mu - \lambda} = W_Q + \frac{1}{\alpha\mu} = \frac{\lambda}{\alpha\mu(\alpha\mu - \lambda)} + \frac{1}{\alpha\mu} =$$

$$= \frac{\cancel{\lambda} + \alpha\mu - \cancel{\lambda}}{\cancel{\alpha\mu}(\alpha\mu - \lambda)} = \frac{1}{\alpha\mu - \lambda}$$

③ 2 fregueses



$\lambda = 3/\text{hora}$

$E(T_S) = \frac{1}{4}\ \text{hora} = \frac{1}{\mu}$

$\mu = 4/\text{hora}$

(a) $L = ?$

	sai = entra
0	$3 P_0 = 4 P_1$
1	$\cancel{7 P_1 = 3 P_0 + 4 P_2}$
2	$4 P_2 = 3 P_1$

$P_0 + P_1 + P_2 = 1$

$P_1 = \frac{3}{4} P_0$

$P_2 = \frac{3}{4} P_1 = \frac{9}{16} P_0$

$$\left(1 + \frac{3}{4} + \frac{9}{16}\right) P_0 = \frac{16+12+9}{16} P_0 = 1$$

$$P_0 = \frac{16}{37}, \quad P_1 = \frac{3 \times 4}{37} = \frac{12}{37}, \quad P_2 = \frac{9}{37}$$

$$L = \frac{12}{37} + 2 \times \frac{9}{37} = \frac{30}{37}$$

(b) $1 - P_2 = 1 - \frac{9}{37} = \frac{28}{37}$

proporção dos fregueses que entram

(c) $L = \lambda_a W \;, \; \lambda_a = (1 - P_2)\, \lambda = \frac{28}{37} \cdot 3 = \frac{84}{37}$

$$W = \frac{\frac{30}{\cancel{37}}}{\frac{84}{\cancel{37}}} = \frac{15}{42} = \frac{5}{14}$$

(d) $\cancel{\mu} = 2\mu = 8/\text{hora}$

4

$3P_0 = 8P_1 \Rightarrow P_1 = \frac{3}{8}P_0$

$8P_2 = 3P_1 \Rightarrow P_2 = \frac{3}{8}P_1 = \frac{9}{64}$

$\left(1+\frac{3}{8}+\frac{9}{64}\right)P_0 = \frac{64+24+9}{64}P_0 = \frac{97}{64}P_0 = 1$

$P_0 = \frac{64}{97}, \quad P_1 = \frac{24}{97}, \quad P_2 = \frac{9}{97}$

% clientes que entram $= P_0 + P_1 =$

$\mu=4$: $\frac{16+12}{37} = \frac{28}{37}$

$\mu=8$: $\frac{64+24}{97} = \frac{88}{97}$

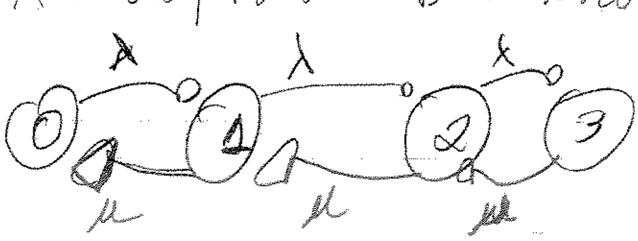
$\mu=8$

$\frac{88/97}{28/37} = \frac{37}{97} \times \frac{88}{28} = 1.198$

$\mu=4$

vai atender mais 19,80%

4 $\lambda = 20/\text{hora} \rightarrow$ máximo 3 carros



$E(T_s) = 5$ minutos $= \frac{5}{60} = \frac{1}{12}$ hora

$\mu = 12/\text{hora}$

a) $20P_0 = 12P_1$

$(20+12)P_1 = 20P_0 + 12P_2$

$12P_3 = 20P_2$

$P_1 = \frac{20}{12}P_0 = \frac{10}{6}P_0 = \frac{5}{3}P_0$

$P_3 = \frac{20}{12}P_2 = \left(\frac{5}{3}\right)^2 P_2$

$P_2 = \frac{32}{12}P_1 - \frac{20}{12}P_0 = \left(\frac{8}{3}\cdot\frac{5}{3} - \frac{5}{3}\right)P_0 = \left(\frac{8}{3}-1\right)\frac{5}{3}P_0 = \frac{25}{9}P_0$

