A MECÂNICA DE SISTEMAS DE MASSA VARIÁVEL

MECÂNICA II
PME 3200

18/ 06 / 2015

Horários:
07:30 – 09:10
09:20 – 11:10

Prof. Dr. Celso P. Pesce

Escola Politécnica

Summary

The subject has not always been deeply discussed in Engineering Mechanics Education and, even worse, not always properly included in many modern Engineering Courses curricula, at both undergraduate and graduate levels.

The purpose of the present talk is to re-address such an important matter, aiming at contributing to Engineering Mechanics Education, by discussing under a historical perspective some theoretical aspects involved in variable mass systems dynamics which are usually hidden behind many derivations.

Motivation

Variable mass systems have been the focus of a large number of problems in classical mechanics. However, despite the classic nature and importance of variable mass systems dynamics, many misinterpretations were done on the correct application of Newton's second law, even in a not so distant past. Such misinterpretations sometimes give rise to apparent paradoxes in Classical Mechanics.

For instance, motivated by the rocket problem, a long debate on the correct application of Newton's law took place during the 1960's, among American scholars and educators.

Even subtler may be the proper application and interpretation of the Lagrangian formalism to systems presenting mass dependence on time, position (and velocity).

Facts on Newton's law application

1960's American scholars debate

"…this basic law of mechanics is currently being seriously misinterpreted. This misinterpretation appears under conditions where the mass of a body is a function of time."

Meriam J.L. 1960 *J. Eng. Ed.* **51** 243

Facts on Newton's law application

1960's American scholars debate

"There exists considerable confusion and disagreement among professional physicists concerning the correct classical equations of motion for systems of changing mass…"

Tiersten M.S. 1969 *Am. J. Phys.* **40** 183

Facts on Newton's law application

1990's

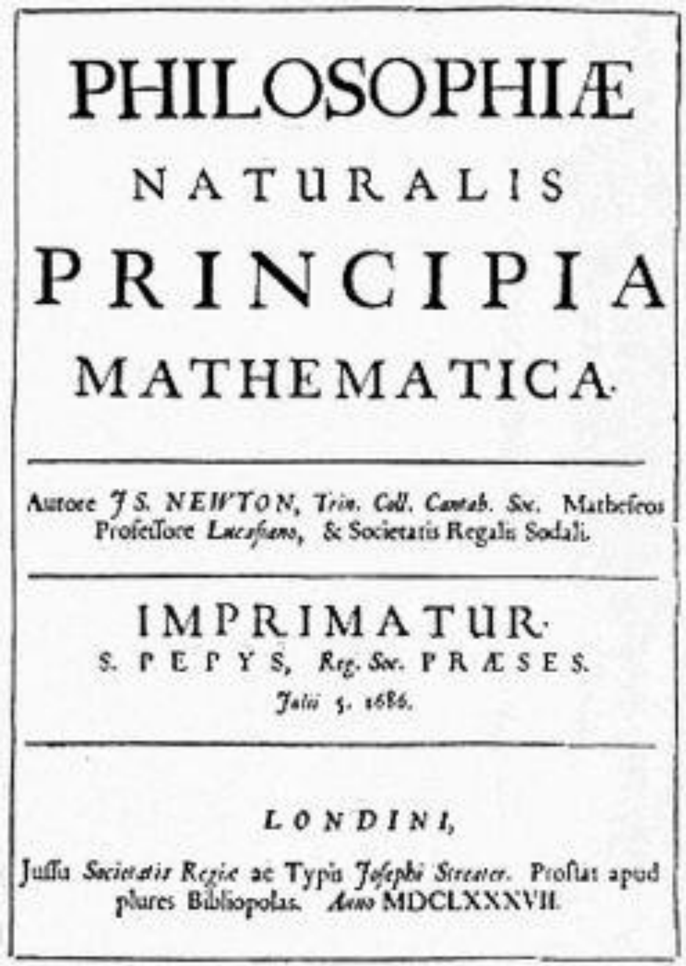
Despite the fact that variable mass dynamics has been an active research field for many years, we still find in the literature wrong applications of Newton's second law in this context. For example, Shrivastava and Ishwar (1983), Singh and Ishwar (1984), and Das *et al.* (1989), who analyzed the restricted three-body problem when the mass of the infinitesimal body varies, and Saslaw (1985), who discussed the virial theorem for a collection of bodies of variable mass, incorrectly applied Newton's second law (or the equivalent Lagrange's equations) to deal with the variable masses and obtained erroneous results.

ON THE USE AND ABUSE OF NEWTON'S SECOND LAW FOR VARIABLE MASS PROBLEMS

Plastino, A.R. & Muzzio, J.C. 1992

Celestial Mechanics and Dynamical Astronomy **53:** 227–232, 1992.

Brief history



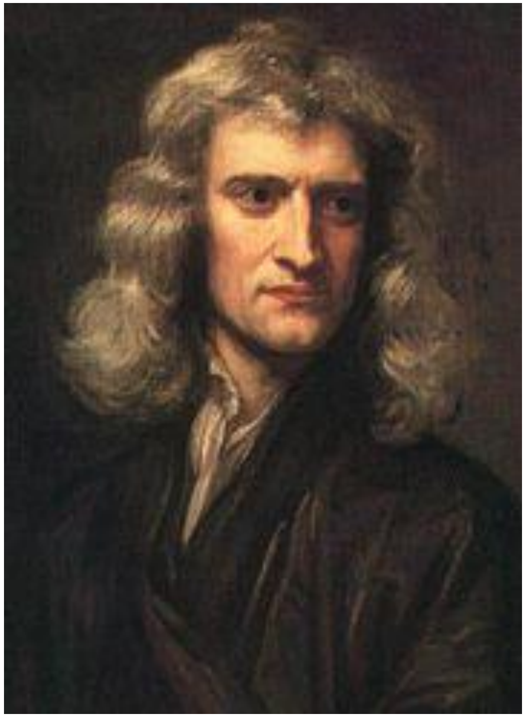
PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.

Autore J S. NEWTON, Trin. Coll. Cantab. Soc. Matheseos Professore Lucasiano, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, Reg. Soc. PRÆSES.
Julii 5. 1686.

LONDINI,
Jussu Societatis Regiæ ac Typis Josephi Streater. Prostat apud plures Bibliopolas. Anno MDCLXXXVII.

(1687)



Sir Isaac Newton
(1689, by Godfrey Kneller)

* January 4th 1643, in Woolsthorpe; † March 31st 1727 in London

Newton's Laws

Lex I: Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.

Lex II: Mutationem motis proportionalem esse vi motrici impressae, etfieri secundum lineam rectam qua vis illa imprimitur.

Lex III: Actioni contrariam semper et aequalem esse reactionem: sine corporum duorum actiones in se mutuo semper esse aequales et in partes contrarias dirigi.

Newton's Laws

"Law I – Every body perseveres in its state of rest or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon";

Law of inertia; establishes inertial frames of reference

"Law II - The alteration of [the quantity] of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed";

Law of acceleration

"Law III - To every action there is always opposed an equal reaction – or the mutual actions of the two bodies upon each other are always equal, and directed to contrary parts";

Action-reaction Principle

Dugas, page 206.

Dugas, R., 1955, A History of Mechanics, Dover ed.,1988, 662 pp.

Newton's Definition of mass

The term 'law of motion' was introduced in the 17th century by Descartes. After stating the law of inertia in an essentially modern form, Descartes stated a law of conservation of momentum with respect to its magnitude only, and not to direction, and continued with a list of 'laws of impact' which involved the impact between solid bodies.

At the beginning of his famous *Philosophiae Naturalis Principia Mathematica*, Newton asserted the so-called 'laws of motion', which is more than a result of an appreciation of previous works.

"Rather than dealing with relations between initial and final conditions in an interacting system, as done by Descartes, Newton dealt directly with the effect of the forces acting on individual bodies…"

Arons and Bork (1964)

Newton's Definition of mass

When Newton discussed the motion of bodies, the trajectories were conic sections and not straight lines or other paths, and the forces considered by him were central forces.

Apparently, he showed no interest in the mechanical problems usually found in today textbooks, particularly the variable mass ones.

Paradoxically, according to Dugas (1951), "Newton introduced the notion of mass into Mechanics(*)", even though "this notion had appeared in Huyghen's work, but only in an impermanent form".

(*) "Definition I – The Quantity of Matter is the measure of the same, arising from its density and bulk conjunctly"; Dugas (1951), page 201.

Variable mass problems

Early 19th Century

According to Šíma and Podolský (2005), the Czech scientist and inventor von Buquoy "was the first to investigate systems with a varying mass".

Georg Franz August de Longueval, Baron von Vaux, Graf von Buquoy

(* Brussel, September 7th 1781; † Prag, April 19th 1851)

"In 1812 von Buquoy explicitly formulated the correct dynamical equation of motion for the case when the mass of a moving object is changing".

von Buquoy, G., 1812, "Analytische Bestimmung des Gesetzes der Virtuellen Geschwindigkeiten in Mechanischer und Statischer Hinsicht", Leipzig: Breitkopf und Hartel.

Variable mass problems

Early 19th Century

von Buquoy's work was presented in 1815, at the Paris Academy of Sciences.

"Apart from a single short article by Poisson (1819), his ideas did not attract attention, and they gradually become forgotten."

Šíma and Podolský (2005).

Poisson, S.D., 1819, "Sur le Mouvement d'un Système de Corps, en supposant le Masses Variables", Bull. Sci. Soc. Philomat. Paris, avril, pp. 60-62

Siméon Denis Poisson

* Pithviers, June 21^{st} 1781; † Paris, April 25^{th} 1840

Variable mass problems

19th Century

"Buquoys's general equation of motion and other explicit examples were later formulated independently by various authors;

to be mentioned, Tait and Steele (1856) and Meshchersky (1897),"

Šíma and Podolský (2005).

Peter Guthrie **Tait**, 1831-1901;

Scottish, topology and mathematical physics

Variable mass problems

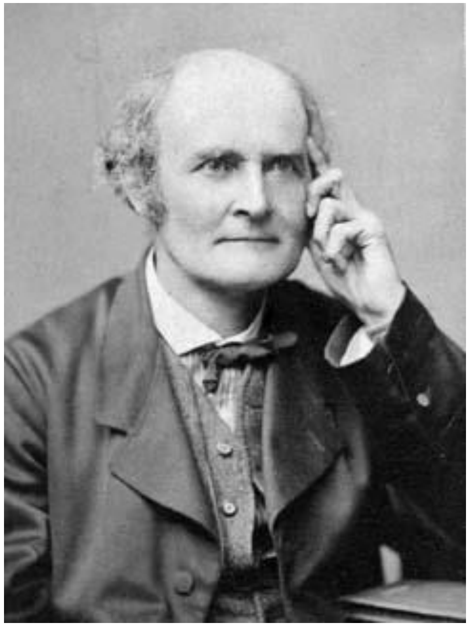
Mid 19th Century

1857, Cayley: *suspending chain coiled up at a table*

- Cayley, A., 1857, "On a Class of Dynamical Problems". *Proceedings of Royal Society of London,* Vol VIII, pp 506-511.

