PGF5003: Classical Electrodynamics I
Problem Set 4

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(Due to June 1, 2021)

Guidelines: *write down the most relevant passages in your calculations, not only the final results. Do not forget to write the mathematical expressions that you are using in order to solve the questions. We strongly recommended the use of the International System of Units.*

1 Question (1 point)

In three dimensions the solution to the wave equation

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = -4\pi f(\mathbf{r}, t), \tag{1}$$

where $\Psi(\mathbf{r}, t)$ is the wave function and $f(\mathbf{r}, t)$ is a known source distribution, for a point source in space and time (a light flash at $t' = 0$ and $\mathbf{r}' = 0$) is a spherical shell disturbance of radius $R = ct$, namely the **retarded Green function**[1]

$$G(\mathbf{r}, t; \mathbf{r}', t') = \frac{\delta\left(t' - \left[t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right]\right)}{|\mathbf{r} - \mathbf{r}'|}. \tag{2}$$

The solutions for fewer dimensions than three can be found by superposition in the superfluous dimension(s), to eliminate the dependence on such variable(s). For example, a flashing line source of uniform amplitude is equivalent to a point source in two dimensions.

a) *Starting with the retarded solution to the three-dimensional wave equation*

$$\Psi(\mathbf{r}, t) = \int d^3r' \frac{[f(\mathbf{r}', t')]_{ret}}{|\mathbf{r} - \mathbf{r}'|}, \tag{3}$$

where $[\ ]_{ret}$ means that the time t' is to be evaluated at the retarded time $t' = t - |\mathbf{r} - \mathbf{r}'|/c$, show that the source $f(\mathbf{r}', t') = \delta(x')\delta(y')\delta(t')$ (which is equivalent to a $t = 0$ point source at the origin in two spatial dimensions), produces a two-dimensional wave

$$\Psi(x, y, t) = \frac{2c\Theta(ct - \rho)}{\sqrt{c^2t^2 - \rho^2}}, \tag{4}$$

where $\rho^2 = x^2 + y^2$ and $\Theta(\chi)$ is the unit step function $\Theta(\chi) = \begin{cases} 0, \chi < 0 \\ 1, \chi > 0 \end{cases}$.

[1] Just remember that this name is due the fact that this function exhibits the causal behavior associated to a wave disturbance.

b) *Show that a "sheet" source, equivalent to a point pulsed source at the origin in one space dimension, produces a one-dimensional wave proportional to*

$$\Psi(x,t) = 2\pi c\Theta(ct - |x|). \tag{5}$$

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2 Question (1 point)

a) *Find the fields ($\mathbf{E}$ and $\mathbf{B}$), the charge ρ and the current distribution $\mathbf{J}$ corresponding to*

$$V(\mathbf{r},t) = 0, \mathbf{A}(\mathbf{r},t) = \frac{-1}{4\pi\epsilon_0}\frac{qt}{r^2}\hat{r}. \tag{6}$$

b) *Use the gauge function $\lambda = \frac{-1}{4\pi\epsilon_0}\frac{qt}{r}$ to transform the potentials and comment the result.*

c) *Are the potentials of the item (a) in Coulomb gauge? Are they in Lorentz gauge? (Notice that these gauges are not mutually exclusive).*

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3 Question (2 point)

A very long linear wire has the following current

$$i(t) = \alpha t, t \geq 0.$$

Determine the potentials (scalar and vectorial) and fields (electric and magnetic) generated at a distance ρ from the wire. **Hint:** *use the retarded expressions for the potentials.*

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4 Question (1 point)

a) *Show that the mixed form of the electromagnetic field tensor is given by*

$$[F^{\mu}{}_{\nu}] = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix}. \tag{7}$$

b) *Verify that the equations*

$$\partial_\nu F^{\mu\nu} = \mu_0 j^\mu \tag{8}$$
$$\partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} = 0 \tag{9}$$

are equivalent to the Maxwell equations.