$$P_3 = \left(\frac{5}{3}\right)^3 P_0$$

$$P_0\left(1 + \frac{5}{3} + \left(\frac{5}{3}\right)^2 + \left(\frac{5}{3}\right)^3\right) = \frac{1-\left(\frac{5}{3}\right)^4}{1-\frac{5}{3}} P_0 =$$

$$\frac{1-\left(\frac{625}{81}\right)}{-2/3} P_0 = \frac{625-81}{54} P_0 = \frac{272}{27} P_0 = 1$$

$$P_0 = \frac{27}{272}, \quad P_1 = \frac{5}{3} P_0 = \frac{5}{\cancel{3}} \times \frac{\cancel{27}\,9}{272} = \frac{45}{272}$$

$$P_2 = \frac{25}{9} P_0 = \frac{25}{\cancel{9}} \times \frac{\cancel{27}\,3}{272} = \frac{75}{272}$$

$$P_3 = \frac{125}{\cancel{27}} \times \frac{\cancel{27}}{272} = \frac{125}{272}$$

a) $P_1 + P_2 + P_3 = \frac{245}{272}$

b) $P_3 = \frac{125}{272}$

c) Tempo médio gasto no sistema W:

$$L = \frac{45 + 2\times 75 + 3\times 125}{272} = \frac{570}{272}$$

$$\lambda_a = (1 - P_3)\lambda = \frac{147}{272} \times 20$$

$$W = \frac{L}{\lambda_a} = \frac{\frac{57\cancel{0}}{\cancel{272}}}{\frac{147\times 2\cancel{0}}{\cancel{272}}} = 0.1938 \text{ horas}$$

ou 11.628 minutos

d) na fila $W_Q = W - 5 = 6,628$ minutos

5) $\lambda = 40/\text{hora}$ Máximo de 4 clientes

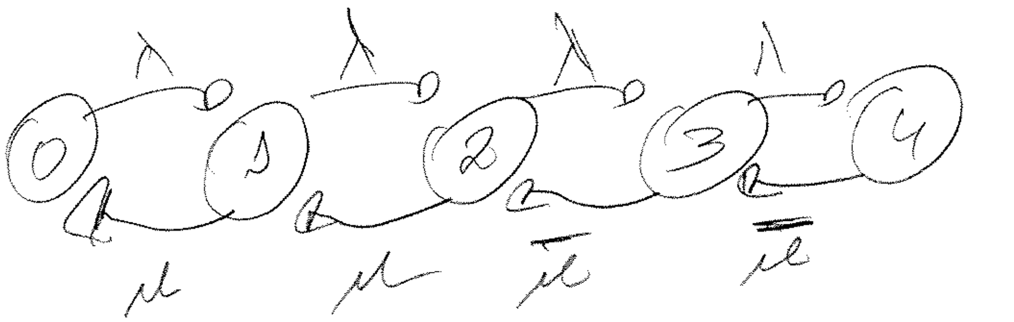
total Clientes ≤ 2 → 1 atendente com média de 2 minutos

total clientes $= 3, 4$ → 2 atendentes trabalham juntos e a média cai para 1 minuto

(a) Proporção do tempo que os 2 atendentes estão livres?

b) Cada atendente recebe proporcional ao tempo atendendo os clientes. Total a ser pago = 100 para os dois. Quanto cada um vai receber?

a) Estado = nº de clientes no sistema



$\lambda = \frac{40}{60} = 2/3$

$\mu = 1/2$

$\bar{\mu} = 1$

todo	sai = entra
0	$\lambda P_0 = \mu P_1$
1	$(\lambda + \mu) P_1 = \lambda P_0 + \mu P_2$
2	$(\lambda + \mu) P_2 = \lambda P_1 + \bar{\mu} P_3$
3	$(\lambda + \bar{\mu}) P_3 = \lambda P_2 + \bar{\mu} P_4$
4	$\bar{\mu} P_4 = \lambda P_3$

$$2/3\, P_0 = 1/2\, P_1$$

$$(2/3 + 1/2) P_1 = 2/3\, P_0 + 1/2\, P_2$$

$$(2/3 + 1/2) P_2 = 2/3\, P_1 + P_3$$

$$(2/3 + 1) P_3 = 2/3\, P_2 + P_4$$

$$P_4 = 2/3\, P_3$$

7

$P_0 + P_1 + P_2 + P_3 + P_4 = 1$

$\Rightarrow P_0 = \frac{81}{493}, \quad P_1 = \frac{108}{493}, \quad P_2 = \frac{144}{493},$

