# PGF5003: Classical Electrodynamics I
# Problem Set 4

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(Due to June 1, 2021)

**Guidelines:** *write down the most relevant passages in your calculations, not only the final results. Do not forget to write the mathematical expressions that you are using in order to solve the questions. We strongly recommended the use of the International System of Units.*

## 1 Question (1 point)

*In three dimensions the solution to the wave equation*

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = -4\pi f(\mathbf{r},t), \tag{1}$$

*where $\Psi(\mathbf{r},t)$ is the wave function and $f(\mathbf{r},t)$ is a known source distribution, for a point source in space and time (a light flash at $t'=0$ and $\mathbf{r}'=0$) is a spherical shell disturbance of radius $R=ct$, namely the* **retarded Green function**[1]

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{\delta\left(t' - \left[t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right]\right)}{|\mathbf{r}-\mathbf{r}'|}. \tag{2}$$

*The solutions for fewer dimensions than three can be found by superposition in the superfluous dimension(s), to eliminate the dependence on such variable(s). For example, a flashing line source of uniform amplitude is equivalent to a point source in two dimensions.*

**a)** *Starting with the retarded solution to the three-dimensional wave equation*

$$\Psi(\mathbf{r},t) = \int d^3r' \frac{[f(\mathbf{r}',t')]_{ret}}{|\mathbf{r}-\mathbf{r}'|}, \tag{3}$$

*where $[\ ]_{ret}$ means that the time $t'$ is to be evaluated at the retarded time $t' = t - |\mathbf{r}-\mathbf{r}'|/c$, show that the source $f(\mathbf{r}',t') = \delta(x')\delta(y')\delta(t')$ (which is equivalent to a $t=0$ point source at the origin in two spatial dimensions), produces a two-dimensional wave*

$$\Psi(x,y,t) = \frac{2c\Theta(ct-\rho)}{\sqrt{c^2t^2-\rho^2}}, \tag{4}$$

*where $\rho^2 = x^2+y^2$ and $\Theta(\chi)$ is the unit step function $\Theta(\chi) = \begin{cases} 0, \chi < 0 \\ 1, \chi > 0 \end{cases}$.*

[1] Just remember that this name is due the fact that this function exhibits the causal behavior associated to a wave disturbance.

**b)** *Show that a "sheet" source, equivalent to a point pulsed source at the origin in one space dimension, produces a one-dimensional wave proportional to*

$$\Psi(x,t) = 2\pi c\Theta(ct - |x|). \tag{5}$$

## 1.1 Solution

**a)** We need to write the retarded solution in 2D. Then, we need first the source in retarded time $t' = t - |\mathbf{r} - \mathbf{r}'|/c$. Taking a flash line source in the $z$ axis, it becomes

$$\begin{aligned} f(\mathbf{r}',t') &= \delta(x')\delta(y')\delta(t') \\ [f(\mathbf{r}',t')]_{ret} &= \delta(x')\delta(y')\delta(t - |\mathbf{r}-\mathbf{r}'|/c). \end{aligned} \tag{6}$$

Putting it in the retarded solution

$$\Psi(\mathbf{r},t) = \int d^3r' \frac{[f(\mathbf{r}',t')]_{ret}}{|\mathbf{r}-\mathbf{r}'|} = \int d^3r' \frac{\delta(x')\delta(y')\delta(t - |\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|} \tag{7}$$

$$= \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz' \frac{\delta(x')\delta(y')\delta(t - \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}/c)}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \tag{8}$$

$$= \int_{-\infty}^{\infty} dz' \frac{\delta(t - \sqrt{x^2+y^2+(z-z')^2}/c)}{\sqrt{x^2+y^2+(z-z')^2}} = \int_{-\infty}^{\infty} dz' \frac{\delta(t - \sqrt{\rho^2+(z-z')^2}/c)}{\sqrt{\rho^2+(z-z')^2}}. \tag{9}$$

To solve this integral we can use

$$\delta[f(x)] = \sum_i \frac{\delta(x-x_i)}{|df(x_i)/dx|}, \tag{10}$$

where $x_i$ are the zeros of $f(x)$. In our case:

$$x = z' \tag{11}$$

$$f(x) = t - \sqrt{\rho^2+(z-z')^2}/c \tag{12}$$

$$\frac{df}{dx} = \frac{d}{dz'}\left[t - \sqrt{\rho^2+(z-z')^2}/c\right] = \frac{1}{c}\frac{(z-z')}{\sqrt{\rho^2+(z-z')^2}} \tag{13}$$

$$f(x) = 0 \Rightarrow z' = z \pm \sqrt{c^2t^2-\rho^2}, \tag{14}$$

in the way that

$$\delta(t - \sqrt{\rho^2+(z-z')^2}/c) = \frac{\delta\left[z' - \left(z + \sqrt{c^2t^2-\rho^2}\right)\right]}{\frac{1}{c}\frac{\sqrt{c^2t^2-\rho^2}}{\sqrt{\rho^2+(z-z')^2}}} + \frac{\delta\left[z' - \left(z - \sqrt{c^2t^2-\rho^2}\right)\right]}{\frac{1}{c}\frac{\sqrt{c^2t^2-\rho^2}}{\sqrt{\rho^2+(z-z')^2}}}. \tag{15}$$

Coming back to $\Psi(\mathbf{r},t)$ we have

$$\Psi(\mathbf{r},t) = \int_{-\infty}^{\infty} dz' \frac{\delta(t - \sqrt{\rho^2+(z-z')^2}/c)}{\sqrt{\rho^2+(z-z')^2}} \tag{16}$$

$$= \int_{-\infty}^{\infty} dz' c \frac{\left\{\delta\left[z' - \left(z + \sqrt{c^2t^2-\rho^2}\right)\right] + \delta\left[z' - \left(z - \sqrt{c^2t^2-\rho^2}\right)\right]\right\}}{\sqrt{c^2t^2-\rho^2}} \tag{17}$$

$$= \begin{cases} \frac{2c}{\sqrt{c^2t^2-\rho^2}}, c^2t^2 > \rho^2 \\ 0, \text{otherwise} \end{cases} = \frac{2c\Theta(ct-\rho)}{\sqrt{c^2t^2-\rho^2}} \square. \tag{18}$$

This represents a cylindrical expanding shell.

