PTC-3440 Modelos Probabilísticos

Oswaldo Luiz do Valle Costa

Aulas 19 - 21 - 2021

PTC-EPUSP

Fila M/M/1/N

Seja $\rho = \frac{\lambda}{\mu}$. Mostre que

$$P_n = \rho^n \frac{1-\rho}{1-\rho^{N+1}}, \;\; n = 0, 1, \ldots, N$$

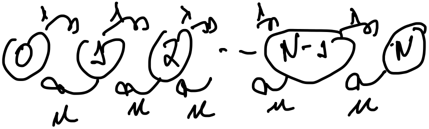
Percentual de utilização do servidor:

$$U = 1 - P_0 = \rho \frac{1-\rho^{N}}{1-\rho^{N+1}}$$

Probabilidade de Bloqueio:

$$P_B = P_N = \rho^N \frac{1-\rho}{1-\rho^{N+1}}$$

$M|M|1|N$



Estado	Sai = Entra
0	$\lambda P_0 = \mu P_1$
$\vdots$	$\vdots$
n	$(\lambda+\mu)P_n = \lambda P_{n-1} + \mu P_{n+1}$
$\vdots$	$\vdots$
N	$\mu P_N = \lambda P_{N-1}$

$$\lambda P_0 = \mu P_1$$

$$(\lambda+\cancel{\mu})P_1 = \cancel{\lambda P_0} + \mu P_2 \Rightarrow \lambda P_1 = \mu P_2$$

$$(\lambda+\cancel{\mu})P_2 = \cancel{\lambda} P_1 + \mu P_3 \Rightarrow \lambda P_2 = \mu P_3$$

$$\vdots$$

$$\lambda P_{N-1} = \mu P_N$$

$$\lambda P_i = \mu P_{i+1} \Rightarrow P_{i+1} = \left(\frac{\lambda}{\mu}\right) P_i \Rightarrow P_i = \left(\frac{\lambda}{\mu}\right)^i P_0 \quad i=0,1,N$$

$$1 = \sum_{i=0}^{N} P_i = P_0 \sum_{i=0}^{N} \left(\frac{\lambda}{\mu}\right)^i \Rightarrow P_0 = \frac{1}{\sum_{i=0}^{N} \left(\frac{\lambda}{\mu}\right)^i}$$

$$\lambda \neq \mu \Rightarrow \sum_{i=0}^{N} \left(\frac{\lambda}{\mu}\right)^i = \frac{1-\left(\frac{\lambda}{\mu}\right)^{N+1}}{1-\lambda/\mu}$$

$$P_0 = \frac{1-\lambda/\mu}{1-\left(\frac{\lambda}{\mu}\right)^{N+1}}, \quad \Rightarrow P_i = \left[\frac{1-\lambda/\mu}{1-\left(\frac{\lambda}{\mu}\right)^{N+1}}\right]\left(\frac{\lambda}{\mu}\right)^i$$

$$i = 0, \dots, N$$

$$L = \lambda_a W, \quad L_Q = \lambda_a W_Q \qquad \lambda_a = (1-P_N)\lambda$$

$$\begin{cases} L = \sum_{i=0}^{N} i P_i, \quad W = \frac{L}{\lambda(1-P_N)}, \quad W_Q = W - \frac{1}{\mu} \\ L_Q = (1-P_N)\lambda W_Q \end{cases}$$

$$\rho = \frac{\lambda}{\mu} \quad \Rightarrow P_i = \left(\frac{1-\rho}{1-\rho^{N+1}}\right)\rho^i$$

$$\Rightarrow U = 1 - P_0 = 1 - \frac{1-\rho}{1-\rho^{N+1}} = \frac{\cancel{1}-\rho^{N+1}\cancel{-1}+\rho}{1-\rho^{N+1}} = \frac{\rho(1-\rho^N)}{1-\rho^{N+1}}$$

$$P_B = P_N = \rho^N \frac{(1-\rho)}{1-\rho^{N+1}}$$

Exemplo - Fila M/M/1/N

Determine o menor N em uma fila M/M/1/N de modo que não mais que 10% de chegadas sejam bloqueadas. Considere $\lambda = 1$ por minuto, taxa de processamento μ com custo C_μ. Custo de espaço na fila é 80. Logo desejamos minimizar

$$\min \;\; C_\mu + 80N.$$

Considere 3 casos:

1. $\mu_1 = 0,5$ por min com $C_{\mu_1} = 100$
2. $\mu_2 = 1,2$ por min com $C_{\mu_2} = 300$
3. $\mu_3 = 2$ por min com $C_{\mu_3} = 500$

$$P_B \leq 0,1$$

① $u_1 = 0,5 \Rightarrow$ não existe N que atenda a essa restrição

② $u_2 = 1,2 \Rightarrow N = 6$

$$g(u_2, 6) = 300 + 80 \cdot 6 = \underline{\underline{780}}$$

$$P_B = 7,74\%$$

③ $u_3 = 2 \Rightarrow p = \frac{1}{u} = \frac{1}{2}$

$$P_B = P_N = \left(\frac{1}{2}\right)^N \left(\frac{1 - 1/2}{1 - (1/2)^{N+1}}\right) \leq 0,1$$

$$\left(\frac{1}{2}\right)^{N}\left(\frac{1}{2}\right) \le 0{,}1\left(1-\left(\frac{1}{2}\right)^{N+1}\right) \Rightarrow (1+0{,}1)\left(\frac{1}{2}\right)^{N+1} \le 0{,}1$$

$$1{,}1\left(\frac{1}{2}\right)^{N+1} \le 0{,}1 \Rightarrow \left(\frac{1}{2}\right)^{N+1} \le \frac{1}{11}$$

$$(N+1)\underbrace{\ln(1/2)}_{-0{,}693} \le \underbrace{\ln(1/11)}_{-2{,}3972} \Rightarrow$$

$$-0{,}693(N+1) \le -2{,}3972 \Rightarrow N+1 \ge \overset{3{,}17}{\frac{2{,}3972}{0{,}693}}$$

$$N \ge 2{,}17 \Rightarrow N=3$$

$$f(\mu_3, 3) = 500 + 3\cdot 80 = 740$$ mínimo

$$P_B = \frac{\left(\frac{1}{2}\right)^3\cdot 1/2}{1-(1/2)^4} = \frac{(1/2)^4}{1-(1/2)^4} = 6{,}67\%$$

$$U = \rho\,\frac{1-\rho^N}{1-\rho^{N+1}} = \frac{1}{2}\cdot\frac{1-(1/2)^3}{1-(1/2)^4} = 46{,}67\,\% \doteq 7/15$$

$N=3$

Calcule L, W, L_Q, W_Q.

$$P_n = \rho^n\,\frac{1-\rho}{1-\rho^{N+1}}$$

$$P_0 = \frac{1-1/2}{1-(1/2)^4} = \frac{1/2}{1-1/16} = \frac{8}{15}, \quad P_1 = \frac{1}{\cancel{2}}\cdot\frac{\overset{4}{\cancel{8}}}{15} = 4/15$$

$$P_2 = \frac{1}{\cancel{2}}\cdot\overset{2}{\cancel{4}}/15 = \frac{2}{15}, \quad P_3 = \frac{1}{\cancel{2}}\,\frac{\cancel{2}}{15} = 1/15$$

$$\lambda_a = \underbrace{(1-P_3)}_{14/15}\underbrace{\lambda}_{1}$$

$$L = 1.P_1 + 2.P_2 + 3P_3 = \frac{4+4+3}{15} = \frac{11}{15} = 0{,}73$$

$$W = \frac{L}{\lambda_a} = \frac{11}{\cancel{15}\cdot\frac{14}{\cancel{15}}} = \frac{11}{14}\text{ min} = 0{,}786\text{ min}$$

$$W_Q = W - \frac{1}{\mu} = \frac{11}{14} - \frac{1}{2} = \frac{4}{14} = \frac{2}{7}\text{ min}$$

$$L_Q = \underbrace{(1-P_3)\lambda}_{\lambda_a} W_Q = \frac{\overset{2}{\cancel{14}}}{15}\cdot\frac{2}{\cancel{7}} = \frac{4}{15} = 0{,}267$$