$P_3 = \frac{96}{493}, \quad P_4 = \frac{64}{493}$

2 atendentes livres = estado 0

a) $P_0 = \frac{81}{493}$

b) proporção do tempo que atendente titular trabalha =

$1 - P_0 = 1 - \frac{81}{493} = \frac{412}{493}$

proporção do tempo que auxiliar trabalha =

$P_3 + P_4 = \frac{96+64}{493} = \frac{160}{493}$

atendente titular recebe $\frac{412}{412+160} = \frac{412}{572} \cong 72\%$

auxiliar recebe $\frac{160}{412+160} = \frac{160}{572} \cong 28\%$

atendente $\rightarrow$ 72, auxiliar $\rightarrow$ 28

(6) 3 Máquinas → taxa de falha $\lambda = 1/10$ (média = 10)

2 Operários → taxa de reparos $\mu = 1/8$ (média = 8)

(a) número médio de máquinas fora de uso?

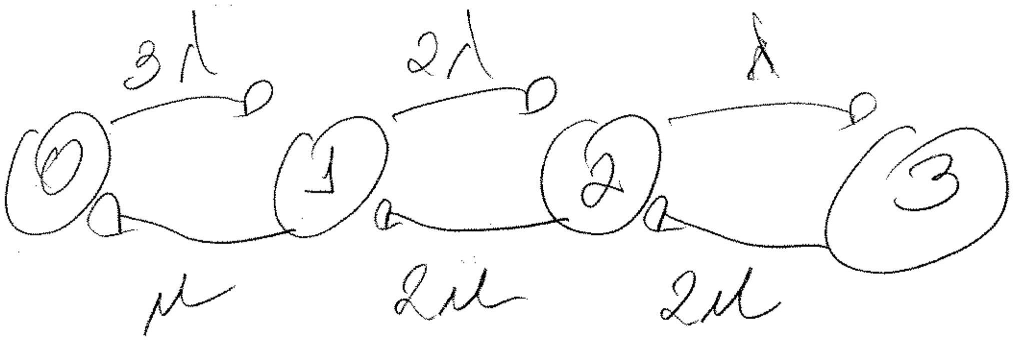
(b) proporção do tempo que os 2 operários estão ocupados?

a) Estados: 0 → 0 máquinas fora de uso

1 → 1 " fora de uso

2 → 2 " " " "

3 → 3 " " " "



estado	sai = entra
0	$3\lambda P_0 = \mu P_1$
1	$(2\lambda+\mu) P_1 = 3\lambda P_0 + 2\mu P_2$
2	$(2\mu+\lambda) P_2 = 2\mu P_3 + 2\lambda P_1$
3	$2\mu P_3 = \lambda P_2$

$\frac{3}{10} P_0 = \frac{1}{8} P_1$

$\left(\frac{2}{10}+\frac{1}{8}\right) P_1 = \frac{3}{10} P_0 + \frac{2}{8} P_2$

$\left(\frac{2}{8}+\frac{1}{10}\right) P_2 = \frac{2}{8} P_3 + \frac{2}{10} P_1$

$\frac{2}{8} P_3 = \frac{1}{10} P_2$

$$P_0 + P_1 + P_2 + P_3 = 1$$

$$P_1 = \frac{24}{10} P_0 = \frac{12}{5} P_0$$

$$P_3 = \frac{4}{10} P_2 = \frac{2}{5} P_2$$

$$\cancel{\left(\frac{1}{4} + \frac{1}{10}\right)} P_2 = \frac{1}{4} \cancel{\frac{1}{10} P_2} + \frac{\overset{}{2}}{\underset{5}{\cancel{10}}} \, \frac{\overset{12}{\cancel{24}}}{\cancel{10}5} P_0$$

$$\frac{1}{4} P_2 = \frac{12}{25} P_0 \Rightarrow P_2 = \frac{48}{25} P_0$$

$$P_2 = \frac{48}{25} P_0$$

$$\frac{16}{73} = \frac{56}{273}$$

$$\left(1 + \frac{12}{5} + \frac{48}{25} + \frac{2}{5} \times \frac{48}{25}\right) P_0 = 1$$

$$P_0 \left[\frac{125 + 300 + 240 + 96}{125}\right] = P_0 \left[\frac{761}{125}\right] = 1 \Rightarrow P_0 = \frac{125}{761}$$

$$P_0 = \frac{125}{761}, \quad P_1 = \frac{12}{\cancel{5}} \, \frac{\overset{25}{\cancel{125}}}{761} = \frac{300}{761}, \quad P_2 = \frac{48}{\cancel{25}} \, \frac{\overset{5}{\cancel{125}}}{761} = \frac{240}{761}$$