**b)** Following the same idea from the previous item, we can write a flashing sheet source at $x = 0$ in the retarded time as

$$
\begin{aligned}
f(\mathbf{r}', t') &= \delta(x')\delta(t') \\
[f(\mathbf{r}', t')]_{ret} &= \delta(x')\delta(t - |\mathbf{r} - \mathbf{r}'|/c).
\end{aligned}
\tag{19}
$$

And, again, taking the three dimensional retarded solution

$$
\Psi(\mathbf{r}, t) = \int d^3r' \frac{[f(\mathbf{r}', t')]_{ret}}{|\mathbf{r} - \mathbf{r}'|} = \int d^3r' \frac{\delta(x')\delta(t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|} \tag{20}
$$

$$
= \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz' \frac{\delta(x')\delta(t - \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}/c)}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \tag{21}
$$

$$
= \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz' \frac{\delta(t - \sqrt{x^2 + (y - y')^2 + (z - z')^2}/c)}{\sqrt{x^2 + (y - y')^2 + (z - z')^2}}. \tag{22}
$$

To solve this integral we need to take care that, due to symmetry, we can set the origin of the observation point anywhere we want. Choosing "smartly" we can set that on $y = z = 0$. In this way, we can even change it for polar coordinates such that $\rho' = (y')^2 + (z')^2$, obtaining

$$
\Psi(\mathbf{r}, t) = \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz' \frac{\delta(t - \sqrt{x^2 + (y')^2 + (z')^2}/c)}{\sqrt{x^2 + (y - y')^2 + (z - z')^2}} \tag{23}
$$

$$
= \int_0^{2\pi} d\theta \int_0^{\infty} d\rho' \rho' \frac{\delta(t - \sqrt{x^2 + (\rho')^2}/c)}{\sqrt{x^2 + (\rho')^2}} \tag{24}
$$

$$
= 2\pi \int_0^{\infty} d\rho' \rho' \frac{\delta(t - \sqrt{x^2 + (\rho')^2}/c)}{\sqrt{x^2 + (\rho')^2}}. \tag{25}
$$

Here, guess what, we can use the Delta property again looking the terms:

$$
x = \rho' \tag{26}
$$

$$
f(x) = t - \sqrt{x^2 + (\rho')^2}/c \tag{27}
$$

$$
\frac{df}{dx} = \frac{d}{d\rho'}\left[t - \sqrt{x^2 + (\rho')^2}/c\right] = -\frac{1}{c}\frac{\rho'}{\sqrt{x^2 + (\rho')^2}} \tag{28}
$$

$$
f(x) = 0 \Rightarrow \rho' = \pm\sqrt{c^2t^2 - x^2}. \tag{29}
$$

Putting all this together, we have

$$
\delta[t - \sqrt{x^2 + (\rho')^2}/c] = \frac{\delta\left[\rho' - \sqrt{c^2t^2 - x^2}\right]}{\frac{1}{c}\frac{\sqrt{c^2t^2 - x^2}}{\sqrt{x^2 + (\rho')^2}}} + \cancel{\frac{\delta\left[\rho' + \sqrt{c^2t^2 - x^2}\right]}{\frac{1}{c}\frac{\sqrt{c^2t^2 - x^2}}{\sqrt{x^2 + (\rho')^2}}}}, \tag{30}
$$

because the radial coordinate can only be positive! Therefore,

$$
\Psi(\mathbf{r}, t) = 2\pi \int_0^{\infty} d\rho' \rho' \frac{\delta(t - \sqrt{x^2 + (\rho')^2}/c)}{\sqrt{x^2 + (\rho')^2}} \tag{31}
$$

$$
= 2\pi c \int_0^{\infty} d\rho' \rho' \frac{\delta\left[\rho' - \sqrt{c^2t^2 - x^2}\right]}{\sqrt{c^2t^2 - x^2}} \tag{32}
$$

$$
= \left\{ \begin{array}{c} 2\pi c, c^2t^2 > |x| \\ 0, \text{otherwise} \end{array} \right. = 2\pi c\Theta(ct - |x|)\square. \tag{33}
$$

This represents a plane traveling in the positive $x$ direction and a plane traveling in the negative $x$ direction, both starting at $x = 0$.

# 2 Question (1 point)

**a)** *Find the fields ($\mathbf{E}$ and $\mathbf{B}$), the charge $\rho$ and the current distribution $\mathbf{J}$ corresponding to*

$$V(\mathbf{r},t)=0, \mathbf{A}(\mathbf{r},t)=\frac{-1}{4\pi\epsilon_0}\frac{qt}{r^2}\hat{r}. \tag{34}$$

**b)** *Use the gauge function $\lambda=\frac{-1}{4\pi\epsilon_0}\frac{qt}{r}$ to transform the potentials and comment the result.*

**c)** *Are the potentials of the item (a) in Coulomb gauge? Are they in Lorentz gauge? (Notice that these gauges are not mutually exclusive).*

## 2.1 Solution

**a)** Given the potentials, we can find the fields $\mathbf{E}$ and $\mathbf{B}$ as follows

$$\mathbf{E}=-\cancel{\vec{\nabla}V}^0-\frac{\partial\mathbf{A}}{\partial t}=\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \tag{35}$$

$$\mathbf{B}=\vec{\nabla}\times\mathbf{A}=0. \tag{36}$$

By this we can already argument what is this object ("it is a kind of obvious"), but, computing the charge and distribution, we have

$$\rho=\epsilon_0\vec{\nabla}\cdot\mathbf{E}=\frac{q}{4\pi}\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)=q\delta(\mathbf{r}) \tag{37}$$

$$\mathbf{J}=\frac{1}{\mu_0}\vec{\nabla}\times\mathbf{B}-\epsilon_0\frac{\partial\mathbf{E}}{\partial t}=0. \tag{38}$$