Falling chain: still a very intriguing problem…

- *Davis, Am. J. Phys., 1952*
- *Prato & Gleiser, Am. J. Phys, 1982*
- *Calkin & March, Am. J. Phys., 1989*
- *Vrbik, Am. J. Phys., 1993*
- *Keiffer, Am. J. Phys., 2001*
- *Tomaszewski & Pieranski, Am. J. Phys., 2006*
- *Wong & Yasui, Am. J. Phys., 2006*
- *Wong, Youn & Yasui, Eur. J Phys, 2007*
- *Casetta, Doctoral Thesis, EPUSP, 2008*
- *Grewal, Johnson & Ruina, Am. J. Phys., 2011*
- *Irschik, J Theo & Appl Mech, 2012*



Arthur **Cayley**, 1821-1895
British mathematician

Variable mass problems

Late 19th Century

Ivan Vsevolodovich Meshchersky, 1859 - 1935

Meshchersky (1897) Master Thesis, and his subsequent work written in 1904, have been ever since recognized - in the Russian technical literature - as the limestone in the study of variable mass systems in the context of Classical Mechanics; see, e.g., Targ (1976), page 394 or Starjinski (1980), page 498.

Meshchersky, I.V., 1897, Dinamika tochki Peremnoj Massy (*), St Petesburg, Akademia Nauk, Peterburskij Universitet.
(*) Динамика точки переменной массы

Meshchersky, I.V, 1904, "Equations of Motion of a Variable Mass Point in the General Case" (in Russian), St. Petersburg Polytechnic University News, Vol.1, pp. 77-118.

Ivan Vsevolodovich Meshchersky

1859 – 1935

(http://www-gap.dcs.st-and.ac.uk/~history/Biographies/Meshchersky.html)

Meshchersky taught in St Petersburg for 58 years. Obtained a Master's Degree in applied mathematics in 1889 and was appointed as a dozent at the university in 1890.

Meshchersky is best known for his work on the motion of bodies of variable mass which he described in January 1893 at a meeting of the St Petersburg Mathematical Society.

He continued to develop his work on this topic for his dissertation entitled *The dynamics of a point of variable mass,* submitted in 1897.

Meshcheysky's examples:

the increase of the mass of the earth caused by falling meteorites;

the increase of the mass of a freezing iceberg and the decrease of a thawing one;

the increase of the mass of the sun gathering of cosmic dust and its decrease with radiation;

***the decrease of the mass of a rocket as its fuel is consumed*.**

Ivan Vsevolodovich Meshchersky

1859 – 1935

(http://www-gap.

Even before that, in 1903, the Russian scientist Tsiolkovski - who had invented a kind of rocket-aircraft around 1883 - applied Meshchersky's Equation to solve the rocket problem in two versions: (i) *gravity-free* and (ii) *non-gravity-free*. Those two problems are sometimes referred to as the first and the second problems of Tsiolkovski.

He applied his theory
being the first to stud
problem of determini
mass from a knowled
and the acting forces

His work on the motio
of variable mass formed th
for rocket technology developed
after 1945.

Even before his dissertation
Meshchersky had shown another
major interest in his life: **teaching**.

He published *The teaching of mechanics in certain institutions of higher education in Italy, France, Switzerland and Germany* in 1895.

aising
g of
t
tute was

r role in
He was
appointed as Head of Applied Mathematics at the Polytechnic Institute in 1902.

Curiously, at the time of the American Scholars Debate

ics course
g in 1914 to
ection of problems of m cs".

This fam s book had reached its 26th Russ an edition by 1960, translated into English by Pergamon Press in 1965.

Other worth mentioning studies

Early 20th Century

(lost or gained at

1928; Levi-Civita: ble mass problem

"Sul Moto di un ti delle Sedute della Reale Accademia Nazi Aggiunta alla nota, pp 621-622.

"Ancora sul Moto ute della Reale Accademi

ost or gained ll velocity

1936; Agostinelli: with variable mass

"Sui Sistemi Din ale Accademia delle Scienze di Torino Naturali, 71, I, pp. 254-272.

1873-1941, Italy

1940-50's; several a *roblems.*

Leitmann, G., 1957, "On the Equation of Rocket Motion", Journal of the British Interplanetary Society, Vol.16, No. 3, pp. 141-147.

American Scholars debate

The 1960's

1960, Meriam: *Engineering Education*

"Variable-Mass Dynamics", Journal of Engineering Education, Vol.51, No. 3, pp. 240-243.

1962, Thorpe: *Engineering Education*

"On the Momentum Theorem for a Continuous System of Variable Mass", American Journal of Physics, Vol.30, No. 9, pp. 637-640.

....

1969, Tiersten: *Engineering Education*

"Force, Momentum Change and Motion", American Journal of Physics, Vol.37, No. 1, pp. 82-87.

Recent studies on variable mass systems

Late 60's to 80's

1968, Weber, H.I.

Supervisor:
Bevilacqua, L.

y contribution
generalizing
Reynolds Transport
Theorem

"Vibração de Vigas com Massa Variavel no T

1973 McIver: *general open systems*

"Hamilton's Principle for Systems of Changing Mass", Journal of Engineering Mathematics, Vol.7, No. 3, pp. 249-261.

1975, Mikhailov: history of mechanics.

"On the history of variable-mass system dynamics". Mechanics of Solids, 10(5), 32-40

1982 Copeland work-energy theorem

"Work-energy theorem for variable mass systems", A. J. Phys., 50(7), 599-601.

Recent studies on variable mass systems

Early 80's to 90's

1982, 84, Ge:	*non-holonomic variable mass systems*
1984, 89, 92, 93, 94, 2001: (Cveticanin)	*industrial systems (textile, lifting-crane) vibration problems due to variable mass;*
1995, 97 (Crellin et al):	*tethered satellites;*
1999, Mušicki:	*general open systems*

"General Energy Change Law for Systems with Variable Mass", European Journal of Mechanics A/Solids, Vol.18, pp. 719-730.

Nowadays studies: 2000-2004

2000, Mušicki: *general open systems*

"Generalization of a New Parametric Formulation of Mechanics for Systems with Variable Mass", Eur Journal of Mechanics A/Solids, Vol.19, pp. 1059-1076.

2002, Eke & Mao: *Engineering Education*

"On the Dynamics of Variable Mass Systems", International Journal of Mechanical Engineering Education, Vol.30, No. 2, pp. 123-137.

2003, Pesce: *Lagrange Equation and variable mass systems*

"The Application of Lagrange Equations to Mechanical Systems with Mass Explicitly Dependent on Position", Journal of Applied Mechanics, Vol. 70, pp. 751-756.

2004, Irschik & Holl: *general open systems*

"The Equations of Lagrange Written for a Non-Material Volume", Acta Mechanica, Vol.153, pp. 231-248.

2004, Irschik & Holl: *general open systems*

"Mechanics of Variable-Mass Systems – Part 1: Balance of Mass and Linear Momentum", Applied Mechanics Review, Vol.57, No. 2, pp. 145-160.

2005-2007

2005, Mušicki: *general open systems*

“Extended Lagrangian Formalism and Main General Principles of Mechanics”, European Journal of Mechanics A/Solids, Vol.24, pp. 227-242

2006, Wong & Yasui: Engineering Education

“Falling chains”. American Journal of Physics, v. 6, 490-496.

2006, Pesce, Casetta, Tannuri: *ocean engineering applications*

“The Lagrange Equations for Systems with Mass Varying explictly with Position: Some Applications to Offshore Engineering”, JBSMSE, vol. 28, 496-504.

2007, Wong, Youn & Yasui: Engineering Education

“The falling chain of Hopkins, Tait, Steele and Cayley”. European Journal of Physics, v. 28, 385-400.

2007, Bazant & Verdure: *mechanics of progressive collapse*

“Mechanics of progressive collapse: learning from World Trade center and Building Demolitions”, Journal of Engineering Mechanics, ASCE Vol.133 (3), pp. 308-319

2007, Casetta & Pesce: *hydrodynamic impact*

“Hamilton’s Principle for Dissipative Systems and Wagner’s Problem”, 2nd International Workshop on Water Waves and Floating Bodies 15th–18th April 2007, Plitvice, Croatia.

2008-2010

2008, Seffen: *mechanics of progressive collapse*

“Progressive Collapse of the World Trace Center: simple analysis”, Journal of Engineering Mechanics, ASCE Vol.134 (2), pp. 125-132

2008, Casetta: mechanics of variable mass systems

“Contribuições à Mecânica dos Sistemas de Massa Variável”, EPUSP, Tese de Doutorado, 185 pp

http://www.teses.usp.br/teses/disponiveis/3/3152/tde-05082009-100852/

2009, Cveticanin: *multi body dynamics*

“Dynamics of Body Separation – analytical procedure”, Nonlinear Dynamics, Vol. 55, pp. 269-278

2009, Schwarzbart et al: *tethered satellites*

“Tethered satellite systems: a challenge for mechanics and applied mathematics. GAMM-Mitteilungen, v. 32, n. 1, p. 105-20.

2010, Bažant, Le, Greening, & Benson: *mechanics of progressive collapse*

“What did and did not cause collapse of World Trade Center twin towers in New York?”, Journal of Engineering Mechanics, ASCE vol. 134 (10). 892-906

2011

2011, Casetta & Pesce: *variational principles in hydrodynamics*

"On Seliger and Whitham's variational principle for hydrodynamic systems from the point of view of fictitious particles", Acta Mechanica, vol. 219, 181-184.

2011, Le & Bažant: *mechanics of progressive collapse*

"Why the observed motion history of World Trade Center towers is smooth". Journal of Engineering Mechanics, ASCE, 137 (1), 82-84.

2011, Casetta, Pesce, Santos : *hydrodynamic impact*

"On the Hydrodynamic Vertical Impact Problem: an Analytical Mechanics Approach", Marine Systems and Ocean Technology, 6(1), 47-57.

2011, Grewal, Johnson and Ruina: *falling chains*

"A Chain that speeds up, rather tan slows, due to collisions: how compression can cause tension", Am. J .Phys., 79(7), 723-729.

2011, Jeltsema & Dòria-Cerezo: *systems modeling*

"Modeling of systems with position-dependent mass revisited: a Port-Hamiltonian approach", Journal of Applied Mechanics, Vol. 78 / 061009-1.

2011-2012

2011, Bedoustani et al: *robotics*, *cable-driven manipulators*

"Lagrangian dynamics of cable-driven parallel manipulators: a variable mass formulation". Transactions Canadian Soc. Mech. Engineers, 35(4), 529-542.

2011, Holl & Hammelmuller: *coiling processes*

"Analysis of the vibrations due to thermal deflection of the drum in the coiling process. Proc. Appl. Math. Mech. 11, 317-318

2012, Cveticanin: *nonlinear oscillators*

"Oscillator with non-integer order nonlinearity and time variable parameters". Acta Mechanica, 223 (7):1417-1429.

2012, Cveticanin & Pogany: *nonlinear oscillators*

"Oscillator with a sum of non-integer order non-linearities". Journal of Applied Mathematics, vol. 2012, art. no. 649050.

2012, Casetta & Pesce: *general open systems*

"On the generalized canonical equations of Hamilton for a time-dependent mass particle", Acta Mechanica, vol. 223, 2723-2726.

2012, Irschik: *continuous impact and open systems*

"The Cayley variational principle for continuous-impact problems: a continuum mechanics based version in the presence of a singular surface", J of Theoretical and Appl Mech, 50 (3), 717-727.

2013

2013, Cruz y Cruz & Rosa-Ortiz: *position dependent mass & Poisson algebra*

“Generating Algebras of Mechanical Systems with Position-Dependent Mass”. Symmetry, Integrability and Geometry: Methods and Applications, Special issue.

2013, Cveticanin: *nonlinear oscillators*

“Van der Pol oscillator with time variable parameters”, Acta Mechanica, Vol. 224(5), 945-955.

2013, Casetta & Pesce: *gener*

“The generalized Hamilton’s principle for a non-mater
224, 919-924.

> *Generalizes McIver’s Hamiltonian approach to Reynolds transport theorem*

2013, Casetta & Pesce: *discrete systems and inverse problems*

“The inverse problem of Lagrangian mechanics for Meshchersky’s equation”, Acta Mechanica, vol. 225, 1607-1623.