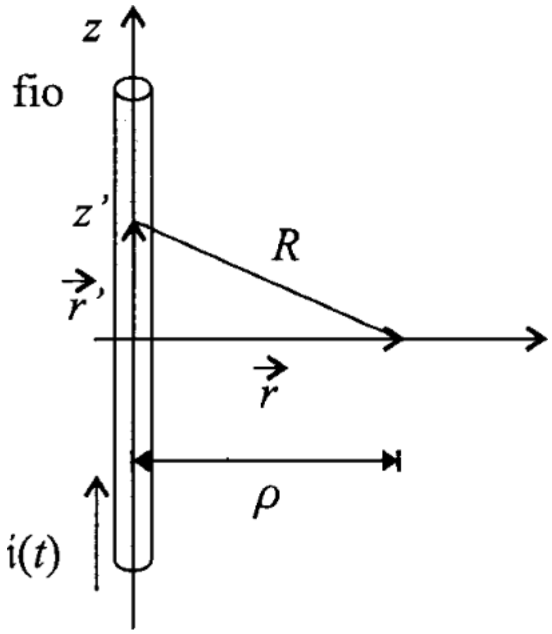

Figure 1: A long linear wire with current varying with time. It is represented a point of observation at a distance ρ from the wire.

c) *Verify that the Lorentz force*

$$\frac{d\mathbf{p}}{dt} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \tag{10}$$

and the rate at which the electromagnetic field imparts energy E to the particles

$$\frac{dE}{dt} = \mathbf{F} \cdot \mathbf{v} = q\mathbf{E} \cdot \mathbf{v} \tag{11}$$

may be brought together in the single equation

$$\frac{dp^\mu}{d\tau} = qF^\mu_{\ \nu} u^\nu. \tag{12}$$

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5 Question (2 points)

a) *Show that under a boost in the x direction the components of the electric field intensity $\mathbf{E}$ and the magnetic field induction $\mathbf{B}$ transform according to*

$$\begin{cases} E'_x &= E_x \\ E'_y &= \gamma\left(E_y - vB_z\right) \\ E'_z &= \gamma\left(E_z + vB_y\right) \end{cases} \qquad \begin{cases} B'_x &= B_x \\ B'_y &= \gamma\left(B_y + vE_z/c^2\right) \\ B'_z &= \gamma\left(B_z - vE_y/c^2\right) \end{cases} \tag{13}$$

b) *In a certain inertial frame S, the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are neither parallel nor perpendicular, at a particular space-time point. Show that in a different inertial system $\bar{S}$, moving relative to S with velocity $\mathbf{v}$ is given by*

$$\frac{\mathbf{v}}{1 + v^2/c^2} = \frac{\mathbf{E} \times \mathbf{B}}{B^2 + E^2/c^2}, \tag{14}$$

the fields $\bar{\mathbf{E}}$ and $\bar{\mathbf{B}}$ are parallel at that point. Is there a frame in which the two are perpendicular? **Hint:** *choose the directions in the inertial frame S so that $\mathbf{E}$ points in the z direction and $\mathbf{B}$ in the yz plane with ϕ angle for z.*

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6 Question (1 point)

A uniform charge distribution of proper density ρ_0 is at rest in an inertial frame K. Show that and observer with a velocity $\mathbf{v}$ relative to K sees a charge density $\gamma\rho_0$ and a current density $-\gamma\rho_0\mathbf{v}$.

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7 Question (1 point)

Verify that Ohm's law $\mathbf{J} = \sigma\mathbf{E}$ can be written as

$$j^\mu + \frac{1}{c^2}u^\mu u_\nu j^\nu = \sigma u_\nu F^{\mu\nu}$$

where σ is the conductivity and u^μ is the 4-velocity.

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8 Question (1 point)

An electromagnetic plane wave of frequency ω is traveling in the x direction through the vacuum. It is polarized in the y direction and the amplitude of the electric field is E_0.

a) *Write down the electric $\mathbf{E}(\mathbf{r}, t)$ and magnetic fields $\mathbf{B}(\mathbf{r}, t)$.*

b) *This same wave is observed from an inertial system $\bar{S}$ moving in the x direction with speed v relative to the original system S. Find the electric and magnetic fields in $\bar{S}$ and express them in terms of the $\bar{S}$ coordinates: $\bar{\mathbf{E}}(\bar{\mathbf{r}}, \bar{t})$ and $\bar{\mathbf{B}}(\bar{\mathbf{r}}, \bar{t})$.*

c) *What is the frequency $\bar{\omega}$ of the wave in $\bar{S}$? Interpret this results. What is the wavelength $\bar{\lambda}$ of the wave in $\bar{S}$? From $\bar{\omega}$ and $\bar{\lambda}$, determine the speed of the waves in $\bar{S}$. Is it what you expected? Why?*

d) *What is the ratio of the intensity in $\bar{S}$ to the intensity in S? And what about the amplitude, frequency and intensity of the wave, as v approaches c?*