**b)** Using $\lambda$ as our gauge function we obtain

$$V'=\cancel{V}^0-\frac{\partial\lambda}{\partial t}=\frac{1}{4\pi\epsilon_0}\frac{q}{r} \tag{39}$$

$$\mathbf{A}'=\mathbf{A}+\vec{\nabla}\lambda=\frac{-1}{4\pi\epsilon_0}\frac{qt}{r^2}\hat{r}+\left(\frac{-qt}{4\pi\epsilon_0}\right)\left(\frac{\hat{r}}{r^2}\right)=0. \tag{40}$$

Thus, the potential $\lambda$ transforms the potentials of the item (a) into just the potentials for a stationary point of charge $q$.

**c)** To say anything about the chosen gauge for the potentials of the item (a) we need to compute the following quantities:

$$\vec{\nabla}\cdot\mathbf{A}=-\frac{qt}{4\pi\epsilon_0}\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)=-\frac{qt}{\epsilon_0}\delta(\mathbf{r}), \tag{41}$$

$$\nabla^2V=0, \tag{42}$$

$$\frac{\partial V}{\partial t}=0. \tag{43}$$

As all these quantities are different of Coulomb ($\vec{\nabla}\cdot\mathbf{A}=0$ and $\nabla^2V=-\rho/\epsilon_0$) and Lorentz ($\vec{\nabla}\cdot\mathbf{A}=-\mu_0\epsilon_0\frac{\partial V}{\partial t}$), they do not follow any one of these gauges.

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# 3 Question (2 point)

*A very long linear wire has the following current*

$$i(t) = \alpha t, t \geq 0.$$

*Determine the potentials (scalar and vectorial) and fields (electric and magnetic) generated at a distance $\rho$ from the wire.* **Hint:** *use the retarded expressions for the potentials.*

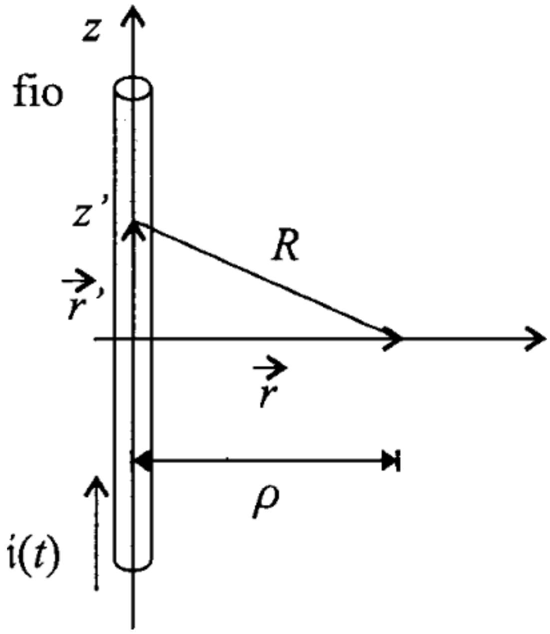


Figure 1: A long linear wire with current varying with time. It is represented a point of observation at a distance $\rho$ from the wire.

## 3.1 Solution

As we have a temporal dependence with the current, we can determine the potentials using the retarded potentials

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{[\rho(\mathbf{r}', t')]_{ret}}{|\mathbf{r} - \mathbf{r}'|}, \tag{44}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int d^3r' \frac{[\mathbf{J}(\mathbf{r}', t')]_{ret}}{|\mathbf{r} - \mathbf{r}'|}. \tag{45}$$

As the exercise does not mention any charge, the wire is electrically neutral, then, $\rho = 0$ and, as consequence

$$\Phi(\mathbf{r}, t) = 0.$$

Computing the vector potential we can write

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int d^3r' \frac{[\mathbf{J}(\mathbf{r}', t')]_{ret}}{|\mathbf{r} - \mathbf{r}'|} \tag{46}$$

$$= \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz' \frac{[i(t')\delta(x')\delta(y')\hat{z}]_{ret}}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}|_{z=0} \tag{47}$$

$$= \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} dz' \frac{[i(t')\hat{z}]_{ret}}{\sqrt{x^2 + y^2 + (z')^2}} = \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} dz' \frac{[i(t')\hat{z}]_{ret}}{\sqrt{\rho^2 + (z')^2}}, \tag{48}$$

where, for simplicity we take $z = 0$ (and to compute the quantities according to the coordinate $\rho$). Here we can consider that the retarded time could be written as

$$t_{ret} \geq 0 \Rightarrow t_{ret} = t - \frac{R}{c} \geq 0 \tag{49}$$

$$R = \sqrt{\rho^2 + (z')^2} \Rightarrow |z'| \leq \sqrt{c^2t^2 - \rho^2}. \tag{50}$$