Leading to the

Advanced International Course

Dynamics of Mechanical Systems with Variable Mass

CISM

International Centre for Mechanical Sciences

Udine, Italia

24-28 Sep 2012

http://www.cism.it/courses/C1212

TIME TABLE

TIME	Monday September 24	Tuesday September 25	Wednesday September 26	Thursday September 27	Friday September 28
9.00 - 9.45	Registration	Pesce	Indeitsev	Irschik	Indeitsev
9.45 - 10.30	Irschik	Pesce	Indeitsev	Irschik	Indeitsev
11.00 - 11.45	Irschik	Zilian	Cveticanin	Zilian	Pesce
11.45 - 12.30	Indeitsev	Zilian	Cveticanin	Zilian	Pesce
14.00 - 14.45	Belyaev	Belyaev	Zilian	Cveticanin	
14.45 - 15.30	Belyaev	Belyaev	Zilian	Cveticanin	
16.00 - 16.45	Cveticanin	Irschik	Pesce	Belyaev	
16.45 - 17.30	Cveticanin	Irschik	Pesce	Belyaev	

ADMISSION AND ACCOMMODATION

Applicants must contact CISM Secretariat at least one month before the beginning of the course. Application forms should be sent on-line through our web site: http://www.cism.it or by post.

A message of confirmation will be sent to accepted participants. If you need assistance for registration please contact our secretariat.

The 700,00 Euro registration fee includes a complimentary bag, four fixed menu buffet lunches (Friday not included), hot beverages, on-line/downloadable lecture notes and wi-fi internet access.

A limited number of participants from universities and research centres who are not supported by their own institutions can be offered board and/or lodging in a reasonably priced hotel. Requests should be sent to CISM Secretariat by **July 24, 2012** along with the applicant's curriculum and a letter of recommendation by the head of the department or a supervisor confirming that the institute cannot provide funding. Preference will be given to applicants from countries that sponsor CISM.

Information about travel and accommodation is available on our web site, or can be mailed upon request.

Please note that the centre will be closed for summer vacation the first three weeks in August.

For further information please contact:

CISM
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Piazza Garibaldi 18
33100 Udine (Italy)
tel. +39 0432 248511 (6 lines)
fax +39 0432 248550
e-mail: cism@cism.it

ACADEMIC YEAR 2012
The Nowacki Session

Centre International des Sciences Mécaniques
International Centre for Mechanical Sciences

DYNAMICS OF MECHANICAL SYSTEMS WITH VARIABLE MASS

Advanced School
coordinated by

Hans Irschik
University of Linz
Austria

Alexander K. Belyaev
Russian Academy of Sciences
St. Petersburg
Russia

Udine, ***September 24 - 28***, 2012

DYNAMICS OF MECHANICAL SYSTEMS WITH VARIABLE MASS

The fundamental equations of classical mechanics were originally formulated for situations where mass is conserved in the mechanical system under consideration. Mass is generally not conserved when a supply of mass is present, or when open systems with a flow of mass through their surface are to be considered. Mass of the mechanical system then is said to be variable. In such a situation, the general methodological approaches of mechanics have to be properly modified. In fluid mechanics, open systems are encountered when studying a non-material control volume. In solid mechanics, systems with a variable mass appear as the result of a problem-oriented modeling, e.g. when mass is expelled or captured by a structure or machine. This again leads to the treatment as an open system, or to the assumption that that mass is explicitly dependent on the position. In solid mechanics, as well as in fluid mechanics, it is often appropriate to model the exchange of mass between the system under consideration and the environmental world by means of a supply of mass in the interior. This is of particular interest in the continuum theory of mixtures, for which mass and other entities are exchanged between the various components. It is the goal of the proposed course to present up-to-date and unifying formulations for treating the dynamics of different types of mechanical systems with variable mass. We start with an overview of the continuum mechanics relations of balance and jump for open systems, from which extended Lagrange and Hamiltonian formulations will be derived, as a basis of current numerical procedures. Corresponding approaches will be stated at the level of the analytical mechanics, with emphasis on systems with a position-dependent mass, and applications to offshore engineering, as well as at the level of structural mechanics. Special emphasis will be laid upon axially moving structures, like belts and chains, and on pipes with an axial flow of fluid. Constitutive relations appearing in the dynamics of mechanical systems with variable mass will be studied with particular reference to the modeling of multi-component mixtures. Damage of steel structures in the form of hydrogen embrittlement will be addressed in this context. The dynamics of machines with a variable mass will be treated in detail and, in this context, conservation laws and the stability of motion will be analyzed. Novel finite element formulations for open systems in coupled fluid and structural dynamics will be presented. Moreover, the course will provide mathematical models directly related to methods of automatic control, and therefore should be of interest in the fields of Civil and Mechanical Engineering, as well as in Mechatronics.

PRELIMINARY SUGGESTED READINGS

Irschik, H., Holl, H.J., Mechanics of variable-mass systems - part 1: balance of mass and linear momentum". Applied Mechanics Review, 57, 145-160, 2004.

Irschik, H., Holl, H., The equations of Lagrange written for a non-material volume. Acta Mechanica, 153, 231-248, 2002.

Cveticanin, L., Dynamics of Machines with Variable Mass, Gordon and Breach Sc. Publishers, London, 1998.

Cveticanin, L., Dynamics of body separation - Analytical procedure, Nonlinear Dynamics, 55, 269-278, 2009.

Cveticanin, L., Djukic, Dj., Dynamic properties of a body with discontinual mass variation, Nonlinear Dynamics, 52, 249-261, 2008.

Cveticanin, L., Kovacic, I., On the dynamics of bodies with continual mass variation, Trans ASME, Journal of Applied Mechanics, 74, 810-815, 2007.

Indeitsev, D.A., Semenov, B. N., About one model of structural-phase transformations under hydrogen influence. Acta Mechanica, 195, 295-304, 2008.

Indeitsev, D.A., Naumov, V.N., Semenov, B.N., Belyaev, A.K., Thermoelastic waves in a continuum with complex structure. ZAMM, 89, 279-287, 2009.

Pesce, C.P., The application of Lagrange equations to mechanical systems with mass explicitly dependent on position. Journal of Applied Mechanics, 70, 751-6, 2003.

McIver, D.B., Hamilton's principle for systems of changing mass. Journal of Engineering Mathematics, 7, 249-261, 1973.

Musicki, D., General energy change law for systems with variable mass. European Journal of Mechanics A/ Solids, 18, 719-730, 1999.

Bazant, Z.P., Verdure, M., Mechanics of progressive collapse: learning from world trade center and building demolitions. Journal of Engineering Mechanics, ASCE, 133, 308-19, 2007.

E. Naudascher, E., Rockwell, D., Flow-induced vibrations: an engineering guide. A.A. Balkema, Rotterdam, 1994.

INVITED LECTURERS

Alexander K. Belyaev - Russian Acad. of Sciences, St. Petersburg, Russia
6 lectures on: General formulations: Structural mechanics of systems with variable mass; the Rayleigh-Ritz method for vibrating structures with variable mass; dynamics and stability of axially moving structures; Lagrange and Hamiltonian formulations for axially moving strings and beams. Engineering applications: Transmission processes, such as those by belts and chains; pipes with an axial flow of fluid.

Livija Cveticanin - University of Novi Sad, Serbia
6 lectures on: General formulations: General principles and dynamics of machines with continual and discontinual mass variation; chaos in systems with variable mass; conservation laws and stability of motion for machines with variable mass. Engineering applications: Dynamics and stability of machines with rotating elements and variable mass.

Dmitry Indeitsev - State University of St. Petersburg, Russia
5 lectures on: General formulations: Mechanics of multi-component media with an exchange of mass and non-classical supplies; inelastic constitutive relations modeled in the framework of multi-component media. Engineering applications: Analytical and numerical formulations for damage in steel; hydrogen embrittlement of alloys.

Hans Irschik - University of Linz, Austria
6 lectures on: General formulations: Continuum mechanics based relations of balance and jump for systems with variable mass and non-classical supplies; derivation of Lagrange and Hamiltonian formulations; equations of Lagrange for systems with variable mass written in the Euler and Lagrange description of continuum mechanics. Engineering applications: Industrial coiling processes.

Celso P. Pesce - University of Sao Paulo, Brasil
6 lectures on: General formulations: Analytic mechanics of systems with mass explicitly dependent on position; corresponding Lagrange and Hamiltonian formulations. Engineering applications: Offshore engineering problems; problems of the falling and unfolding chain type; collapsing dynamics of buildings.

Andreas Zilian - Technical University of Braunschweig, Germany
6 lectures on: General formulations: Mechanics of coupled systems with mass dependent on structural motion and/or deformation; effects of added mass/damping/stiffness; models to fluid-structure interaction of discrete and distributed mass systems. Numerical schemes for complex fluid-structure interaction and associated reduced-order models. Engineering applications: Aeroelasticity and hydroelasticity.

LECTURES

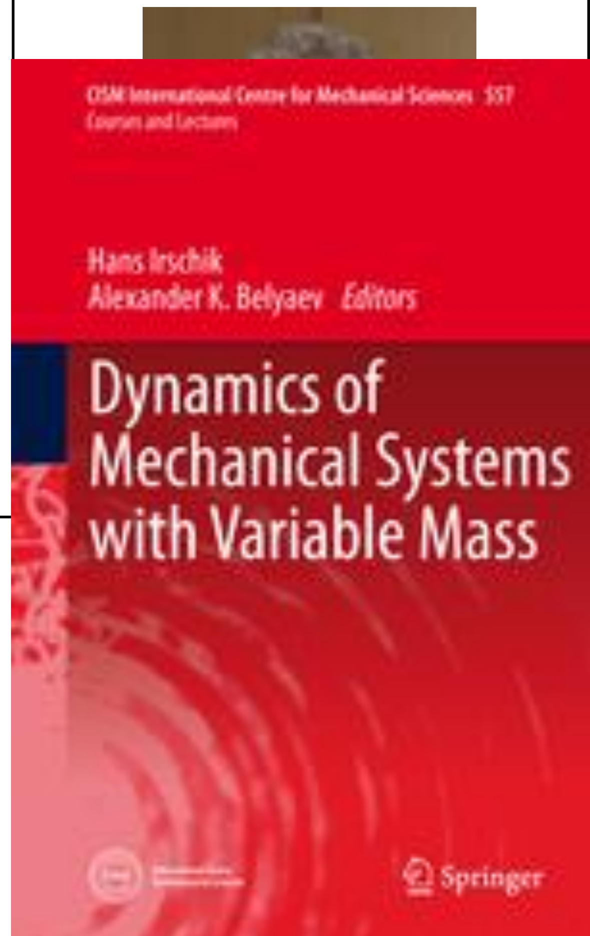
All lectures will be given in english. lecture notes can be downloaded from cism web site, instructions will be sent to accepted participants.