$$P_3 = \frac{2}{\cancel{5}} \times \frac{48}{\cancel{25}} \times \frac{\cancel{125}}{761} = \frac{96}{761}$$

$$L = P_1 + 2P_2 + 3P_3 = \frac{300 + 480 + 288}{761} = \frac{1068}{761}$$

(b) Prop. do tempo 2 operários ocupados =

Prop. do tempo em 2 e em 3 $= P_2 + P_3$

$$P_2 + P_3 = \frac{240}{761} + \frac{96}{761} = \frac{336}{761} \cong 44\%$$

(c) Proporção do tempo que cada máquina fica em uso?

$$P(Z(t)=1) = \sum_{i=0}^{3} P(Z(t)=1 \mid X(t)=i) \cdot P(X(t)=i)$$

máquina 1 em uso

i fora de uso

$$P(Z(t)=1 \mid X(t)=i) = \frac{3-i}{3}$$

$$P(Z(t)=1) = \overset{i=0}{\frac{125}{761}} + \overset{i=1}{\frac{\overset{1}{\cancel{300}}}{761} \times \frac{2}{\cancel{3}}} + \overset{i=2}{\frac{\overset{80}{\cancel{240}}}{761} \times \frac{1}{\cancel{3}}}$$

$$= \frac{125 + 200 + 80}{761} = \frac{405}{761} \cong 53{,}2\%$$

Cada máquina funciona 53,2% do tempo.

7) A → $\mu_A = 4/\text{hora}$ $\lambda = 2/\text{hora}$

B → $\mu_B = 2/\text{hora}$

cliente entra somente se A estiver livre

A acabou → B livre → vai para B

↳ B ocupado → vai embora

a) Proporção de clientes que entra no sistema?

b) Proporção dos que entram que recebem serviço de B?

c) Número médio de clientes no sistema?

d) Tempo médio gasto no sistema pelos clientes que entram?

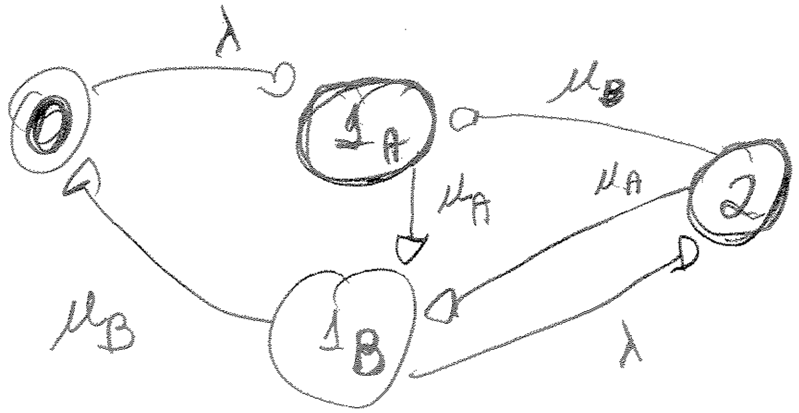
Estados:

(0) → A livre, B livre

(1_B) → A livre, B ocupado

(1_A) → A ocupado, B livre

(2) → A ocupado, B ocupado



Estados	sai = entra
0	$\lambda P_0 = \mu_B P_{1B}$
1_A	$\mu_A P_{1A} = \lambda P_0 + \mu_B P_2$
1_B	$(\lambda+\mu_B) P_{1B} = \mu_A P_2 + \mu_A P_{1A}$
2	$(\mu_A+\mu_B) P_2 = \lambda P_{1B}$

12

$$\begin{cases} 2P_0 = 2P_{1B} \\ 4P_{1A} = 2P_0 + 2P_2 \\ 4P_{1B} = 4P_2 + 4P_{1A} \\ 6P_2 = 2P_{1B} \end{cases}$$