Therefore,

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} dz' \frac{[i(t')\hat{z}]_{ret}}{\sqrt{\rho^2 + (z')^2}} = \frac{\mu_0}{4\pi} \int_{-\sqrt{c^2t^2-\rho^2}}^{\sqrt{c^2t^2-\rho^2}} dz' \frac{\alpha\left(t - \frac{\sqrt{\rho^2+(z')^2}}{c}\right)}{\sqrt{\rho^2 + (z')^2}} \hat{z} \tag{51}$$

$$= \frac{\mu_0 \alpha}{4\pi} \left[ \int_{-\sqrt{c^2t^2-\rho^2}}^{\sqrt{c^2t^2-\rho^2}} dz' \frac{t}{\sqrt{\rho^2 + (z')^2}} - \int_{-\sqrt{c^2t^2-\rho^2}}^{\sqrt{c^2t^2-\rho^2}} \frac{dz'}{c} \right] \hat{z} \tag{52}$$

$$= \frac{\mu_0 \alpha}{4\pi} \left[ 2 \int_{0}^{\sqrt{c^2t^2-\rho^2}} dz' \frac{t}{\sqrt{\rho^2 + (z')^2}} - \frac{2\sqrt{c^2t^2 - \rho^2}}{c} \right] \hat{z} \tag{53}$$

$$= \frac{\mu_0 \alpha}{2\pi} \left[ \int_{0}^{\sqrt{c^2t^2-\rho^2}} dz' \frac{t}{\sqrt{\rho^2 + (z')^2}} - \frac{\sqrt{c^2t^2 - \rho^2}}{c} \right] \hat{z} \tag{54}$$

$$\tag{55}$$

We can solve the first integral performing the change of variables

$$\tan\theta = \frac{z'}{\rho} \Rightarrow \frac{dz'}{\rho} = \sec^2\theta d\theta \tag{56}$$

$$I = \int \frac{dz'}{\sqrt{\rho^2 + (z')^2}} = \int \frac{\sec^2\theta d\theta}{\sec\theta} = \int d\theta \sec\theta \tag{57}$$

$$= \ln(\sec\theta + \tan\theta) = \ln\left[ \frac{\sqrt{(z')^2 + \rho^2}}{\rho} + \frac{z'}{\rho} \right]. \tag{58}$$

Replacing this result in the above integral we finally got

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 \alpha}{2\pi} \left[ \int_{0}^{\sqrt{c^2t^2-\rho^2}} dz' \frac{t}{\sqrt{\rho^2 + (z')^2}} - \frac{\sqrt{c^2t^2 - \rho^2}}{c} \right] \hat{z} \tag{59}$$

$$= \frac{\mu_0 \alpha}{2\pi} \left\{ t \left[ \ln\left( \frac{\sqrt{(z')^2 + \rho^2}}{\rho} + \frac{z'}{\rho} \right) \right]_{z'=0}^{z'=\sqrt{c^2t^2-\rho^2}} - \frac{\sqrt{c^2t^2 - \rho^2}}{c} \right\} \hat{z} \tag{60}$$

$$= \frac{\mu_0 \alpha}{2\pi} \left\{ t \ln\left[ \frac{ct}{\rho} + \frac{\sqrt{c^2t^2 - \rho^2}}{\rho} \right] - \ln 1 - \frac{\sqrt{c^2t^2 - \rho^2}}{c} \right\} \hat{z} \tag{61}$$

$$= \frac{\mu_0 \alpha}{2\pi} \left\{ t \ln\left[ \frac{ct}{\rho} + \frac{\sqrt{c^2t^2 - \rho^2}}{\rho} \right] - \frac{\sqrt{c^2t^2 - \rho^2}}{c} \right\} \hat{z}. \tag{62}$$

And we compute the fields by definition

$$
\begin{aligned}
\mathbf{E} &= -\cancel{\vec{\nabla}\Phi^0} - \frac{\partial \mathbf{A}}{\partial t} && (63)\\
&= \frac{\mu_0\alpha}{2\pi}\left\{\ln\left[\frac{ct}{\rho} + \frac{\sqrt{c^2t^2-\rho^2}}{\rho}\right] + t\frac{\frac{c}{\rho} + \frac{2c^2t}{2\rho\sqrt{c^2t^2-\rho^2}}}{\left[\frac{ct}{\rho} + \frac{\sqrt{c^2t^2-\rho^2}}{\rho}\right]} - \frac{2c^2t}{2c\sqrt{c^2t^2-\rho^2}}\right\}\hat{z} && (64)\\
&= \frac{\mu_0\alpha}{2\pi}\left\{\ln\left[\frac{ct}{\rho} + \frac{\sqrt{c^2t^2-\rho^2}}{\rho}\right] + \frac{ct + \frac{c^2t^2}{\sqrt{c^2t^2-\rho^2}}}{\left[ct + \sqrt{c^2t^2-\rho^2}\right]} - \frac{ct}{c\sqrt{c^2t^2-\rho^2}}\right\}\hat{z} && (65)\\
&= \frac{\mu_0\alpha}{2\pi}\left\{\ln\left[\frac{ct + \sqrt{c^2t^2-\rho^2}}{\rho}\right] + \cancel{\frac{ct}{c\sqrt{c^2t^2-\rho^2}}} - \cancel{\frac{ct}{c\sqrt{c^2t^2-\rho^2}}}\right\}\hat{z} && (66)\\
&= \frac{\mu_0\alpha}{2\pi}\left[\ln\left(\frac{ct + \sqrt{c^2t^2-\rho^2}}{\rho}\right)\right]\hat{z} && (67)\\
\mathbf{B} &= \vec{\nabla}\times\mathbf{A} && (68)\\
&= \left[\frac{1}{\rho}\cancel{\frac{\partial A_z}{\partial\theta}} - \cancel{\frac{\partial A_\theta}{\partial z}}\right]\hat{\rho} + \left[\cancel{\frac{\partial A_\rho}{\partial z}} - \frac{\partial A_z}{\partial\rho}\right]\hat{\theta} + \left[\cancel{\frac{\partial(\rho A_\theta)}{\partial\rho}} - \cancel{\frac{\partial A_\rho}{\partial\theta}}\right]\frac{\hat{z}}{\rho} && (69)\\
&= -\frac{\mu_0\alpha}{2\pi}\left\{t\frac{-\frac{ct}{\rho^2} + \frac{-\frac{\rho^2}{\sqrt{c^2t^2-\rho^2}} - \sqrt{c^2t^2-\rho^2}}{\rho^2}}{\left[\frac{ct}{\rho} + \frac{\sqrt{c^2t^2-\rho^2}}{\rho}\right]} + \frac{\rho}{c\sqrt{c^2t^2-\rho^2}}\right\}\hat{\theta} && (70)\\
&= -\frac{\mu_0\alpha}{2\pi}\left\{t\frac{-\frac{ct}{\rho^2} + \frac{-\frac{\rho^2}{\sqrt{c^2t^2-\rho^2}} - \sqrt{c^2t^2-\rho^2}}{\rho^2}}{\left[\frac{ct}{\rho} + \frac{\sqrt{c^2t^2-\rho^2}}{\rho}\right]} + \frac{\rho}{c\sqrt{c^2t^2-\rho^2}}\right\}\hat{\theta} && (71)\\
&= \frac{\mu_0\alpha}{2\pi}\left[\frac{t\rho}{\sqrt{c^2t^2-\rho^2}(ct + \sqrt{c^2t^2-\rho^2})} + \frac{t}{\rho} - \frac{\rho}{c\sqrt{c^2t^2-\rho^2}}\right]\hat{\theta}. && (72)
\end{aligned}
$$