Springer book (2014):

Dynamics of Mechanical Systems with Variable Mass

http://www.springer.com/br/book/9783709118085

Hans Irschik,
University of Linz, Austria



Livija Cveticanin,
University of Novi Sad, Serbia

Celso Pesce,
University of S. Paulo, Brazil

Dmitry Indeitsev,
Univ of St Petersburg, Russia

Andreas Zilian,
Tech Univ of Braunshweig, Germany

Springer book (2014):
Dynamics of Mechanical Systems with Variable Mass

http://www.springer.com/br/book/9783709118085

Chapter 1 - H. Irschik and A. Humer
A rational treatment of the relations of balance for mechanical systems with a time-variable mass and other nonclassical supplies

Chapter 2 - C.P. Pesce and L. Casetta
Systems with mass explicitly dependent on position

Chapter 3 - L. Cveticanin
Dynamics of the Mass Variable Body

Chapter 4 - D. Indeitsev and Yu. Mochalova
Mechanics of multi-component media with exchange of mass and non-classical supplies;

Chapter 5 - A. Zilian
Modelling of Fluid-Structure Interaction – Effects of Added Mass, Damping and Stiffness;

Chapter 6 - A.K. Belyaev
Dynamics and Stability of Engineering Systems with Moving Continua;

2014-2015

2014, Casetta: *general open s*

Generalizes the inverse problem of Lagrangian mechanics for continuous open sytems

"The inverse problem of Lagrangian mec
Mechanica, Vl. 225 (6) DOI.

2014, Cveticanin: *multi body dynamics*

"Principle of generalized velocities in dynamics of planar separation of a rigid body".
Acta Mechanica, 226 2511-2525

2015, Irschik & Holl: *general open systems*

"Lagrange´s equations for open systems, derived via the method of fictitious particles, and written in the Lagrange description of continuum mechanics", Acta Mechanica, Vol. 226 (1), 63-79, 2015

2015, Casetta & Pesce: *discrete syste*

Solves special cases of Meshchersky's equation analytically, including Cayley's problem

"A brief note on the analytical solution of Mesh
problem of Lagrangian mechanics", Acta Mechani

2015

2015, Cveticanin: *multi body dynamics*

"Principle of generalized velocities in dynamics of planar separation of a rigid body". Acta Mechanica, Vol. 226, 2511-2525

2015, Garcia-Ferieta & Casas: *celestial mechanics*

"Simulación interactiva del problema de dos cuerpos perturbados por un objeto de masa variable dependiente de la posición: un ilustrativo ejemplo para el estudio de la cinemática de cometas", Revista de Ciencias, Vol. 6, No. 3 de 2015.

2015, Bartkowiak, Grabski & Kołodziej: *discrete systems*

"Numerical and experimental investigations of the dynamics of a variable mass pendulum", J of Mechanical Engineering Science, DOI: 10.1177/0954406215590454

2015, Casetta, Irschik & Pesce: *open systems and conservation laws*

"A generalization of Noether's theorem for a non-material volume", ZAMM - Zeitschrift fur Angewandte Mathematik und Mechanik, approved, to appear.

and much more to be done…

Meshchersky's equation

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{F} + \mathbf{\Phi} = \mathbf{F} + \frac{\mathrm{d}m}{\mathrm{d}t}\mathbf{v}_{rel}$$

Meshchersky force

$$\mathbf{\Phi} = \dot{m}\mathbf{v}_{rel}$$

$$\mathbf{v}_{rel} = \mathbf{w} - \mathbf{v}$$

$\mathbf{w}$ is the velocity of the accreted or lost mass with respect to the same inertial frame of reference

Particular case

$$\mathbf{F} = \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v})$$

Levi-Civita Case

It is not generally valid, for a single particle.

It is only valid if mass is gained or lost at null velocity!

More over, it is not invariant with respect to the choice of inertial frames of reference, except when mass is constant.

Therefore, it does not satisfy the Galilean relativity principle.

On the other hand, Meshchersky's Equation

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} + \mathbf{\Phi} = \mathbf{F} + \frac{dm}{dt}\mathbf{v}_{rel}$$

Is generally valid!

It is invariant with respect to the choice of inertial frames of reference.

It does satisfy the Galilean relativity principle.

$\dot{\mathbf{v}} = \dot{\mathbf{v}}'$

Galilean Invariance

Consider two inertial frames of reference. One of them, for simplicity and no loss of generality, is supposed fixed and the other one moves with a constant velocity $\mathbf{v}_{ref}$. Let **v** and **v**' be the velocity of a point with respect to those frames of reference. So,

$$\mathbf{v} = \mathbf{v}' + \mathbf{v}_{ref} \qquad \dot{\mathbf{v}} = \dot{\mathbf{v}}'$$

Meshchersky

$$\mathbf{F} = \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}) - \frac{\mathrm{d}m}{\mathrm{d}t}\mathbf{w} = \frac{\mathrm{d}m}{\mathrm{d}t}\mathbf{v} + m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} - \frac{\mathrm{d}m}{\mathrm{d}t}\mathbf{w}$$

$$\mathbf{F} = \frac{\mathrm{d}m}{\mathrm{d}t}(\mathbf{v}' + \mathbf{v}_{ref}) + m\frac{\mathrm{d}\mathbf{v}'}{\mathrm{d}t} - \frac{\mathrm{d}m}{\mathrm{d}t}(\mathbf{w}' + \mathbf{v}_{ref})$$

$$\mathbf{F} = \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}') - \frac{\mathrm{d}m}{\mathrm{d}t}\mathbf{w}'$$

Form is invariant

$\dot{\mathbf{v}} = \dot{\mathbf{v}}'$

On the other hand…

the, particular form

$$\mathbf{F} = \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v})$$

Depends on the choice of the inertial frame:

$$\mathbf{F} = \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}) = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \frac{\mathrm{d}m}{\mathrm{d}t}\mathbf{v} = m\frac{\mathrm{d}\mathbf{v}'}{\mathrm{d}t} + \frac{\mathrm{d}m}{\mathrm{d}t}(\mathbf{v}' + \mathbf{v}_{ref}) = \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}') + \frac{\mathrm{d}m}{\mathrm{d}t}\mathbf{v}_{ref}$$

Except when $\frac{\mathrm{d}m}{\mathrm{d}t} \equiv 0$

Despite all these

- Even though most text books, either at undergraduate or graduate level, mention variable mass systems, there are not so many of them presenting comprehensive and properly didactic treatments of Newton’s second law.
- Examples which do give proper treatments are: Inglis (1951), Targ (1976), Starjinski (1986), José and Saletan (1998).
- In many other good undergraduate and graduate texts, either the problem is simply not addressed (some times just mentioned) or is treated only when dealing with the ‘rocket problem’. Meriam and Kraige (1987) or Boresi and Schmidt (1954) are examples of this last approach.
- Worse, there are even some classics that give wrong treatments to the problem, stating Eq. (1) as generally valid for a single varying mass particle, with no further consideration; see, e.g., Goldstein (1950, 1981), chapter 1, Singe and Griffith (1959), chapter 12.
- The reasons for this are not clear, but certainly influenced the surprising debate occurred among American educators in the 1950’s and 60’s.

Example of course in Brazil

USP — Sistema Administrativo da Pós-Graduação — Janus

Login

Usuário

Senha

Entrar

Apresentação
- Apresentação

Acesso
- Esqueci a senha
- Primeiro acesso

Acesso público
- Período de matrícula
- Disciplinas oferecidas
- Catálogo de disciplinas
- Orientadores

- Egressos USP

Disciplinas oferecidas

Disciplina PME5010-5
Mecânica Analítica

Área de Concentração: 3152

Criação: 08/12/2008

Ativação: 08/12/2008

Nr. de Créditos: 8

Carga Horária:

Teórica (por semana)	Prática (por semana)	Estudos (por semana)	Duração	Total
3	0	7	12 semanas	120 horas

Docentes Responsáveis:

Celso Pupo Pesce

Clovis de Arruda Martins

Objetivos:

Aprofundar conceitos da Mecânica Clássica, sob a ótica da Mecânica Analítica, formando uma base teórica sólida; apresentar e discutir métodos de solução preparando o aluno para resolver problemas avançados da dinâmica.

Justificativa:

Uma formação conceitual sólida em Mecânica Analítica é desejável para todos aqueles que desenvolvem pesquisas em temas relacionados com a Dinâmica. Essa formação teórica não é por si só suficiente, mas deve ser aliada ao desenvolvimento da habilidade de aplicar os conhecimentos adquiridos na solução de problemas da engenharia.

Conteúdo:

Introdução. Graus de Liberdade. Coordenadas Generalizadas. Vínculos. Sistemas Holônomos. Princípio dos Trabalhos Virtuais. Princípio de d´Alembert. Princípio de Hamilton. Equações de Lagrange. Sistemas Não-Holônomos. Multiplicadores de Lagrange. Sistemas Dissipativos e Sistemas com Variação de Massa. Função de Dissipação de Rayleigh. Leis de Conservação. Método de Routh. Equações de Hamilton. Formulação Lagrangeana da Dinâmica do Contínuo.

Forma de Avaliação:

Exercícios, Provas e Trabalho Final.

Observação:

Bibliografia:

Lanczos, C., The Variational Principles of Mechanics, Dover, 1986.
Goldstein, H., Poole, C.P., Safko, J.L., Classical Mechanics, Addison-Wesley, 3r
Meirovitch, L., Methods of Analytical Dynamics, McGraw-Hill, 1988.
Dugas, R., A History of Mechanics, Dover, 1988.
José, J.V., Saletan, E.J., Classical Dynamics: a Contemporary Approach, Cambridge University Press, 1998, reprinted 2002.
Arnold, V.I., Weistein, A. Vogtmann, Mathematical Methods of Classical Mechanics, Springer, 1989.

POLI

Example of course in Brazil

ANO BASE: 2006

PROGRAMA: 31005012012P-1 Engenharia Mecânica - PUC-RIO

DISCIPLINA	Sigla-Número	Nível	Carga Horária			Créditos
			M	D	F	
Mecânica Clássica	MEC-2101	Mestrado/Doutorado	45	45		3

Obrigatória nas Áreas de Concentração

Mecânica Aplicada

Período: 1º Semestre **Carga-Horária:** 45 **Créditos:** 3

Sub-Título:

Docentes	**Categoria**		**Carga Horária**	**%**
Rubens Sampaio Filho	Docente	Permanente	45	100,00
Nº de Docentes: 1			45	100,00

Ementa:

Mecânica newtoniana aplicada a partículas, sistemas de partículas e sistemas de massa variável com ênfase em referências móveis. Formulação de Lagrange e aplicações. Aplicações do cálculo das variações. Princípios de Hamilton e equações de Hamilton. Cinemática e dinâmica dos corpos rígidos e aplicações. Introdução à teoria geométrica e estabilidade de sistemas autônomos.

Bibliografia:

Principles of Dynamics, Greenwood, D.T., Prentice-Hall, 1965; Methods of Analytical Dynamics, Meirovitch, L., McGraw-Hill, 1970.

PUC - RJ

Specific text book

Cveticanin, Livija

Dynamics of Machines with Variable Mass

Gordon and Breach Science Publishers. Series of Books and Monographs in Stability and Control Theory, Methods and Applications, 1998, 236 p.

Much subtler: Lagrange Equation

Recall the usual invariant mass form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

Kinetic Energy

or

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{nc}$$

Lagrangean

Non-conservative generalized forces

Lagrange Equations for Variable Mass Systems

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = \hat{Q}_j$$

Mass with time !

$$m_i = m_i(t) \qquad \hat{Q}_j = \sum_i \left(\mathbf{F}_i + \dot{m}_i \mathbf{u}_i\right) \cdot \frac{\partial P_i}{\partial q_j}$$

Mass with position !!

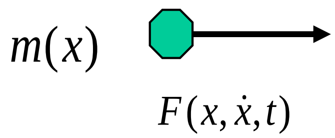
$$m_i = m_i(q_j; t) \qquad \hat{Q}_j = \sum_i \left(\mathbf{F}_i + \dot{m}_i \mathbf{u}_i\right) \cdot \frac{\partial P_i}{\partial q_j} - \sum_i \frac{1}{2} \frac{\partial m_i}{\partial q_j} (\mathbf{v}_i)^2$$

$$m_i = m_i(q_j, \dot{q}_j; t) \qquad \hat{Q}_j = \sum_i \left(\mathbf{F}_i + \dot{m}_i \mathbf{u}_i\right) \cdot \frac{\partial P_i}{\partial q_j} + \sum_i \left\{ \frac{1}{2} \frac{d}{dt}\left(\frac{\partial m_i}{\partial \dot{q}_j} (\mathbf{v}_i)^2 \right) - \frac{1}{2} \frac{\partial m_i}{\partial q_j} (\mathbf{v}_i)^2 \right\}$$

Mass with velocity !!!