$$P_0 + P_{1A} + P_{1B} + P_2 = 1$$

$$P_0 = \frac{3}{9}, \; P_{1A} = \frac{2}{9}, \; P_{1B} = \frac{3}{9}$$

$$P_2 = \frac{1}{9}$$

a) Proporção que entra $= P_0 + P_{1B} = \frac{6}{9} = \frac{2}{3}$

b) RB → receber ~~serviço~~ de B

E → cliente entrou no sistema

$E1_B$ → entrar pelo estado 1_B

$E0$ → " " " 0

$E = E1_B \cup E0$

$E \cap E_{1B} = E1_B$

$E \cap E0 = E0$

$$P(RB|E) = P(RB, E0|E) + P(RB, E1_B|E) =$$

$$\frac{P(RB, E0, \cancel{E})}{P(E)} + \frac{P(RB, E1_B, \cancel{E})}{P(E)} =$$

$$P(RB|E0)\frac{P(E0)}{P(E)} + P(RB|E1_B)\frac{P(E1_B)}{P(E)}$$

13

$$= P(RB|E_0)\frac{P_0}{P_0+P_{1B}} + P(RB|E1_B)\frac{P_{1B}}{P_0+P_{1B}}$$

$$\frac{P_0}{P_0+P_{1B}} = \frac{3/9}{3/9+3/9} = \frac{1}{2} = \frac{P_{1B}}{P_0+P_{1B}}$$

$$P(RB|E_0) = 1 \quad , \quad P(RB|E1_B) = P(T_B < T_A)$$

$$= \frac{\mu_B}{\mu_A+\mu_B} = \frac{2}{6} = \frac{1}{3}$$

$$P(RB|E) = 1\times\frac{1}{2} + \frac{1}{3}\times\frac{1}{2} = \frac{1}{\cancel{2}}\times\frac{\cancel{4}^{2}}{3} = \frac{2}{3}$$

c) $L = 1(1_A + 1_B) + 2P_2 = \frac{5}{9} + \frac{2}{9} = \frac{7}{9}$

d) $W = \frac{L}{\lambda_a}$, $\lambda_a = \lambda(P_0 + P_{1B}) = 2\times\frac{\cancel{6}^{2}}{\cancel{9}3} = \frac{4}{3}$

$$W = \frac{7}{{}_{3}\cancel{9}\times 4/\cancel{3}} = \frac{7}{12}$$

(14)

Outra forma:

$$E(T|E) = E(T|E_0)\underbrace{\frac{P(E_0)}{P(E)}}_{=1/2} + E(T|E_1B)\underbrace{\frac{P(E_1B)}{P(E)}}_{=1/2}$$

$$E(T|E_0) = \frac{1}{\mu_A} + \frac{1}{\mu_B} = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

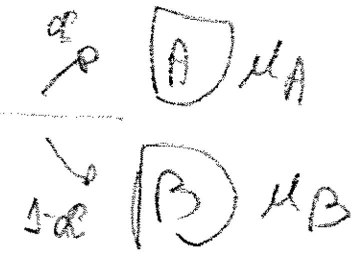
$$E(T|E_1B) = \frac{1}{\mu_A} + \frac{\cancel{\mu_B}}{\mu_A+\mu_B}\left(\frac{1}{\cancel{\mu_B}}\right) = \frac{1}{\mu_A} + \frac{1}{\mu_A+\mu_B}$$

$$= \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

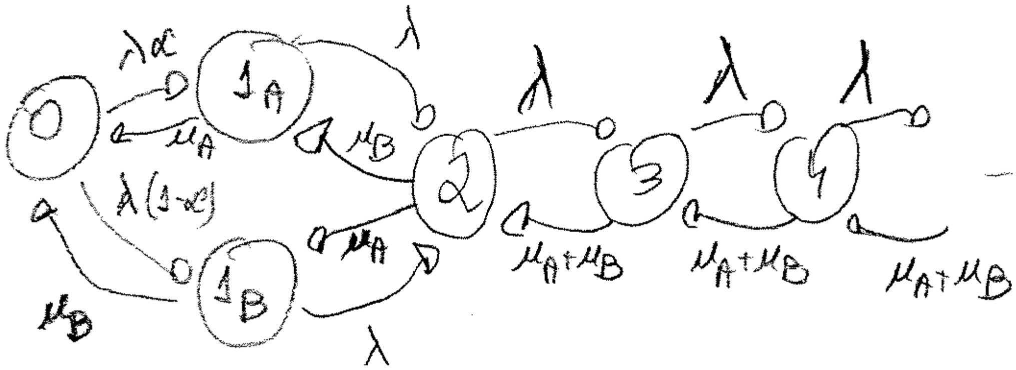
$$E(T|E) = \frac{1}{2}\left(\frac{3}{4} + \frac{5}{12}\right) = \frac{\cancel{14}}{\cancel{2}\times 12} = \frac{7}{12}$$