At the same time, we could use the matnetic field as

$$
\begin{aligned}
\mathbf{B} &= \vec{\nabla}\times\mathbf{A} && (73)\\
&= \left[\frac{1}{\rho}\cancel{\frac{\partial A_z}{\partial\theta}} - \cancel{\frac{\partial A_\theta}{\partial z}}\right]\hat{\rho} + \left[\cancel{\frac{\partial A_\rho}{\partial z}} - \frac{\partial A_z}{\partial\rho}\right]\hat{\theta} + \left[\cancel{\frac{\partial(\rho A_\theta)}{\partial\rho}} - \cancel{\frac{\partial A_\rho}{\partial\theta}}\right]\frac{\hat{z}}{\rho} && (74)\\
&= -\frac{\mu_0\alpha}{2\pi}\left\{\frac{-ct^2}{\rho\sqrt{c^2t^2-\rho^2}} + \frac{\rho}{c\sqrt{c^2t^2-\rho^2}}\right\}\hat{\theta} = \frac{\mu_0\alpha}{2\pi\rho c}\sqrt{c^2t^2-\rho^2}\hat{\theta}. && (75)
\end{aligned}
$$

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# 4 Question (1 point)

**a)** *Show that the mixed form of the electromagnetic field tensor is given by*

$$
[F^\mu{}_\nu] = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix}. \tag{76}
$$

**b)** *Verify that the equations*

$$\partial_\nu F^{\mu\nu} = \mu_0 j^\mu \tag{77}$$

$$\partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} = 0 \tag{78}$$

*are equivalent to the Maxwell equations.*

**c)** *Verify that the Lorentz force*

$$\frac{d\mathbf{p}}{dt} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \tag{79}$$

*and the rate at which the electromagnetic field imparts energy $E$ to the particles*

$$\frac{dE}{dt} = \mathbf{F} \cdot \mathbf{v} = q\mathbf{E} \cdot \mathbf{v} \tag{80}$$

*may be brought together in the single equation*

$$\frac{dp^\mu}{d\tau} = qF^\mu_{\ \nu} u^\nu. \tag{81}$$

## 4.1 Solution

**a)** Starting from the covariant form of the electromagnetic field tensor we can rising the $\mu$ index just using the Minkowski metric as a simple matrix multiplication

$$F^\mu_{\ \nu} = \eta^{\mu\sigma} F_{\sigma\nu} \tag{82}$$

$$[F^\mu_{\ \nu}] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix} \tag{83}$$

$$= \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix} \square. \tag{84}$$

**b)** Taking all the definitions as: the electromagnetic contravariant field tensor, the current and the derivative

$$[F^{\mu\nu}] = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix}. \tag{85}$$

$$j^\mu = (c\rho, \mathbf{J}) \tag{86}$$

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \tag{87}$$

we can start to prove the first property taking each one of the components

$$\mu = 0 \Rightarrow \frac{1}{c}\frac{\partial 0}{\partial t} + \frac{1}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) = \mu_0 c \rho \tag{88}$$

$$\Rightarrow \vec{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{89}$$

$$\mu = 1 \Rightarrow -\frac{1}{c}\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 J_x \tag{90}$$

$$\mu = 2 \Rightarrow -\frac{1}{c}\frac{\partial E_y}{\partial t} - \frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z} = \mu_0 J_y \tag{91}$$

$$\mu = 3 \Rightarrow -\frac{1}{c}\frac{\partial E_z}{\partial t} + \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \mu_0 J_z \tag{92}$$

$$\Rightarrow -\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \vec{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}, \tag{93}$$

where we have used that $c^2 = 1/(\mu_0 \epsilon_0)$. Then, expanding the second property, but now taking the covariant form of the electromagnetic tensor field

$$[F_{\mu\nu}] = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix} \tag{94}$$

we have for the indexes: $(0, 1, 2, 3) = (t, x, y, z)$

$$\partial_2 F_{01} + \partial_0 F_{12} + \partial_1 F_{20} = 0 \Rightarrow -\frac{1}{c}\frac{\partial E_x}{\partial y} + \frac{1}{c}\frac{\partial B_z}{\partial t} + \frac{1}{c}\frac{\partial E_y}{\partial x} = 0 \tag{95}$$

$$\partial_3 F_{01} + \partial_0 F_{13} + \partial_1 F_{30} = 0 \Rightarrow -\frac{1}{c}\frac{\partial E_x}{\partial z} - \frac{1}{c}\frac{\partial B_y}{\partial t} + \frac{1}{c}\frac{\partial E_z}{\partial x} = 0 \tag{96}$$

$$\partial_3 F_{02} + \partial_0 F_{23} + \partial_2 F_{30} = 0 \Rightarrow -\frac{1}{c}\frac{\partial E_y}{\partial z} + \frac{1}{c}\frac{\partial B_x}{\partial t} + \frac{1}{c}\frac{\partial E_z}{\partial y} = 0 \tag{97}$$