Fila M/M/s

Suponha que $\frac{\lambda}{s\mu} < 1$. Mostre que

$$
\begin{aligned}
W_Q &= \frac{P(X \geq s)}{s\mu - \lambda} \\
L_Q &= \lambda W_Q \\
W &= W_Q + \frac{1}{\cancel{\lambda}} \rightarrow \mu \\
L &= \lambda\left(W_Q + \frac{1}{\mu}\right)
\end{aligned}
$$

Fila M/M/s

Defina $\rho = \frac{\lambda}{s\mu}$.

$$P(X \geq s) = P_0 \left(\frac{1}{1-\rho} \right) \left(\frac{\lambda}{\mu} \right)^s \left(\frac{1}{s!} \right)$$

$$P_0 = \frac{1}{\sum_{n=0}^{s-1} \left(\frac{(\frac{\lambda}{\mu})^n}{n!} \right) + \frac{(\frac{\lambda}{\mu})^s}{s!(1-\rho)}}$$

Fila M/M/s

Obtenha W_Q, W, L_Q, L para o caso

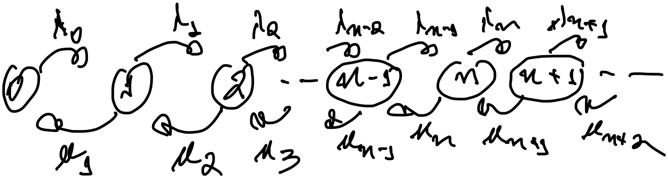
$$s = 2, \ \ \frac{1}{\lambda} = 10 \text{ mins}, \ \ \frac{1}{\mu} = 15 \text{ mins}.$$

Processos de Nascimento e Morte

Seja $\{X(t)\}$ um modelo exponencial tal que $X(t) =$ número de pessoas em sistema no instante t. Temos que quando $X(t) = n$:

A) Novas chegadas ocorrem de acordo com tempo exponencial com parâmetro λ_n.

B) Pessoas deixam o sistema de acordo com um tempo exponencial com parâmetro μ_n.



Exemplos:

1) Processos de Poisson

$$\mu_n = 0, \;\; \lambda_n = \lambda, \;\; n = 0, 1, 2, \ldots$$

2) Processos de Nascimento com taxa de nascimento linear

$$\mu_n = 0, \;\; \lambda_n = n\lambda, \;\; n = 0, 1, 2, \ldots$$

Exemplos:

3) Fila M/M/1

$$\mu_n = \mu, \quad n = 1, 2, \ldots,$$
$$\lambda_n = \lambda, \quad n = 0, 1, 2, \ldots$$

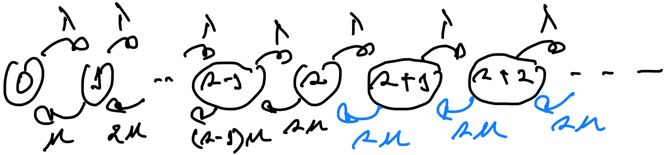
4) Fila M/M/1/N

$$\mu_n = \mu, \quad n = 1, 2, \ldots,$$
$$\lambda_n = \lambda, \quad n = 0, 1, 2, \ldots, N-1,$$
$$\lambda_n = 0, \quad n = N, N+1, \ldots$$

Exemplos:

5) Fila M/M/s (fila com s servidores)

$$
\left.
\begin{aligned}
\mu_n &= n\mu, \quad n = 1, 2, \ldots, s \\
\mu_n &= s\mu, \quad n = s, s+1, \ldots \\
\lambda_n &= \lambda, \quad n = 0, 1, 2, \ldots
\end{aligned}
\right\}
$$

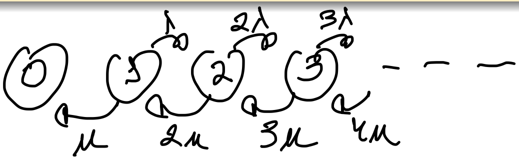


Exemplos:

6) Modelo de Crescimento Linear

$$\mu_n = n\mu, \quad n = 1, 2, \ldots$$

$$\lambda_n = n\lambda, \quad n = 0, 1, 2, \ldots$$

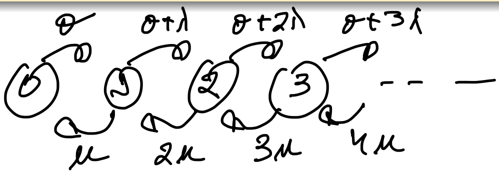


Exemplos:

7) Modelo de Crescimento com Imigração

$$\mu_n = n\mu, \quad n = 1, 2, \ldots$$

$$\lambda_n = \theta + n\lambda, \quad n = 0, 1, 2, \ldots$$

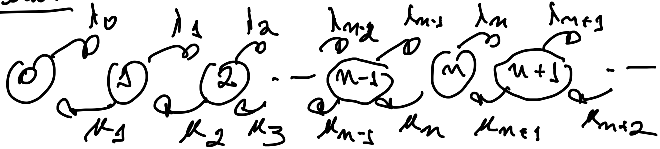


Exemplos:

8) Modelo de Imigração/Emigração

$$\mu_n = n\mu, \quad n = 1, 2, \ldots$$
$$\lambda_n = \lambda, \quad n = 0, 1, 2, \ldots$$

Caso Geral



Equações de Balanço

Sai = Entra

$$\lambda_0 P_0 = \mu_1 P_1$$

$$(\lambda_n + \mu_n) = \lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1}, \quad n = 1, 2, \ldots$$

Estado	Sai = Entra
0	$\lambda_0 P_0 = \mu_1 P_1$
1	$(\lambda_1 + \mu_1) P_1 = \lambda_0 P_0 + \mu_2 P_2$
⋮	
n	$(\lambda_n + \mu_n) P_n = \lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1}$

Equações de Balanço

Segue que

$$\lambda_n P_n = \mu_{n+1} P_{n+1}, \quad n = 0, 1, 2, \ldots$$

Caso das máquinas:

$$(M-n)\lambda P_n = \mu P_{n+1}$$

$$\lambda_0 P_0 = \mu_1 P_1 \Rightarrow P_1 = \frac{\lambda_0}{\mu_1} P_0$$

$$(\lambda_1 + \cancel{\mu_1}) P_1 = \cancel{\lambda_0 P_0} + \mu_2 P_2 \Rightarrow \lambda_1 P_1 = \mu_2 P_2 \Rightarrow P_2 = \frac{\lambda_1}{\mu_2} P_1$$

$$(\lambda_2 + \cancel{\mu_2}) P_2 = \cancel{\lambda_1 P_1} + \mu_3 P_3 \Rightarrow \lambda_2 P_2 = \mu_3 P_3 \Rightarrow P_3 = \frac{\lambda_2}{\mu_3} P_2$$

$$\vdots \qquad\qquad \vdots$$

Logo $$P_{n+1} = \frac{\lambda_n}{\mu_{n+1}} P_n = \left[\frac{\lambda_n \lambda_{n-1} \cdots \lambda_0}{\mu_{n+1} \mu_n \cdots \mu_1}\right] P_0$$

$$1 = \sum_{n=0}^{\infty} P_n = P_0 \sum_{n=0}^{\infty} \left[\frac{\lambda_{n-1} \cdots \lambda_0}{\mu_n \cdots \mu_1}\right]$$

Convenção $\frac{\lambda_{-1}}{\mu_0} = 1$

Condição de Equilíbrio: $$\boxed{\sum_{j=1}^{\infty} \frac{\lambda_{j-1} \cdots \lambda_0}{\mu_j \cdots \mu_1} < \infty}$$

$$P_0 = \frac{1}{\sum_{j=0}^{\infty} \frac{\lambda_{j-1} \cdots \lambda_0}{\mu_j \cdots \mu_1}}$$

$$P_n = \left[\frac{\lambda_{n-1} - \lambda_0}{\mu_n - \mu_1}\right] \left[\frac{1}{\sum_{j=0}^{\infty} \frac{\lambda_{j-1} - \lambda_0}{\mu_j - \mu_1}}\right]$$

$$\left.\begin{array}{l} \lambda_n = \lambda \\ \mu_n = \mu \end{array}\right\} \quad \sum_{j=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^j < \infty \iff \frac{\lambda}{\mu} < 1$$