8

15



a)



$$
\begin{array}{l|l}
0 & \lambda P_0 = \mu_A P_{1A} + \mu_B P_{1B} \\
1_A & (\lambda+\mu_A) P_{1A} = \lambda\alpha P_0 + \mu_B P_2 \\
1_B & (\lambda+\mu_B) P_{1B} = \lambda(1-\alpha) P_0 + \mu_A P_2 \\
2 & (\lambda+\mu_A+\mu_B) P_2 = \lambda P_{1A} + \lambda P_{1B} + (\mu_A+\mu_B) P_3 \\
\vdots & \\
n & (\lambda+\mu_A+\mu_B) P_n = \lambda P_{n-1} + (\mu_A+\mu_B) P_{n+1}
\end{array}
$$

b) $L = 1.(P_{1A}+P_{1B}) + \sum_{n=2}^{\infty} n P_n$

(servidores ociosos): $L_{ociosos} = 2.P_0 + 1(P_{1A}+P_{1B})$

c) $P(\text{servido por A}) = P(\text{servido por A} \mid X=0).P_0 + P(\text{servido por A} \mid X=1).P_1 +$

$P(\text{servido por A} \mid X \geq 2)(1-P_0-P_1) = \alpha P_0 + P_{1B} + \frac{\mu_A}{\mu_A+\mu_B}(1-P_0-P_1)$

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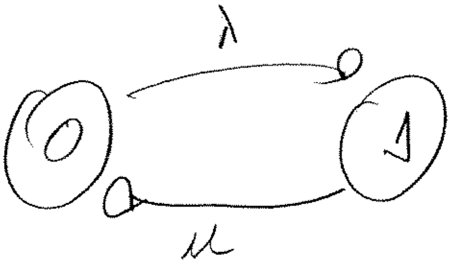

Equações de Avanço

$$\dot{P}_{ij}(t) = \sum_{k \neq j} v_k \, p_{kj} \, P_{ik}(t) - v_j P_{ij}(t)$$

$v_0 = \lambda$, $p_{01} = 1$

$v_1 = \mu$, $p_{10} = 1$

$$\begin{cases} \dot{P}_{01}(t) = v_0 \, p_{01} P_{00}(t) - v_1 P_{01}(t) \\ \quad = \lambda P_{00}(t) - \mu P_{01}(t) \end{cases} \qquad P_{00}(t) + P_{01}(t) = 1$$

$$\begin{cases} \dot{P}_{10}(t) = v_1 \, p_{10} P_{11}(t) - v_0 P_{10}(t) \\ \quad = \mu P_{11}(t) - \lambda P_{10}(t) \end{cases} \qquad P_{10}(t) + P_{11}(t) = 1$$

$$\begin{cases} \dot{P}_{01}(t) = \lambda P_{00}(t) - \mu P_{01}(t) \\ P_{00}(t) + P_{01}(t) = 1 \end{cases}$$

$$P_{00}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

$$\dot{P}_{01}(t) = \lambda \left(1 - P_{01}(t)\right) - \mu P_{01}(t)$$

$$= \lambda - (\lambda+\mu) P_{01}(t) \quad , \quad P_{01}(0) = 0$$

$$P_{01}(t) = \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} \quad , \quad P_{10}(t) = \frac{\mu}{\lambda+\mu} - \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

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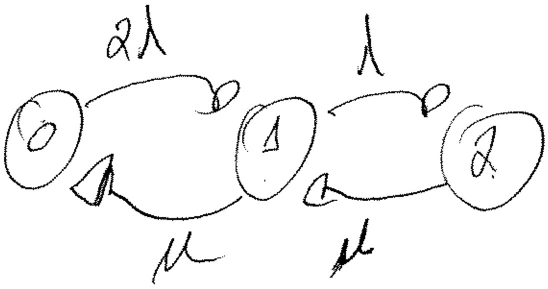

n = nº de máquinas fora de uso

Equações de Avanço: $P_{00}(0)=1,\ P_{01}(0)=0,\ P_{02}(0)=0$

$$\begin{cases} \dot{P}_{01}(t) = 2\lambda P_{00}(t) + \mu P_{02}(t) - (\lambda+\mu) P_{01}(t) \\ \dot{P}_{00}(t) = \mu P_{01}(t) - 2\lambda P_{00}(t) \\ \dot{P}_{02}(t) = \lambda P_{01}(t) - \mu P_{02}(t) \end{cases}$$

$$t \to \infty \Rightarrow \begin{cases} (\lambda+\mu) P_1 = 2\lambda P_0 + \mu P_2 \\ 2\lambda P_0 = \mu P_1 \\ \mu P_2 = \lambda P_1 \end{cases}$$