$$\Rightarrow \vec{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{98}$$

$$\partial_3 F_{12} + \partial_1 F_{23} + \partial_2 F_{31} = 0 \Rightarrow \frac{\partial B_z}{\partial z} + \frac{\partial B_x}{\partial x} + \frac{\partial B_z}{\partial z} = 0 \tag{99}$$

$$\Rightarrow \vec{\nabla} \cdot \mathbf{B} = 0. \tag{100}$$

**c)** To write this equation we need to take in mind the definitions of each quantity, which are: the 4-momentum, the 4-velocity, the derivative of the 4-momentum with respect to $\tau$ and use the mixed form of the electromagnetic tensor field:

$$p^\mu = \left(\frac{E}{c}, \mathbf{p}\right) \tag{101}$$

$$u^\mu = \frac{dt}{d\tau} v^\mu = \gamma\,(c, \mathbf{v}) \tag{102}$$

$$\frac{dp^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dp^\mu}{dt} = \gamma \frac{dp^\mu}{dt} = \gamma \left(\frac{E}{c}, \frac{dp}{dt}\right). \tag{103}$$

Thus, we can open the expression that we want to show that we can write, in the reverse process

$$\frac{dp^\mu}{d\tau} = qF^\mu_{\ \nu}u^\nu \tag{104}$$

$$= -q\gamma \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix} \begin{bmatrix} c \\ v_x \\ v_y \\ v_z \end{bmatrix} \tag{105}$$

$$= q\gamma \begin{bmatrix} (E_x v_x + E_y v_y + E_z v_z)/c \\ E_x + B_z v_y - B_y v_z \\ E_y - B_z v_x + B_x v_z \\ E_z + B_y v_x - B_x v_y \end{bmatrix} \tag{106}$$

$$\frac{dp^0}{d\tau} = \gamma\frac{dp^0}{dt} = \frac{\gamma}{c}\frac{dE}{dt} = q\gamma\frac{\mathbf{E}\cdot\mathbf{v}}{c} \Rightarrow \frac{dE}{dt} = \mathbf{F}\cdot\mathbf{v} = q\mathbf{E}\cdot\mathbf{v}\square \tag{107}$$

$$\frac{d\mathbf{p}}{d\tau} = \gamma\frac{d\mathbf{p}}{dt} = q\gamma\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right) \Rightarrow \frac{d\mathbf{p}}{dt} = \left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)\square \tag{108}$$

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# 5 Question (2 points)

**a)** *Show that under a boost in the $x$ direction the components of the electric field intensity $\mathbf{E}$ and the magnetic field induction $\mathbf{B}$ transform according to*

$$\begin{cases} E'_x & = E_x \\ E'_y & = \gamma\left(E_y - vB_z\right) \\ E'_z & = \gamma\left(E_z + vB_y\right) \end{cases} \qquad \begin{cases} B'_x & = B_x \\ B'_y & = \gamma\left(B_y + vE_z/c^2\right) \\ B'_z & = \gamma\left(B_z - vE_y/c^2\right) \end{cases} \tag{109}$$

**b)** *In a certain inertial frame $S$, the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are neither parallel nor perpendicular, at a particular space-time point. Show that in a different inertial system $\bar{S}$, moving relative to $S$ with velocity $\mathbf{v}$ is given by*

$$\frac{\mathbf{v}}{1 + v^2/c^2} = \frac{\mathbf{E}\times\mathbf{B}}{B^2 + E^2/c^2}, \tag{110}$$

*the fields $\bar{\mathbf{E}}$ and $\bar{\mathbf{B}}$ are parallel at that point. Is there a frame in which the two are perpendicular?* **Hint:** *choose the directions in the inertial frame $S$ so that $\mathbf{E}$ points in the $z$ direction and $\mathbf{B}$ in the $yz$ plane with $\phi$ angle for $z$.*

## 5.1 Solution

**a)** The Lorentz transformation (boost in $x$) can be represented by

$$[\Lambda^{\mu'}_{\nu}] = \begin{bmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{111}$$

In this way, we can write

$$F^{\mu'\nu'} = F^{\alpha\beta}\Lambda^{\mu'}_{\alpha}\Lambda^{\nu'}_{\beta} = \Lambda^{\mu'}_{\alpha}F^{\alpha\beta}\Lambda^{\nu'}_{\beta} \tag{112}$$

$$[F^{\mu'\nu'}] = [\Lambda][F^{\alpha\beta}][\Lambda^T] \tag{113}$$

$$= \begin{bmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & E_x/c & \gamma(E_y - vB_z)/c & \gamma(E_z + vB_y)/c \\ -E_x/c & 0 & \gamma(-vE_y/c^2 + B_z) & -\gamma(vE_z/c^2 - B_y) \\ \gamma(-E_y + B_z v)/c & \gamma(vE_y/c^2 - B_z) & 0 & B_x \\ -\gamma(E_z + B_y v)/c & \gamma(vE_z/c^2 + B_y) & -B_x & 0 \end{bmatrix} \tag{114}$$

what exactly is the relations that we want to prove looking at each component of this result and comparing to the electromagnetic field contracovariant tensor.