Equações de Balanço

Condição de Equilíbrio:

$$\sum_{j=1}^{\infty} \left(\frac{\lambda_{j-1} \dots \lambda_0}{\mu_j \dots \mu_1} \right) < \infty$$

Solução:

$$P_n = \left(\frac{\lambda_{n-1} \dots \lambda_0}{\mu_n \dots \mu_1} \right) \frac{1}{\sum_{j=0}^{\infty} \left(\frac{\lambda_{j-1} \dots \lambda_0}{\mu_j \dots \mu_1} \right)}$$

Exemplos

Teste do Raio: Considere $a_n > 0$. Se $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} < 1$ então $\sum_{i=0}^{\infty} a_n < \infty$.

1. Modelo 5: Equilíbrio se $\frac{\lambda}{s\mu} < 1$
2. Modelo 7: Equilíbrio se $\frac{\lambda}{\mu} < 1$.

Modelo 5:
$$\begin{cases} \mu_n = n\mu, & n = 0, \ldots, s \\ \mu_n = s\mu & n = s+1, \ldots \\ \lambda_n = \lambda, & n = 0, 1, \ldots \end{cases}$$

finito $\sum_{n=1}^{s} \frac{\lambda^n}{n! \cdot \mu^n} + \sum_{n=s+1}^{\infty} \frac{\lambda^n}{s! \cdot \mu^s \cdot (s\mu)^{n-s}}$

$n \geq s+1$

$$a_n = \frac{\lambda^n}{s! \cdot \mu^s (s\mu)^{n-s}} , \quad a_{n+1} = \frac{\lambda^{n+1}}{s! \cdot \mu^s (s\mu)^{n+1-s}}$$

$$\frac{a_{n+1}}{a_n} = \frac{\lambda}{s\mu} < 1$$

Condição de equilíbrio

Exemplos

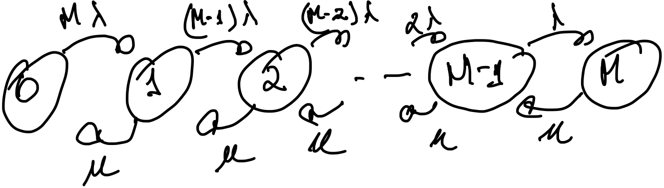
M máquinas funcionam em paralelo, com 1 oficina de reparo. Cada máquina funciona antes de quebrar por um período de tempo com distribuição exponencial e taxa $\lambda > 0$. Um reparo leva um período de tempo exponencial com taxa $\mu > 0$.

1. Em média quantas máquinas não estão em uso? Resposta:

$$L = \sum_{n=1}^{M} n\left(\frac{\frac{M!}{(M-n)!}\left(\frac{\lambda}{\mu}\right)^n}{1+\sum_{j=1}^{M}\left(\frac{\lambda}{\mu}\right)^j\left(\frac{M!}{(M-j)!}\right)}\right)$$

2. Qual é a proporção de tempo que cada máquina fica em uso? Resposta: $\frac{1}{M}\left(\frac{\mu}{\lambda}\right)(1-P_0)$

$X(t)$ = número de máquinas fora de uso no instante t.



$$\mu_n = \mu, \quad \lambda_n = (M-n)\lambda, \quad n = 0, \dots, M$$

$$\lambda_n = 0, \quad n \geq M$$

$$P_0 = \frac{1}{1 + \sum_{n=1}^{M} \frac{M\lambda (M-1)\lambda \cdots (M-n+1)\lambda}{\mu^n}}$$

$$= \frac{1}{1 + \sum_{n=1}^{M} \left(\frac{\lambda}{\mu}\right)^n \frac{M!}{(M-n)!}}$$

