PTC-3440 Modelos Probabilísticos

Oswaldo Luiz do Valle Costa

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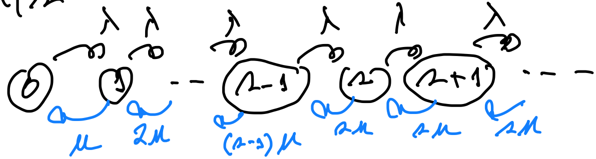
PTC-EPUSP

Fila M/M/s

Suponha que $\frac{\lambda}{s\mu} < 1$. Mostre que

$$\left\{ \begin{aligned} W_Q &= \frac{P(X \geq s)}{s\mu - \lambda} \\ L_Q &= \lambda W_Q \\ W &= W_Q + \frac{1}{\cancel{\lambda}} \rightarrow \mu \\ L &= \lambda\left(W_Q + \frac{1}{\mu}\right) \end{aligned} \right.$$

M|M|s



$$\mu_n = \begin{cases} n\mu, & 1 \le n \le s \\ s\mu, & n \ge s \end{cases}, \quad \lambda_n = \lambda, \; n \ge 0$$

$$\sum_{n=0}^{\infty} P_n = 1$$

Processo de Nascimento e Morte:

$$P_n = \begin{cases} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} P_0, & 0 \le n \le s \\ \left(\frac{\lambda}{s\mu}\right)^{n-s} \left(\frac{\lambda}{\mu}\right)^s \frac{1}{s!} P_0, & n \ge s \end{cases}$$

$$P_0 \left[\sum_{n=0}^{s-1} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} + \left(\frac{\lambda}{\mu}\right)^s \frac{1}{s!} \sum_{n=s}^{\infty} \left(\frac{\lambda}{s\mu}\right)^{n-s} \right] = 1$$

$$W_Q = \sum_{n=0}^{\infty} E(\underbrace{T_Q}_{\text{tempo gasto na fila}} \mid \underbrace{X = n}_{\text{dado que encontra } n \text{ no sistema na chegada}}) P_n$$

$$E(T_Q \mid X = n) = \begin{cases} 0, & n < s \\ \dfrac{n - s + 1}{s\mu}, & n \geq s \end{cases}$$

$$W_Q = \sum_{n=s}^{\infty} \left(\frac{n - s + 1}{s\mu} \right) P_n \quad , \quad L_Q = \sum_{n=s}^{\infty} (n - s) P_n$$

$$W_Q = \underbrace{\left[\sum_{n=s}^{\infty} (n - s) P_n \right]}_{L_Q} \left(\frac{1}{s\mu} \right) + \underbrace{\sum_{n=s}^{\infty} P_n}_{P(X \geq s)} \left(\frac{1}{s\mu} \right)$$

$$W_Q = \frac{1}{s\mu} L_Q + \frac{1}{s\mu} P(X \geq s)$$

$$L_Q = \lambda W_Q \Rightarrow W_Q = \frac{1}{s\mu} \lambda W_Q + \frac{1}{s\mu} P(X \geq s)$$

$$\Rightarrow \left(1 - \frac{\lambda}{s\mu}\right) W_Q = \frac{1}{s\mu} P(X \geq s)$$

$$(s\mu - \lambda) W_Q = P(X \geq s)$$

Conclusão: $$\boxed{W_Q = \frac{P(X \geq s)}{s\mu - \lambda}}$$

$$W = W_Q + \frac{1}{\mu}, \quad L_Q = \lambda W_Q, \quad L = \lambda W$$

<u>Caso $s = 1$:</u> $P_0 = P(X = 0) = 1 - \frac{\lambda}{\mu}$

$$\Rightarrow P(X \geq 1) = 1 - P_0 = \frac{\lambda}{\mu}$$

$$W_Q = \frac{P(X \geq 1)}{\mu - \lambda} = \frac{\lambda/\mu}{\mu - \lambda} = \frac{\lambda}{\mu(\mu - \lambda)}$$ ✓ OK

Como calcular $P(X \geq s)$?

$$P_n = \begin{cases} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} P_0, & 0 \leq n \leq s \\ \left(\frac{\lambda}{s\mu}\right)^{n-s} \left(\frac{\lambda}{\mu}\right)^s \frac{1}{s!} P_0, & n \geq s \end{cases}$$

$$P(X \geq s) = \sum_{n=s}^{\infty} P_n = \left(\frac{\lambda}{\mu}\right)^s \frac{1}{s!} P_0 \underbrace{\sum_{n=s}^{\infty} \left(\frac{\lambda}{s\mu}\right)^{n-s}}_{3}$$

$\frac{\lambda}{s\mu} < 1$

$$\Rightarrow \sum_{n=s}^{\infty} \left(\frac{\lambda}{s\mu}\right)^{n-s} = \sum_{n=0}^{\infty} \left(\frac{\lambda}{s\mu}\right)^{n} = \frac{1}{1 - \frac{\lambda}{s\mu}}$$

$$P_0 = \left(\frac{\lambda}{\mu}\right)^s \frac{1}{s!} \left(\frac{1}{1 - \lambda/s\mu}\right) P_0$$