**b)** We can choose the directions in the inertial frame $S$ so that, for instance, $\mathbf{E}$ points in the $z$ direction and $\mathbf{B}$ in the $yz$ plane. Then,

$$\mathbf{E} = (0, 0, E), \tag{115}$$

$$\mathbf{B} = [0, B\cos\phi, B\sin\phi]\,. \tag{116}$$

Using the relations from the previous item, the fields in the $\bar{S}$ frame are

$$\bar{\mathbf{E}} = [0, -\gamma vB\sin\phi, \gamma\,(E + vB\cos\phi)]\,, \tag{117}$$

$$\bar{\mathbf{B}} = \left[0, \gamma\left(B\cos\phi + vE/c^2\right), \gamma B\sin\phi\right]. \tag{118}$$

In this way, saying that they are parallel fields is equivalent to say that these components satisfies the fraction

$$\frac{E_x}{B_x} = \frac{E_y}{B_y} \tag{119}$$

$$\frac{-\gamma vB\sin\phi}{\gamma\,(B\cos\phi + vE/c^2)} = \frac{\gamma\,(E + vB\cos\phi)}{\gamma B\sin\phi} \tag{120}$$

$$-vB^2\sin^2\phi = EB\cos\phi + vE^2/c^2 + vB^2\cos^2\phi + v^2BE\cos\phi/c^2 \tag{121}$$

$$0 = vB^2 + EB\cos\phi\left(1 + \frac{v^2}{c^2}\right) + \frac{v^1}{c^2}E^2 \tag{122}$$

$$0 = v\left(B^2 + E^2/c^2\right) + EB\cos\phi\left(1 + \frac{v^2}{c^2}\right) \tag{123}$$

$$\frac{v}{(1 + v^2/c^2)} = -\frac{EB\cos\phi}{(B^2 + E^2/c^2)} = \frac{-\mathbf{E}\times\mathbf{B}}{(B^2 + E^2/c^2)} \tag{124}$$

$$\mathbf{E}\times\mathbf{B} = -EB\cos\phi\hat{x} \tag{125}$$

$$\frac{\mathbf{v}}{1 + v^2/c^2} = \frac{\mathbf{E}\times\mathbf{B}}{B^2 + E^2/c^2}\square. \tag{126}$$

Notice that, in both reference frames we have the quantity

$$\mathbf{E}\cdot\mathbf{B}|_S = EB\sin\phi \tag{127}$$

$$\mathbf{E}\cdot\mathbf{B}|_{\bar{S}} = -\gamma^2 vB\sin\phi\left(\cancel{B\cos\phi} + vE/c^2\right) + \gamma^2 B\sin\phi\,(E + \cancel{vB\cos\phi}) \tag{128}$$

$$= EB\sin\phi \tag{129}$$

is invariant! Therefore, there is no way to exist a reference frame where both fields are perpendicular because $\mathbf{E} \cdot \mathbf{B} \neq 0$!

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# 6 Question (1 point)

*A uniform charge distribution of proper density $\rho_0$ is at rest in an inertial frame $K$. Show that and observer with a velocity $\mathbf{v}$ relative to $K$ sees a charge density $\gamma\rho_0$ and a current density $-\gamma\rho_0\mathbf{v}$.*

## 6.1 Solution

We can write the general 4-current vector as

$$j^\mu = (c\rho, \mathbf{J}) \,.$$

For a charge with $\rho_0$ in $K$ we have

$$j^\mu = \left(c\rho_0, \vec{0}\right)$$

and for an observer in $K'$ with $\mathbf{v}$ we got

$$j^{\mu'} = (c\rho', \mathbf{J}') \,.$$

We can do the Lorentz transformation as

$$j^{\mu'} = \Lambda^{\mu'}_{\nu} j^\nu$$

taking, for simplicity $\mathbf{v} = (v, 0, 0)$:

$$[j^{\mu'}] = \begin{bmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\rho_0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma c\rho_0 \\ -\gamma v\rho_0 \\ 0 \\ 0 \end{bmatrix} . \tag{130}$$

Therefore, the observer sees

$$j^{\mu'} = (\gamma c\rho_0, -\gamma\rho_0\mathbf{v})$$

which means

- *Charge density:* $\gamma\rho_0$
- *Current density:* $-\gamma\rho_0\mathbf{v}$ .

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# 7 Question (1 point)

*Verify that Ohm's law $\mathbf{J} = \sigma\mathbf{E}$ can be written as*

$$j^\mu + \frac{1}{c^2} u^\mu u_\nu j^\nu = \sigma u_\nu F^{\mu\nu}$$

*where $\sigma$ is the conductivity and $u^\mu$ is the 4-velocity.*

## 7.1 Solution

We know that

$$j^{\mu} = (c\rho, \mathbf{J})$$

and that the electromagnetic contravariant tensor field is

$$[F^{\mu\nu}] = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix}. \tag{131}$$

$$\tag{132}$$

Taking the rest frame of the electrons we can write the 4-velocity as

$$u^{\mu} = \left(c, \vec{0}\right) \quad u_{\mu} = \left(-c, \vec{0}\right).$$

In this way, expanding the expression that we need to prove into the components, we get

$$\mu = 0 \Rightarrow j^0 + \frac{1}{c^2}u^0 u_\nu j^\nu = \sigma u_\nu F^{0\nu} \Rightarrow c\rho + \frac{1}{c^2}\left(-c^2\rho c + 0\right) \Rightarrow 0 = 0 \tag{133}$$

$$\mu = i \Rightarrow j^i - \cancel{u^i u_\nu j^\nu}^{0} = \sigma u_\nu F^{i\nu} \Rightarrow j^i = \sigma E^i \tag{134}$$

$$\mathbf{J} = \sigma \mathbf{E} \square. \tag{135}$$

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# 8 Question (1 point)

*An electromagnetic plane wave of frequency $\omega$ is traveling in the $x$ direction through the vacuum. It is polarized in the $y$ direction and the amplitude of the electric field is $E_0$.*

**a)** *Write down the electric $\mathbf{E}(\mathbf{r}, t)$ and magnetic fields $\mathbf{B}(\mathbf{r}, t)$.*

**b)** *This same wave is observed from an inertial system $\bar{S}$ moving in the $x$ direction with speed $v$ relative to the original system $S$. Find the electric and magnetic fields in $\bar{S}$ and express them in terms of the $\bar{S}$ coordinates: $\bar{\mathbf{E}}(\bar{\mathbf{r}}, \bar{t})$ and $\bar{\mathbf{B}}(\bar{\mathbf{r}}, \bar{t})$.*