Example: the simplest problem

particle loosing (gaining) mass, at null velocity, but explicitly with position



Newton: $m'(x)\dot{x}^2 + m(x)\ddot{x} = F(x,\dot{x},t)$ Correct

Usual Lagrange: $\frac{1}{2}m'(x)\dot{x}^2 + m(x)\ddot{x} = F(x,\dot{x},t)$ Incorrect

Correct: $d\left(\partial T/\partial \dot{x}\right)/dt - \partial T/\partial x = F(x,\dot{x},t) - m'(x)\dot{x}^2/2$ Missing term

Apparently Paradoxal Problems :

Falling chain problems:

Buquoy version;

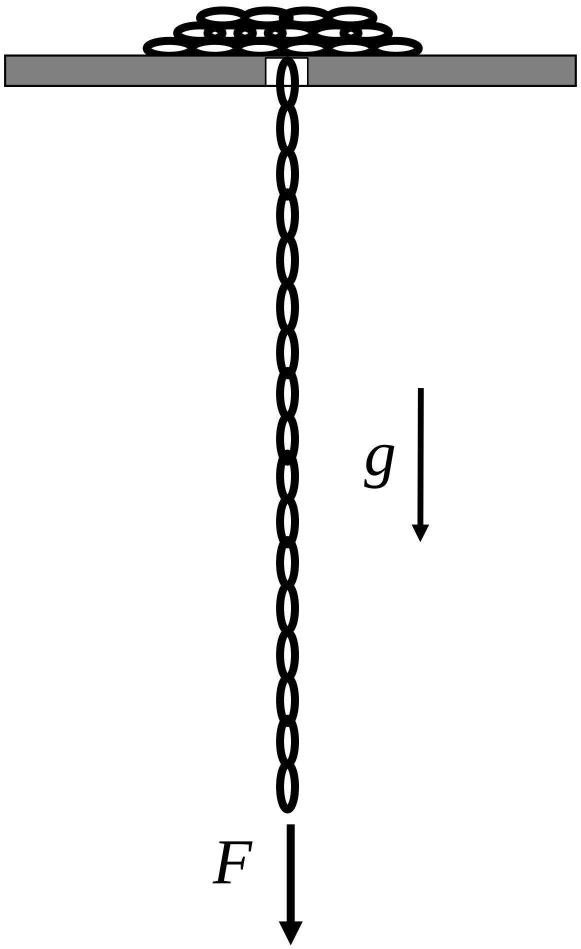
Cayley version;

‘U’ falling chain;

Vertical collapse of buildings

Classical problem

The falling chain of Cayley



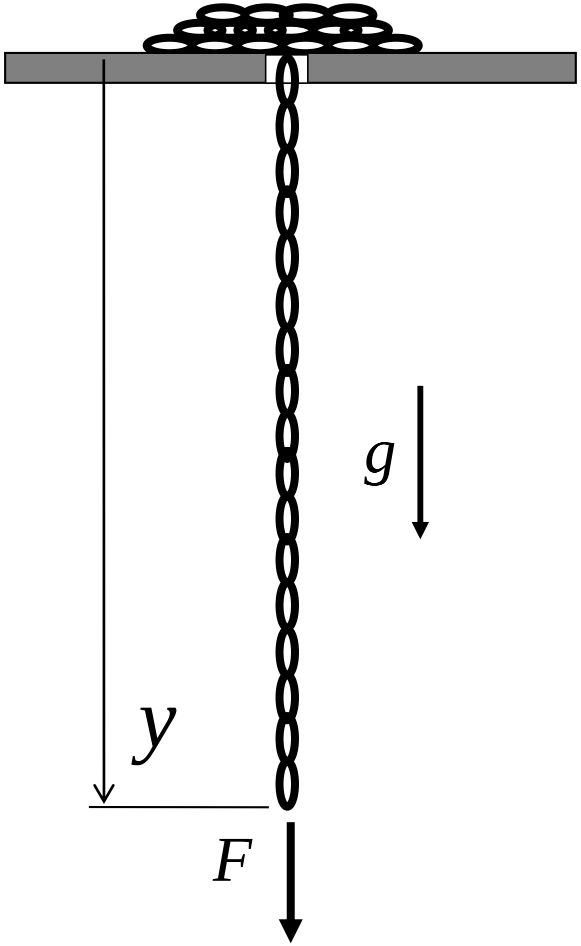
Cayley's 'falling' chain

- Classic idealized problem treated by Cayley, in 1857, similar to Buquoy's.

- Ever since, matter of controversies regarding proper formulation if treated under the Lagrangean approach.

- Recent account, see:

  Grewal, Johnson and Ruina, "A Chain that speeds up, rather than slows, due to collisions: how compression can cause tension", *Am. J. Phys*., 79(7), 723-729, 2011.

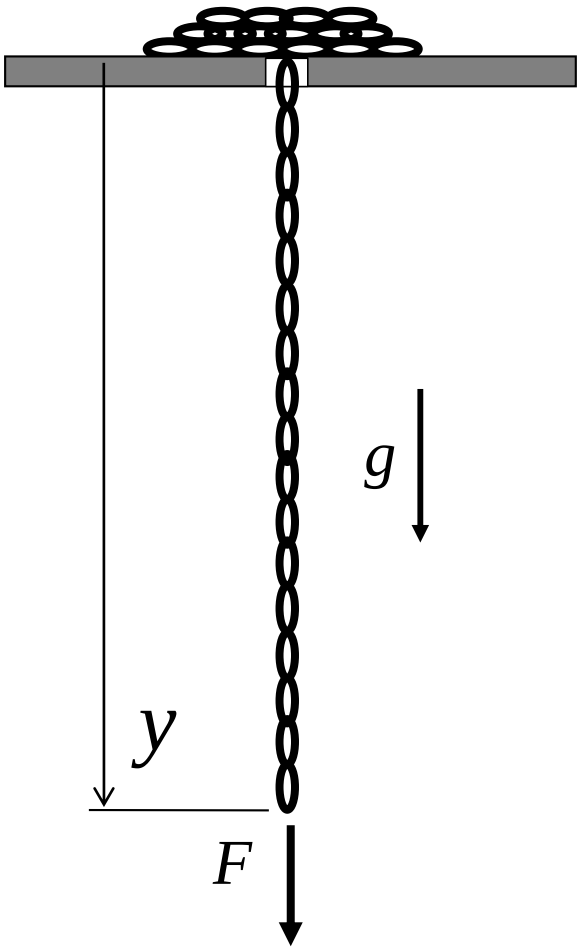


Cayley's 'falling' chain

- Falling (or suspended part) treatable as a position dependent variable mass system.

- Classic **idealized** hypotheses (*):

  a. Falling (suspended) part of the chain treated as a continual vertically moving 'rigid' body (pure translatory motion);
  b. There is no friction force apllied either by the table on slidding links, or by one to each other, or even by the hole internal surface to the leaving link;
  c. Existence of a sudden acceleration (velocity jumps from zero) as the links leave the chain pile, *being the transfer of any angular momentum to linear momentum disregarded*;
  d. Decreasing thickness of chain pile ignored.



(*) Discussion on hypotheses (a), (b) and (c) and other points may be encountered in Grewal, Johnson and Ruina, 2011.

Cayley's 'falling' chain

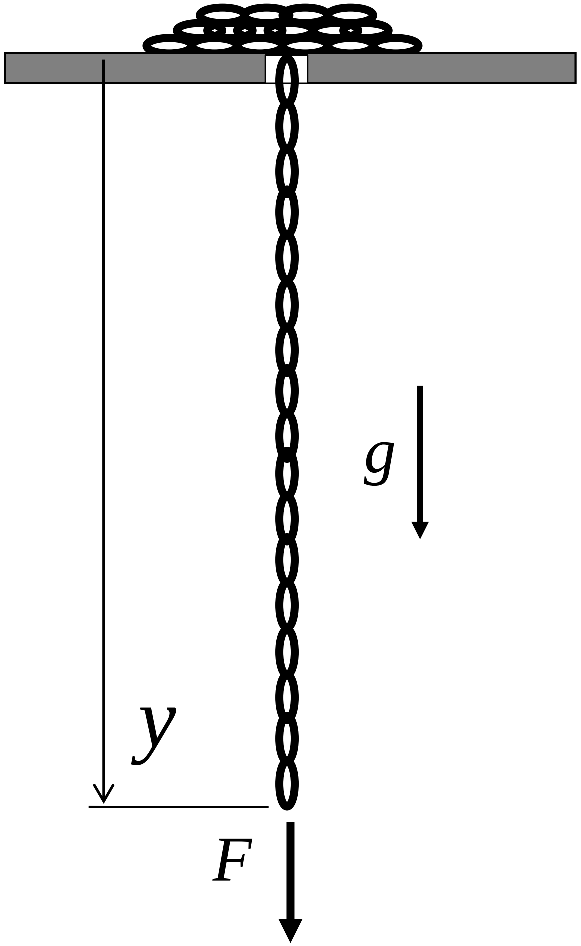
$m = \mu L$ Chain total mass

$m_S = \mu y$ Falling (suspended) mass

$\dot{m}_S = \mu \dot{y}$ Falling mass time rate

$\frac{\partial m_S}{\partial y} = \mu$ Linear density

$\mathrm{q} = \mu \dot{y}^2$ Flux of momentum at the pile



Cayley's 'falling' chain

$$T = T_S = \int_0^y \frac{1}{2}\mu v(\psi)^2 d\psi = \int_0^z \frac{1}{2}\mu \dot{z}^2 d\psi =$$

$$= \frac{1}{2}\mu y \dot{y}^2 = \frac{1}{2} m_S \dot{y}^2$$

Kinetic energy

$$V = V_S = -\int_0^y \mu g \psi d\psi =$$

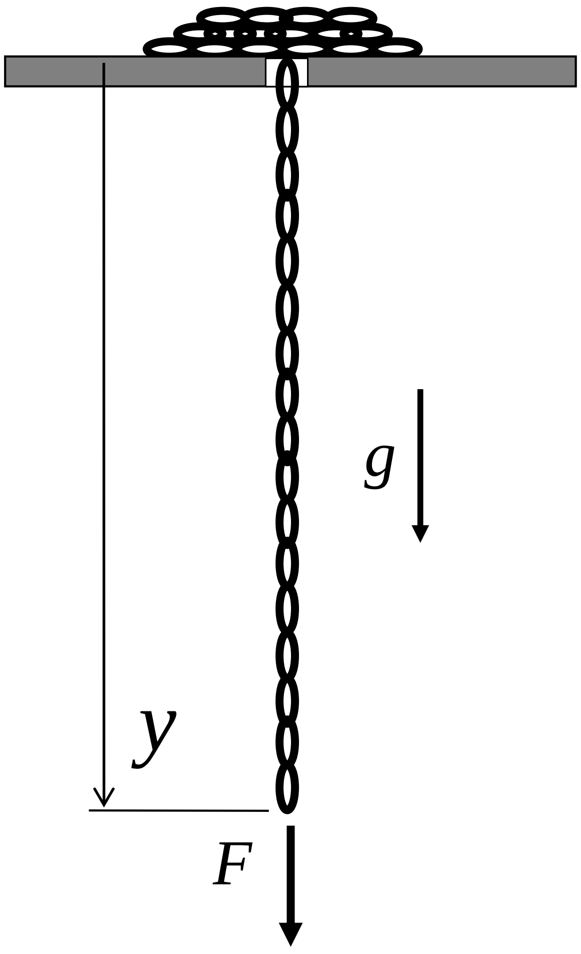
$$= -\frac{1}{2}\mu g y^2 = -\frac{1}{2} m_S g y$$

