Exercício 5

a) $g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} = \frac{\partial x}{\partial x'^\mu}\frac{\partial x}{\partial x'^\nu} + \frac{\partial y}{\partial x'^\mu}\frac{\partial y}{\partial x'^\nu}$

$$g'_{rr} = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 = \cos^2\varphi + \sin^2\varphi = 1$$

$$g'_{r\varphi} = g'_{\varphi r} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial \varphi} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \varphi} = -r\cos\varphi\sin\varphi + r\cos\varphi\sin\varphi = 0$$

$$g'_{\varphi\varphi} = \left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 = r^2\sin^2\varphi + r^2\cos^2\varphi = r^2.$$

Logo: $\boxed{dl^2 = dr^2 + r^2 d\varphi^2}$

- Outra solução:

$$dl^2 = dx^2 + dy^2 = (dr\cos\varphi - r\sin\varphi\, d\varphi)^2 + (dr\sin\varphi + r\cos\varphi\, d\varphi)^2 =$$
$$dr^2\cos^2\varphi \cancel{- 2r\sin\varphi\cos\varphi\, dr\, d\varphi} + r^2\sin^2\varphi\, d\varphi^2$$
$$+ dr^2\sin^2\varphi \cancel{+ 2r\sin\varphi\cos\varphi\, dr\, d\varphi} + r^2\cos^2\varphi\, d\varphi^2$$
$$= dr^2 + r^2 d\varphi^2$$

b)

$$\Gamma^r_{rr} = \frac{1}{2} g^{rr} \cancel{\partial_r g_{rr}}^{0} = 0$$

$$\Gamma^r_{r\varphi} = \Gamma^r_{\varphi r} = \frac{1}{2} g^{rr} \cancel{\partial_\varphi g_{rr}}^{0} = 0$$

$$\Gamma^r_{\varphi\varphi} = -\frac{1}{2} g^{rr}\partial_r g_{\varphi\varphi} = -\frac{1}{2}\cdot 1 \cdot 2r = -r$$

$$\Gamma^\varphi_{rr} = -\frac{1}{2} g^{\varphi\varphi} \cancel{\partial_\varphi g_{rr}}^{0} = 0$$

$$\Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{2} g^{\varphi\varphi}\partial_r g_{\varphi\varphi} = \frac{r^{-2}}{2} 2r = \frac{1}{r}$$

$$\Gamma^\varphi_{\varphi\varphi} = \frac{1}{2} g^{\varphi\varphi} \cancel{\partial_\varphi g_{\varphi\varphi}}^{0} = 0$$

$$\Gamma^\mu_{\alpha\beta} = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -r \end{pmatrix} \\ \begin{pmatrix} 0 & 1/r \\ 1/r & 0 \end{pmatrix} \end{pmatrix}$$

c) $u^\alpha \nabla_\alpha u^\beta = 0 \Leftrightarrow u^\alpha \partial_\alpha u^\beta + \Gamma^\beta_{\alpha\lambda} u^\alpha u^\lambda = 0 \Leftrightarrow \frac{dx^\alpha}{d\lambda}\frac{\partial u^\beta}{\partial x^\alpha} + \Gamma^\beta_{\alpha\lambda} u^\alpha u^\lambda = 0 \Leftrightarrow$

$$\Leftrightarrow \frac{du^\beta}{d\lambda} + \Gamma^\beta_{\alpha\lambda} u^\alpha u^\lambda = 0.$$

→ $\underline{\beta = r}$: $\frac{du^r}{d\lambda} = -\Gamma^r_{\varphi\varphi}(u^\varphi)^2 = r(u^\varphi)^2 \Leftrightarrow \frac{d^2 r}{d\lambda^2} = r\left(\frac{d\varphi}{d\lambda}\right)^2 \Leftrightarrow \ddot{r} = r\dot{\varphi}^2 =: F(r, \dot{\varphi})$

→ $\underline{\beta = \varphi}$: $\frac{du^\varphi}{d\lambda} = -2\Gamma^\varphi_{r\varphi} u^r u^\varphi = -\frac{2}{r} u^r u^\varphi \Leftrightarrow \frac{d^2\varphi}{d\lambda^2} = -\frac{2}{r}\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda} \Leftrightarrow \ddot{\varphi} = -\frac{2}{r}\dot{r}\dot{\varphi} =: G(r, \dot{r}, \dot{\varphi})$