**c)** *What is the frequency $\bar{\omega}$ of the wave in $\bar{S}$? Interpret this results. What is the wavelength $\bar{\lambda}$ of the wave in $\bar{S}$? From $\bar{\omega}$ and $\bar{\lambda}$, determine the speed of the waves in $\bar{S}$. Is it what you expected? Why?*

**d)** *What is the ratio of the intensity in $\bar{S}$ to the intensity in $S$? And what about the amplitude, frequency and intensity of the wave, as $v$ approaches $c$?*

## 8.1 Solution

**a)** In a general way, we can write the plane waves, in the reference $S$ as

$$\mathbf{E}(x, y, z, t) = E_0 e^{i(kx - \omega t)} \hat{y} \tag{136}$$

$$\mathbf{B}(x, y, z, t) = \frac{E_0}{c} e^{i(kx - \omega t)} \hat{z}, \tag{137}$$

where we obtain the direction of $B$ using: $\mathbf{B}_0 = \hat{x} \times \mathbf{E}_0/c$ and $\omega = kc$.

**b)** Using the equations of the exercise 5 we obtain

$$\begin{cases} \bar{E}_x & = E_x \\ \bar{E}_y & = \gamma\left(E_y - vB_z\right) \\ \bar{E}_z & = \gamma\left(E_z + vB_y\right) \end{cases} \qquad \begin{cases} \bar{B}_x & = B_x \\ \bar{B}_y & = \gamma\left(B_y + vE_z/c^2\right) \\ \bar{B}_z & = \gamma\left(B_z - vE_y/c^2\right) \end{cases} \tag{138}$$

where here

$$\bar{E}_x = \bar{E}_z = \bar{B}_x = \bar{B}_y = 0 \tag{139}$$

$$\bar{E}_y = \gamma\left[E_0 e^{i(kx-\omega t)} - \frac{v}{c} E_0 e^{i(kx-\omega t)}\right] \tag{140}$$

$$\bar{B}_z = \gamma\left[E_0 e^{i(kx-\omega t)} - \frac{v}{c} E_0 e^{i(kx-\omega t)}\right]. \tag{141}$$

In other lines,

$$\bar{\mathbf{E}} = \gamma E_0 e^{i(kx-\omega t)}\left(1 - \frac{v}{c}\right)\hat{y} \tag{142}$$

$$\bar{\mathbf{B}} = \gamma E_0 e^{i(kx-\omega t)}\left(1 - \frac{v}{c}\right)\hat{z}. \tag{143}$$

But we still need to change the space and time variables using Lorentz transformations:

$$x = \gamma\left(\bar{x} + v\bar{t}\right) \tag{144}$$

$$t = \gamma\left(\bar{t} + \frac{v}{c^2}\bar{x}\right) \tag{145}$$

$$kx - \omega t = \gamma\left[k\left(\bar{x} + v\bar{t}\right) - \omega\left(\bar{t} + \frac{v}{c^2}\bar{x}\right)\right] \tag{146}$$

$$= \gamma\left[\left(k - \frac{\omega v}{c^2}\right)\bar{x} - \left(\omega - kv\right)\bar{t}\right] \tag{147}$$

$$= \bar{k}\bar{\omega} - \bar{\omega}t, \tag{148}$$

where, of course,

$$\bar{k} = \gamma\left(k - \frac{\omega v}{c^2}\right) = \gamma k\left(1 - \frac{v}{c}\right), \tag{149}$$

$$\bar{\omega} = \gamma\left(\omega - kv\right) = \gamma\omega\left(1 - \frac{v}{c}\right). \tag{150}$$

Finally,

$$\bar{\mathbf{E}} = \gamma E_0 e^{i(\bar{k}\bar{x}-\bar{\omega}\bar{t})}\left(1 - \frac{v}{c}\right)\hat{y} \tag{151}$$

$$\bar{\mathbf{B}} = \gamma E_0 e^{i(\bar{k}\bar{x}-\bar{\omega}\bar{t})}\left(1 - \frac{v}{c}\right)\hat{z}, \tag{152}$$

with the parameters explained above.

**c)** As already showed in the previous item, but doing some beautification

$$\bar{\omega} = \gamma\left(\omega - kv\right) = \gamma\omega\left(1 - \frac{v}{c}\right) \tag{153}$$

$$= \omega\sqrt{\frac{\left(1 - \frac{v}{c}\right)^2}{\left(1 - \frac{v}{c}\right)\left(1 + \frac{v}{c}\right)}} = \omega\sqrt{\frac{\left(1 - \frac{v}{c}\right)}{\left(1 + \frac{v}{c}\right)}}. \tag{154}$$

This is the something like the *Doppler shift* for light!

The wavelength is computed as following

$$\bar{\lambda} = \frac{2\pi}{\bar{k}} = \frac{2\pi}{\gamma k\,(1 - v/c)} = \frac{\lambda}{\gamma\,(1 - v/c)}. \tag{155}$$

In this way, the velocity in this new reference is

$$\bar{v} = \frac{\bar{\lambda}\bar{\omega}}{2\pi} = \frac{\left[\frac{\lambda}{\gamma(1-v/c)}\right]\left[\gamma\omega\left(1 - \frac{v}{c}\right)\right]}{2\pi} = c. \tag{156}$$

It is very good to obtain the result, exactly because we hope that light does not change its value independently of the reference frame!

**d)** The ratio of the intensities if given by

$$R = \frac{\bar{I}}{I} = \frac{(\bar{E})^2}{E^2} = \gamma^2\left(1 - \frac{v}{c}\right)^2 = \frac{\left(1 - \frac{v}{c}\right)}{\left(1 + \frac{v}{c}\right)}. \tag{157}$$

Taking the limit

$$\lim_{v \to c} R = 0. \tag{158}$$

Then, the ratio of the intensity goes to zero in this limit! The same happens to the amplitude and the frequency.