Potential energy

$$L = \left(T - V\right) = L_S = \left(T_S - V_S\right) =$$

$$= \frac{1}{2}\mu y(\dot{y}^2 + gy) = \frac{1}{2} m_S (\dot{y}^2 + gy)$$

Lagrangean



Cayley's 'falling' chain

$$\frac{d}{dt}\left(\frac{\partial T_S}{\partial \dot{y}}\right)-\frac{\partial T_S}{\partial y}=\hat{Q}_y$$

Extended Lagrange equation

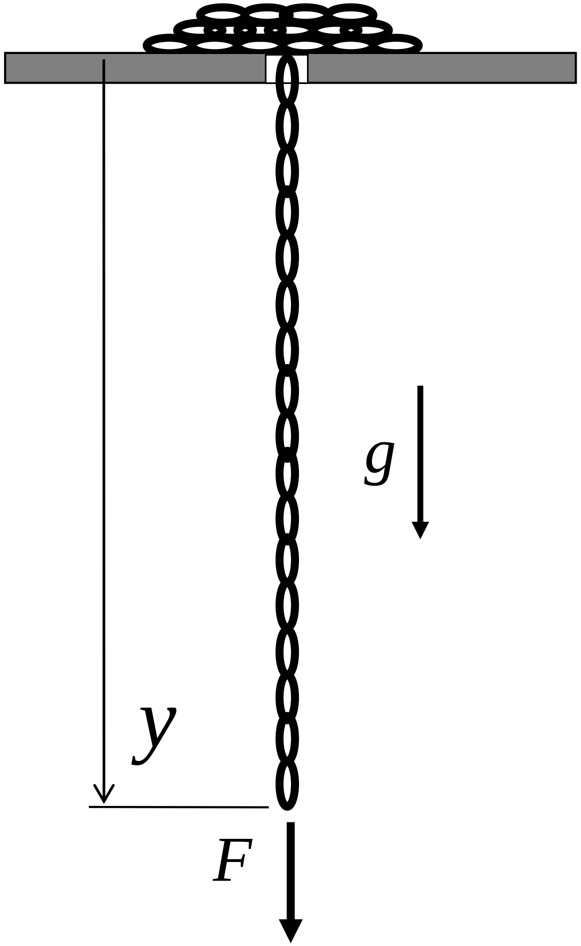
$$\hat{Q}_y=m_S g+F+\dot{m}_S \cancel{w}-\tfrac{1}{2}\frac{\partial m_S}{\partial y}\dot{y}^2$$

or

$$\frac{d}{dt}\left(\frac{\partial L_S}{\partial \dot{y}}\right)-\frac{\partial L_S}{\partial y}=\hat{Q}_y^{nc}$$

In terms of the Lagrangean

$$\hat{Q}_y^{nc}=F+\dot{m}_S w-\tfrac{1}{2}\frac{\partial m_S}{\partial y}\dot{y}^2$$



Recall: $w=0$

Cayley's 'falling' chain

$$\frac{\partial T_S}{\partial y} = \frac{1}{2}\mu\dot{y}^2$$

$$\frac{\partial T_S}{\partial \dot{y}} = \mu y\dot{y} = m_S\,\dot{y}$$

Derivatives

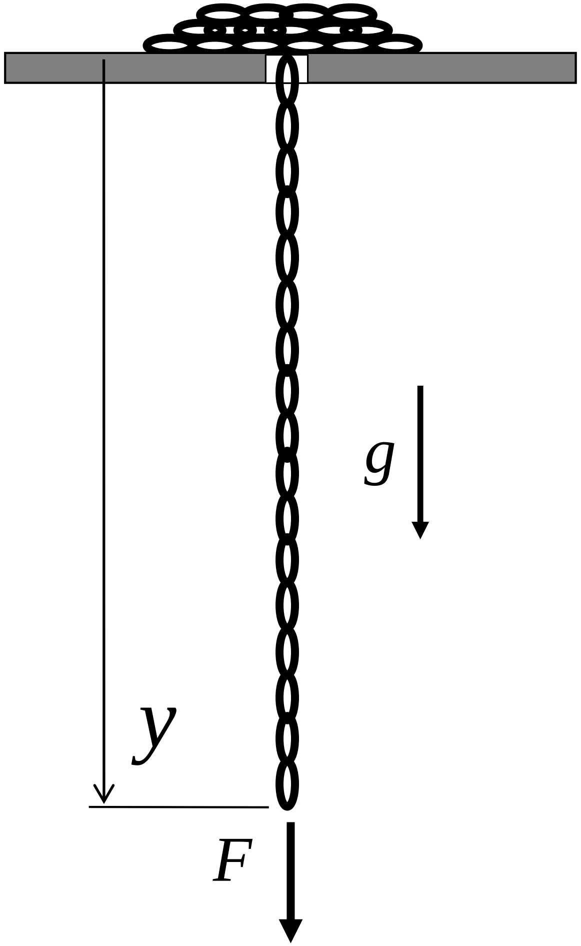
$$\frac{d}{dt}\left(\frac{\partial T_S}{\partial \dot{y}}\right) = \mu y\ddot{y} + \mu\dot{y}^2 = \mu y\ddot{y} + \mathrm{q}$$

Extended Lagrange equation leads to:

$$\mu y\ddot{y} + \mu\dot{y}^2 - \frac{1}{2}\mu\dot{y}^2 = \mu y g + F - \frac{1}{2}\mu\dot{y}^2$$

Finally:

$$\ddot{y} + \frac{\dot{y}^2}{y} = g + \frac{F}{\mu y}$$



Cayley's 'falling' chain

$$y^* = \frac{y}{L}$$

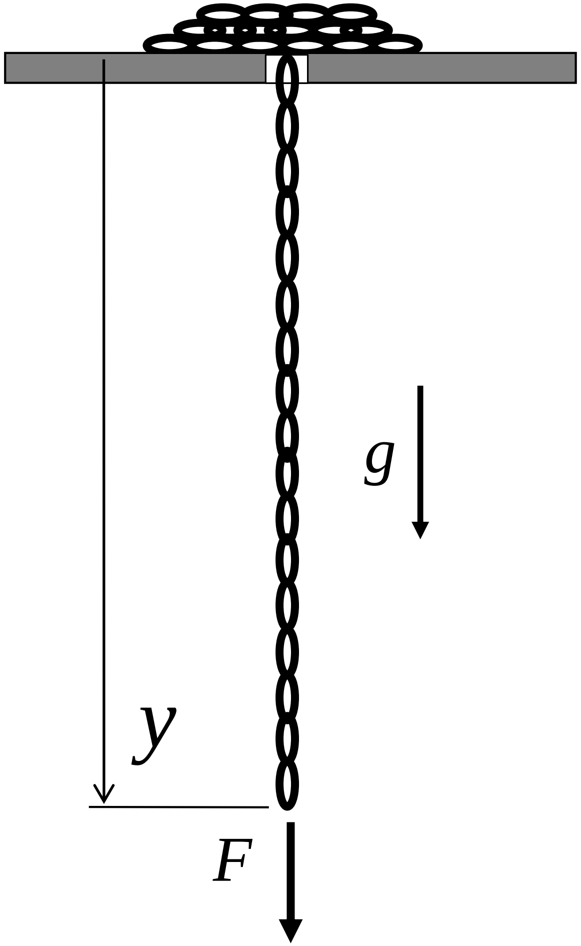
$$t^* = t\sqrt{g/L}$$

$$\dot{y}^* = \frac{\dot{y}}{\sqrt{gL}}$$

$$\ddot{y}^* = \frac{\ddot{y}}{g}$$

$$\Phi = \frac{F}{\mu g L} = \frac{F}{mg}$$

Nondimensional variables



Get:

$$\ddot{y}^* = 1 + \frac{\Phi}{y^*} - \frac{\dot{y}^{*2}}{y^*}$$

Cayley's 'falling' chain

$$\frac{d}{dt}\left(\frac{\partial T_S}{\partial \dot{y}}\right) - \frac{\partial T_S}{\partial y} = \hat{Q}_y$$

Usual Lagrange equation

$$\hat{Q}_y = m_S g + F + \dot{m}_S \cancel{w}$$

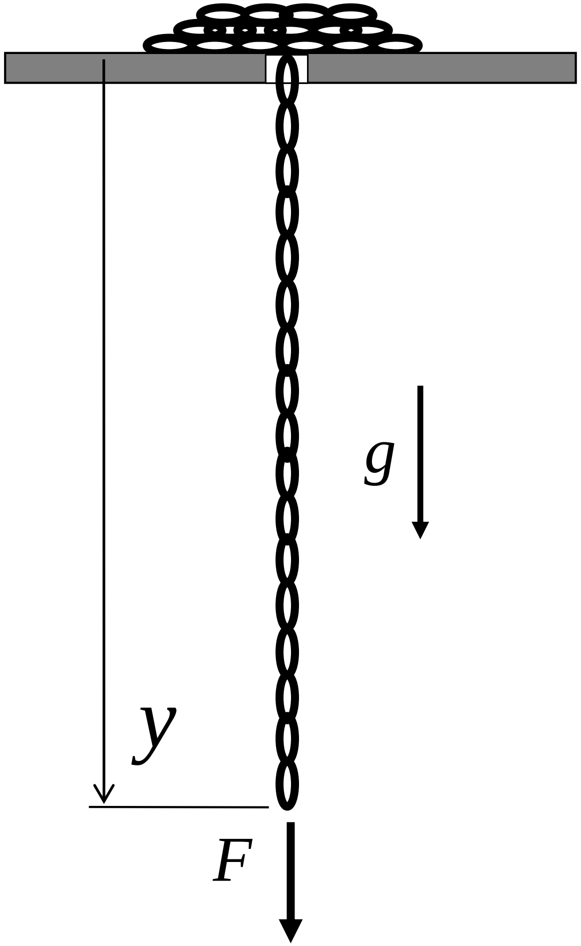
or

$$\frac{d}{dt}\left(\frac{\partial L_S}{\partial \dot{y}}\right) - \frac{\partial L_S}{\partial y} = \hat{Q}_y^{nc}$$

In terms of the Lagrangean

$$\hat{Q}_y^{nc} = F + \dot{m}_S \cancel{w}$$

Recall: $w = 0$



Cayley's 'falling' chain

Get:

$$\mu y\ddot{y} + \mu\dot{y}^2 - \tfrac{1}{2}\mu\dot{y}^2 = \mu y g + F$$

i.e.:

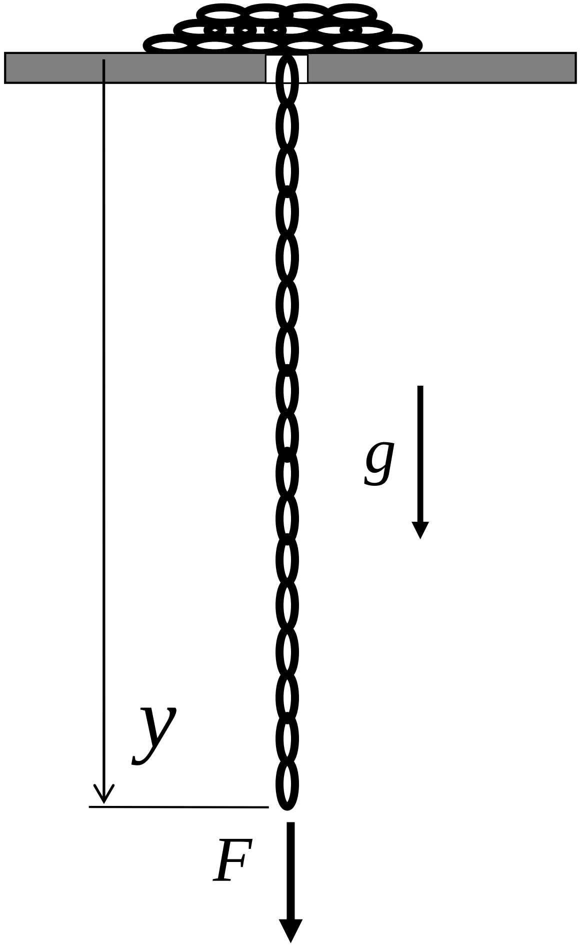
$$\mu y\ddot{y} + \tfrac{1}{2}\mu\dot{y}^2 = \mu y g + F$$

Such that:

$$\ddot{y} = g + \frac{F}{\mu y} - \frac{1}{2}\frac{\dot{y}^2}{y}$$

Or, in nondimensional form:

$$\ddot{y}^* = 1 + \frac{\Phi}{y^*} - \frac{1}{2}\frac{\dot{y}^{*2}}{y^*}$$



Cayley's 'falling' chain

When F=0 (Φ=0), both equations

OK!
Cayley's
result

$$\ddot{y}^* = 1 + \frac{\Phi}{y^*} - \frac{\dot{y}^{*2}}{y^*}$$

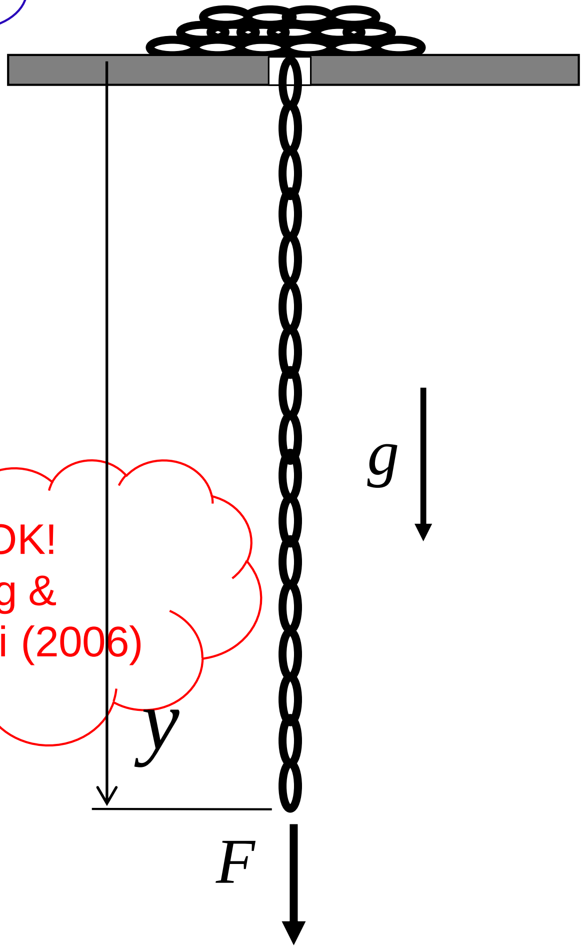
From the extended Lagrange equation

and

$$\ddot{y}^* = 1 + \frac{\Phi}{y^*} - \frac{1}{2}\frac{\dot{y}^{*2}}{y^*}$$

From the invariant mass Lagrange equation

Not OK!
Wong &
Yasui (2006)

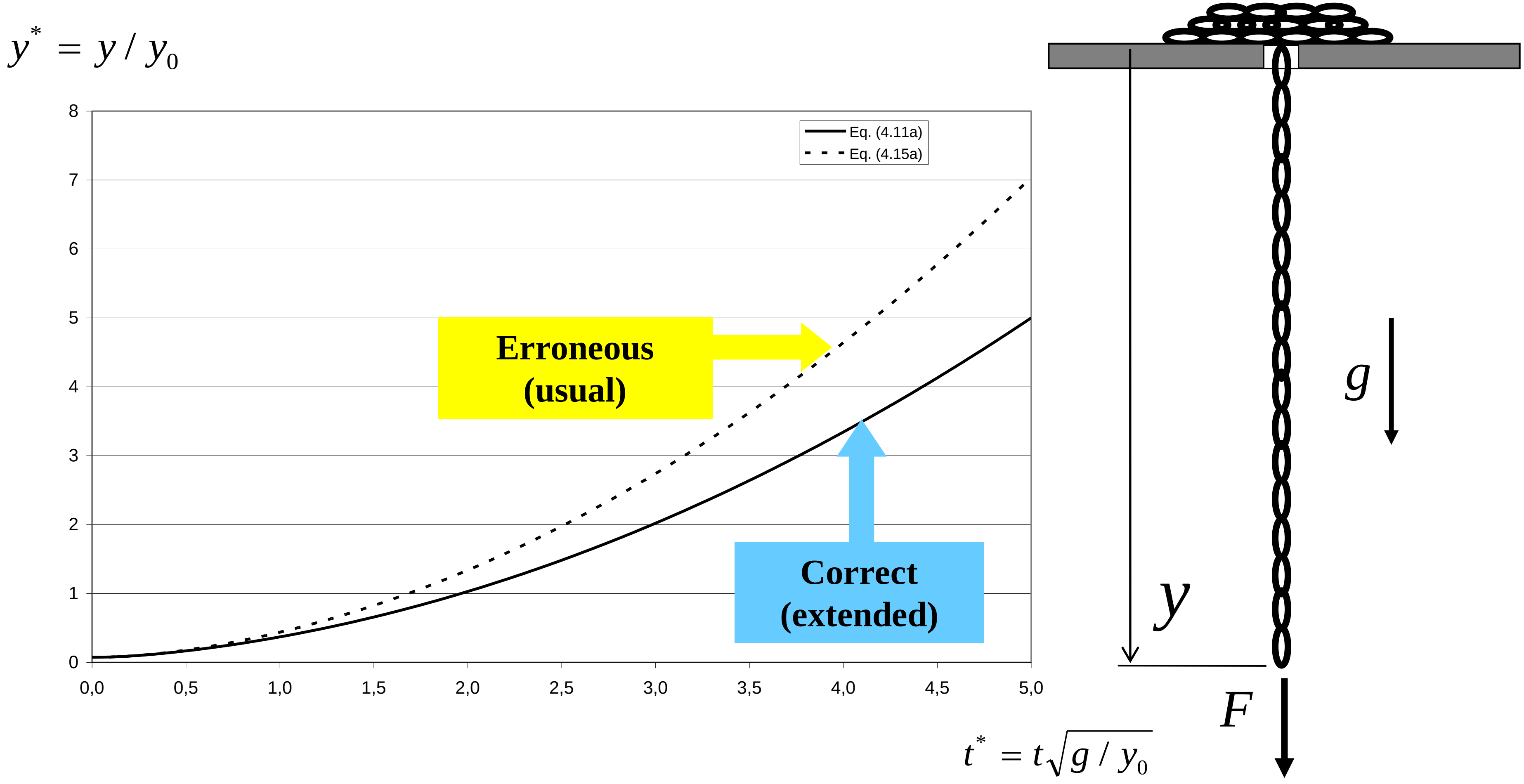


Both predict, from initial rest condition, a 'free-fall' with initial acceleration equal to gravity.

However acceleration decreases monotonically (so is smaller than gravity) tending to different asymptotic limits.

There is no singularity at y=0$^+$!!!

Cayley Problem

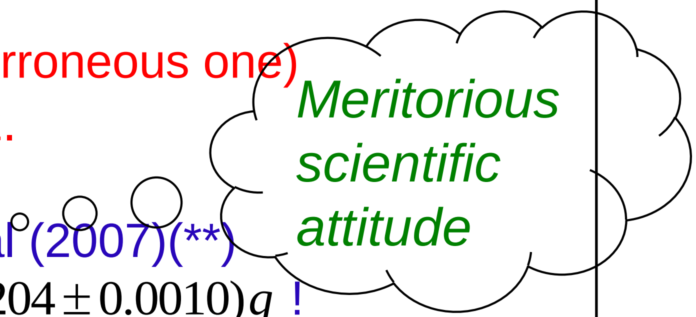


Cayley's 'falling' chain

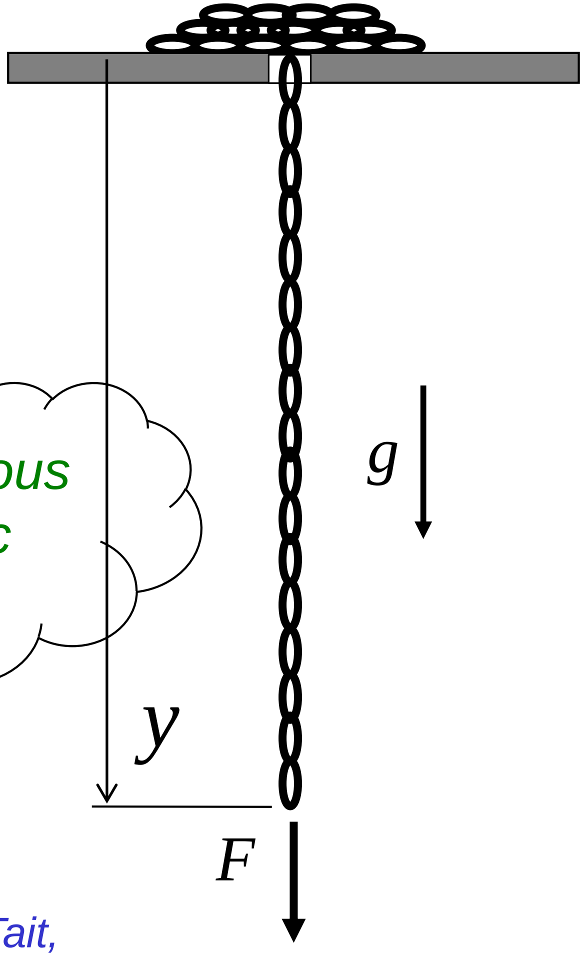
In this case, the puzzling aspect regarding distinct asymptotic limits, is related to the application of either form of the Lagrange equations, rather than to the validity of the idealized hypotheses.

Cayley's solution (***the proper one***) predicts a limit acceleration of $g/3$.

Wong and Yasui (2006)(*) (the erroneous one) predict a limit acceleration of $g/2$.



Experimental work by Wong et al (2007)(**) measured the limit as $\ddot{y}_{\lim} = (0.3204 \pm 0.0010)g$!



(*) Wong &Yasui, "Falling chains". American Journal of Physics, v. 6, 490-496, 2006.

(**) Wong, Youn &Yasui, "The falling chain of Hopkins, Tait, Steele and Cayley". European Journal of Physics, v. 28, 385-400, 2007.