$$P_n = \frac{\dfrac{M!}{(M-n)!}\left(\dfrac{\lambda}{\mu}\right)^n}{1 + \sum_{j=1}^{M}\left(\dfrac{\lambda}{\mu}\right)^j \dfrac{M!}{(M-j)!}}, \quad n = 0, 1, \ldots, M$$

a) $L = \sum_{n=0}^{M} n P_n$

b) $Z(t) = \begin{cases} 1 & \text{se máquina 1 funciona em } t \\ 0 & \text{" " " não funciona em } t \end{cases}$

$$P\{Z(t)=1\} = E\left[P(Z(t)=1 \mid X(t))\right]$$

$$= \sum_{n=0}^{M} P\left(Z(t)=1 \mid X(t)=n\right) P\left(X(t)=n\right)$$

$$\begin{cases} P(Z(t)=1 \mid X(t)=0) = 1 \\ P(Z(t)=1 \mid X(t)=1) = \frac{M-1}{M} \\ P(Z(t)=1 \mid X(t)=2) = \frac{M-2}{M} \\ \vdots \\ P(Z(t)=1 \mid X(t)=n) = \frac{M-n}{M} \\ \vdots \\ P(Z(t)=1 \mid X(t)=M) = 0 \end{cases}$$

$$P_n(t) = P(X(t)=n)$$

$$(M-n)\lambda P_n = \mu P_{n+1}$$

$$\Rightarrow (M-n) P_n = \frac{\mu}{\lambda} P_{n+1}$$

$$P(Z(t)=1) = \sum_{n=0}^{M} \left(\frac{M-n}{M}\right) P_n(t)$$

$$\sum_{n=1}^{M} P_n = 1-P_0$$

$$P(Z(\infty)=1) = \sum_{n=0}^{M-1} \left(\frac{M-n}{M}\right) P_n = \frac{\mu}{\lambda M} \sum_{n=0}^{M-1} P_{n+1} = \frac{1}{M}\left(\frac{\mu}{\lambda}\right)(1-P_0)$$

$$1 = P_0\left(1 + 5\frac{\lambda}{\mu} + 20\left(\frac{\lambda}{\mu}\right)^2 + 60\left(\frac{\lambda}{\mu}\right)^3 + 120\left(\frac{\lambda}{\mu}\right)^4 + 120\left(\frac{\lambda}{\mu}\right)^5\right)$$

Exemplos

Considere $M = 5$. Compare os casos:

1. $\lambda = \mu$
2. $\frac{\lambda}{\mu} = \frac{1}{2}$

$M=5$ 1 $\underline{\underline{\lambda = \mu}}$ $P_0 \left(\underbrace{1 + 5 + 20 + 60 + 120 + 120}_{326}\right) = 1$

$P_0 = \frac{1}{326}$, $P_1 = \frac{5}{326}$, $P_2 = \frac{20}{326}$, $P_3 = \frac{60}{326}$

$P_4 = \frac{120}{326}$, $P_5 = \frac{120}{326}$

$$L_q = 1 \cdot \frac{5}{326} + 2 \cdot \frac{20}{326} + 3 \cdot \frac{60}{326} + 4 \cdot \frac{120}{326} + 5 \cdot \frac{120}{326}$$

número médio de máquinas fora de uso

$$= \frac{1305}{326} \simeq 4$$

% tempo em uso $= \frac{1}{\mu}\left(\frac{\mu}{\lambda}\right)(1-P_0) = \frac{1}{5} \cdot 1 \cdot \left(1 - \frac{1}{326}\right) \simeq 20\%$

② $\frac{\lambda}{\mu} = \frac{1}{2} \Rightarrow \frac{1}{\mu} = \left(\frac{1}{2}\right)\left(\frac{1}{\lambda}\right)$

Mostra que: $P_0 = \frac{4}{109}$, $P_1 = \frac{10}{109}$, $P_2 = \frac{20}{109}$

$P_3 = \frac{30}{109}$, $P_4 = \frac{30}{109}$, $P_5 = \frac{15}{109}$

$L = 3{,}07$ ($\downarrow$ 23,2%)

% tempo em funcionamento = 38,5% ($\uparrow$ 92,7%)