Exercício 3

$$\boxed{ds^2 = -(1-\omega^2 r^2)\,dt^2 + 2\omega r^2\,dt\,d\theta + dr^2 + r^2 d\theta^2 + dz^2}$$

a) $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{00}dt^2 + g_{02}\,dt\,d\theta + g_{20}\,d\theta\,dt + g_{11}dr^2 + g_{22}\,d\theta^2 + g_{33}dz^2$. De onde se infere:

$$\boxed{g_{\mu\nu} = \begin{pmatrix} -(1-\omega^2 r^2) & 0 & \omega r^2 & 0 \\ 0 & 1 & 0 & 0 \\ \omega r^2 & 0 & r^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

b) $u^\mu = \dfrac{dx^\mu}{d\tau} = \dfrac{d}{d\tau}(t, r_0, \theta_0, \varphi_0) = \dfrac{dt}{d\tau}(1,0,0,0)$. Como $g_{\mu\nu}u^\mu u^\nu = -1$, temos:

$$-1 = g_{\mu\nu}u^\mu u^\nu = g_{00}(u^0)^2 = -(1-\omega^2 r^2)\left(\frac{dt}{d\tau}\right)^2 \Leftrightarrow \frac{dt}{d\tau} = \frac{1}{\sqrt{1-\omega^2 r_0^2}}.$$

Logo, vemos que $|\omega r_0| < 1 \Rightarrow \boxed{r_0 < \dfrac{1}{|\omega|}}$ e $\boxed{u^\mu = \dfrac{(1,0,0,0)}{\sqrt{1-\omega^2 r_0^2}}}$

c) Como vimos, $\dfrac{dt}{d\tau} = \dfrac{1}{\sqrt{1-\omega^2 r_0^2}} \Rightarrow \boxed{t = \dfrac{\tau}{\sqrt{1-\omega^2 r_0^2}} + \text{cte}}$