ρ

$$\rho = \frac{\lambda}{s\mu}$$

Fila M/M/s

Defina $\rho = \frac{\lambda}{s\mu}$.

$$P(X \geq s) = P_0\left(\frac{1}{1-\rho}\right)\left(\frac{\lambda}{\mu}\right)^s\left(\frac{1}{s!}\right)$$

$$P_0 = \frac{1}{\sum_{n=0}^{s-1}\left(\frac{(\frac{\lambda}{\mu})^n}{n!}\right) + \frac{(\frac{\lambda}{\mu})^s}{s!(1-\rho)}}$$

$$\sum_{n=0}^{\infty} P_n = 1 \Rightarrow$$

$$P_0\left[\sum_{n=0}^{s-1}\left(\frac{\lambda}{\mu}\right)^n\frac{1}{n!}+\left(\frac{\lambda}{\mu}\right)^s\left(\frac{1}{s!}\right)\underbrace{\sum_{n=s}^{\infty}\left(\frac{\lambda}{s\mu}\right)^{n-s}}_{\frac{1}{1-\lambda/s\mu}}\right]=1$$

$$P_0=\frac{1}{\displaystyle\sum_{n=0}^{s-1}\left(\frac{\lambda}{\mu}\right)^n\frac{1}{n!}+\left(\frac{\lambda}{\mu}\right)^s\left(\frac{1}{s!}\right)\frac{1}{1-\frac{\lambda}{s\mu}}}$$

$$\underline{\underline{s=1}} \Rightarrow P_0=\frac{1}{1+\frac{\lambda}{\mu}\,\frac{1}{1-\lambda/\mu}}=\frac{1}{1+\frac{\lambda}{\mu-\lambda}}=\frac{1}{\mu/(\mu-\lambda)}=1-\frac{\lambda}{\mu}\quad \text{OK}$$

$\underline{s=2}$: $\quad P_0 = \dfrac{1}{1+\frac{\lambda}{\mu}+\left(\frac{\lambda}{\mu}\right)^2 \frac{1}{2} \frac{1}{1-\lambda/2\mu}}$

$$= \frac{1}{1+\frac{\lambda}{\mu}+\left(\frac{\lambda}{\mu}\right)^2\left(\frac{1}{2-\lambda/\mu}\right)}$$

$$= \frac{1}{1+\frac{\lambda}{\mu}+\left(\frac{\lambda}{\mu}\right)^2\left(\frac{\mu}{2\mu-\lambda}\right)} = \frac{1}{1+\frac{\lambda}{\mu}+\frac{\lambda^2}{\mu}\left(\frac{1}{2\mu-\lambda}\right)}$$

$$\frac{\lambda}{\mu} = \frac{3}{2} \Rightarrow P_0 = \frac{1}{1+3/2+(3/2)^2\left(\frac{1}{2-3/2}\right)}$$

$$= \frac{1}{5/2 + 9/4\left(\frac{2}{1}\right)} = \frac{1}{5/2 + 9/2} = \frac{2}{14} = \frac{1}{7}$$

$$P_0 = 1/7$$

$$P(X \geq 2) = \left(\frac{1}{1 - \frac{\lambda}{2\mu}}\right)\left(\frac{\lambda}{\mu}\right)^2 \frac{1}{2} P_0$$

$$= \left(\frac{1}{1 - 3/4}\right)\left(3/2\right)^2 \cdot \frac{1}{2} \cdot \frac{1}{7} = \frac{9}{2} \cdot \frac{1}{7} = 9/14$$

$$P(X \geq 2) = 9/14$$

$$W_Q = \frac{P(X \geq 2)}{2\mu - \lambda} = \frac{9/14}{2/15 - 1/10} =$$

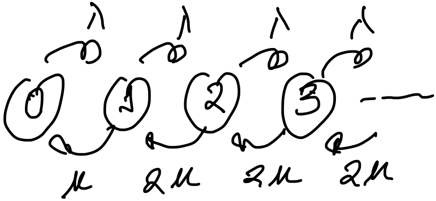
$$= \frac{9/14}{4/30 - 3/30} = \frac{9}{14} \cdot 30 = \frac{135}{7}.$$

$$\cong 19{,}28 \text{ mins}$$

$$W = W_Q + \frac{1}{\mu} = 19{,}28 + 15 = 34{,}28 \text{ mins}$$

$$\rho \; 1/10$$

$$L = \lambda W = \frac{1}{10} \cdot 34{,}28 = 3{,}428 \quad , \quad L_Q = \lambda W_Q = 1{,}928$$



$s = 2$

$\lambda = \frac{1}{10}$ /min

$\mu = \frac{1}{15}$ /min

$$\rho = \frac{\lambda}{2\mu} = \frac{\overset{3}{\cancel{15}}}{2 \cdot \underset{2}{\cancel{10}}} = \frac{3}{4} < 1$$

OK. Condição de estabilidade satisfeita

Fila M/M/s

Obtenha W_Q, W, L_Q, L para o caso

$$s = 2, \quad \frac{1}{\lambda} = 10 \text{ mins}, \quad \frac{1}{\mu} = 15 \text{ mins}.$$