Cayley's 'falling' chain

$$\frac{d}{dt}\left(\frac{\partial L_S}{\partial \dot{y}}\right) - \frac{\partial L_S}{\partial y} + \frac{\partial R}{\partial \dot{y}} = 0$$

Cayley's Lagrange equation, by introducing a special Rayleigh-like function,

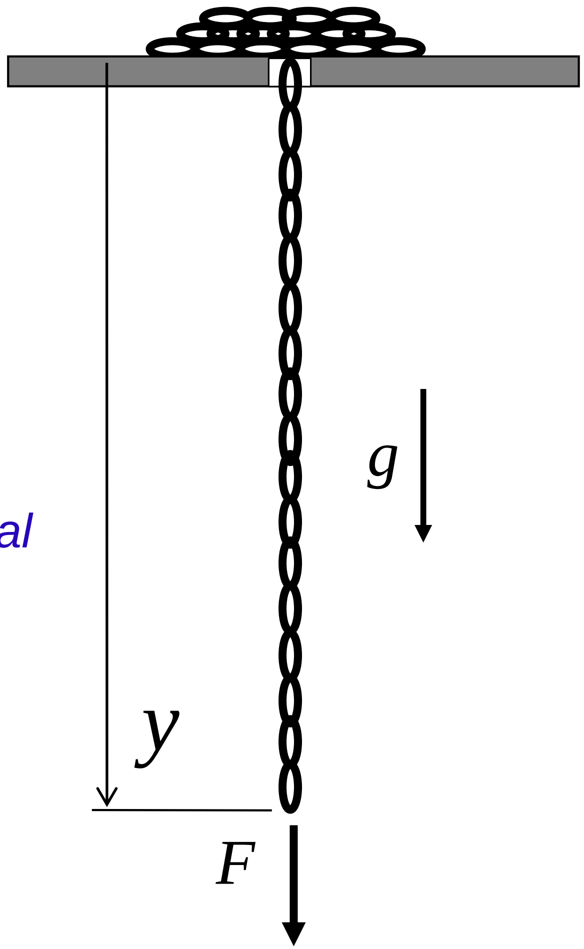
$$R(y,\dot{y}) = \tfrac{1}{2}\left(\tfrac{1}{3}\dot{m}_S \dot{y}^2\right) = \frac{1}{6}\mu\dot{y}^3$$

is equivalent to

$$\frac{d}{dt}\left(\frac{\partial L_S}{\partial \dot{y}}\right) - \frac{\partial L_S}{\partial y} = \hat{Q}_y^{nc}$$

obtained from the general extended Lagrange equation.

$$\hat{Q}_y^{nc} = \not{F} + \dot{m}_S \not{w} - \tfrac{1}{2}\frac{\partial m_S}{\partial y}\dot{y}^2$$



Frequently, the work by Cayley has been not appreciated as it should!

A Civil Engineering Application:

The vertically collapsing tower

Purpose

To highlight the discussion about a still open subject on a simple *single degree of freedom model* (SDOF), addressing a controversial point.

Based on a recently published paper:

Pesce, C.P., Casetta, L., Santos, F.M., 2012, "The equation of motion governing the dynamics of vertically collapsing buildings", http://dx.doi.org/10.1061/(ASCE)EM.1943-7889.0000453

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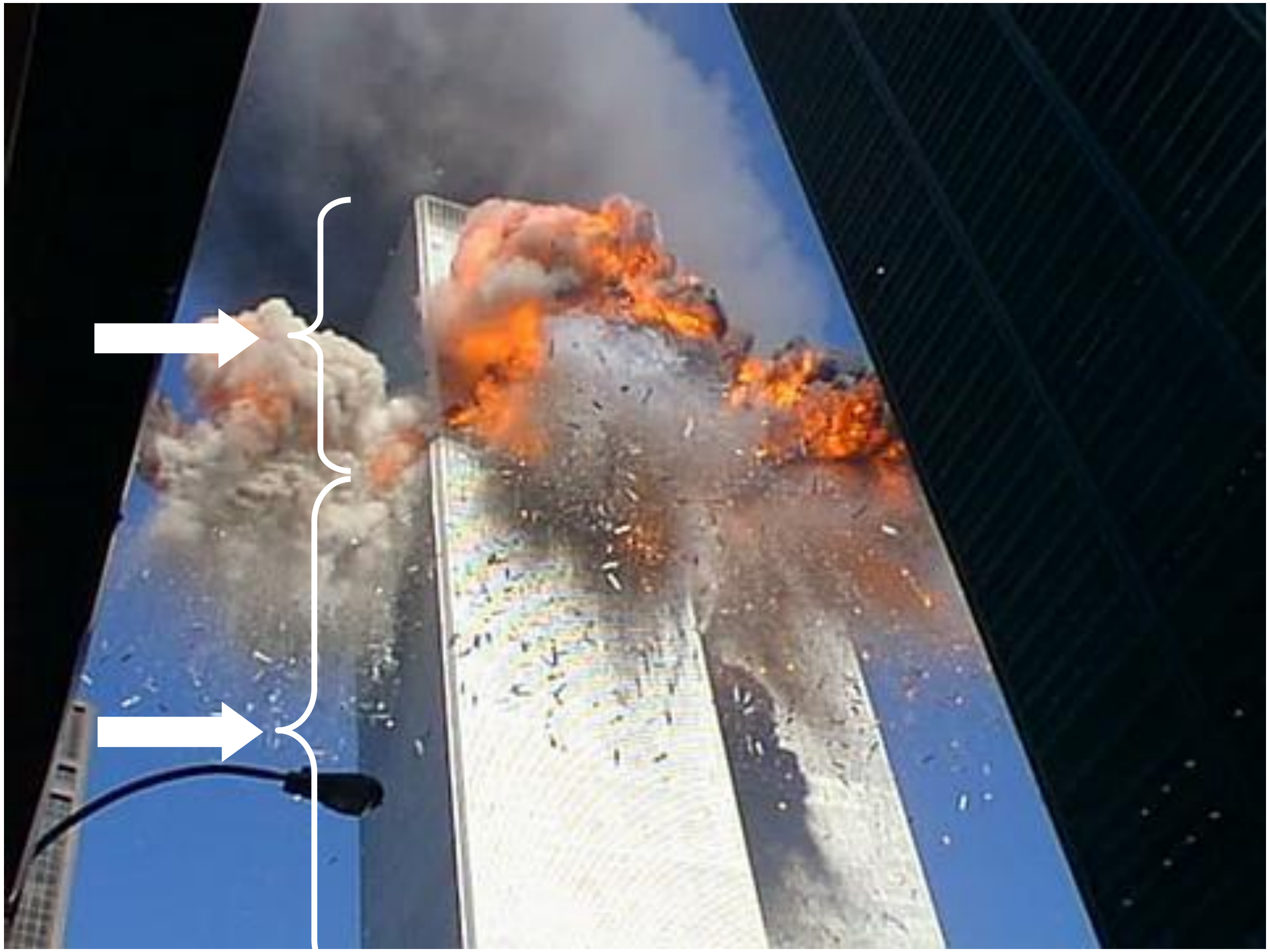
Striking Problem

Vertical collapse of buildings: WTC twin towers

Moving region

Region at rest



Possible approaches

- *Newtonian mechanics*
- *Lagrangian mechanics*
- *Generalized Reynolds' Transport Theorem (McIver, 73, Irschick and Holl, 2004)*
- *'Mass transfer' wave equation with moving boundaries (Bevilacqua, DINAME2011)*
- *Other...*

Motivation:

Can a simple SDOF model represent the dynamics of a vertically collapsing tower?

YES!

Bažant, Z. P., Verdure, M., 2007, "Mechanics of progressive collapse: learning from World Trade Center and building demolitions". Journal of Engineering Mechanics, v. 133, n. 3, pp. 308-19.

Seffen, K. A., 2008, "Progressive collapse of the World Trade Center: simple analysis". Journal of Engineering Mechanics, v. 134, n. 2, p. 125-32.

Motivation:

Such model is able to describe the evolution of the avalanche front of vertically collapsing towers.

However:

The equation of motion derived from the usual Lagrange equation formalism differs from that derived from Newton's law.

An apparent paradox !

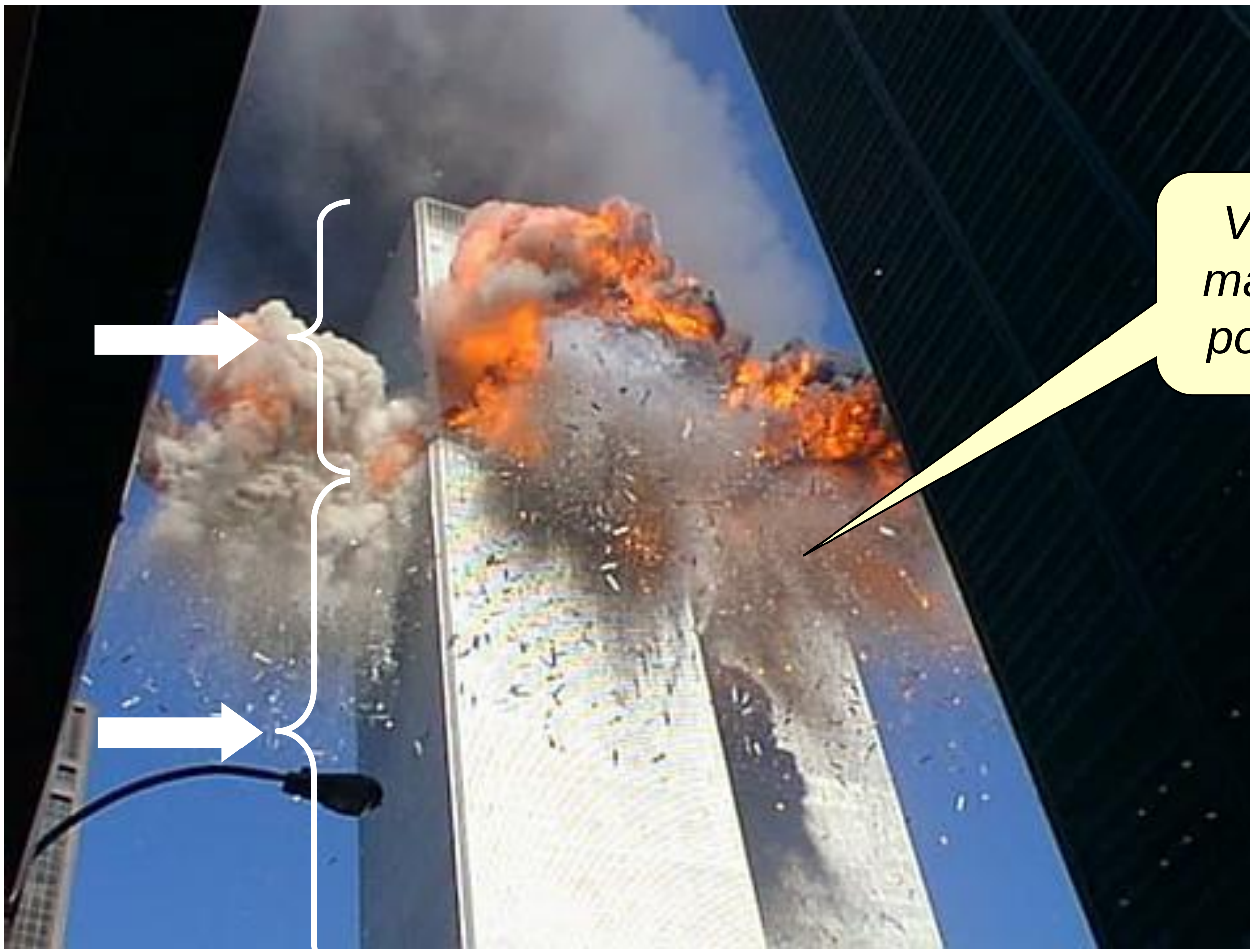
Similar to the *falling chain problem* and likewise controversial.

Neither Bažant & Verdure, or Seffen are conclusive on **which one should be the proper equation!**

The Vertical Collapse

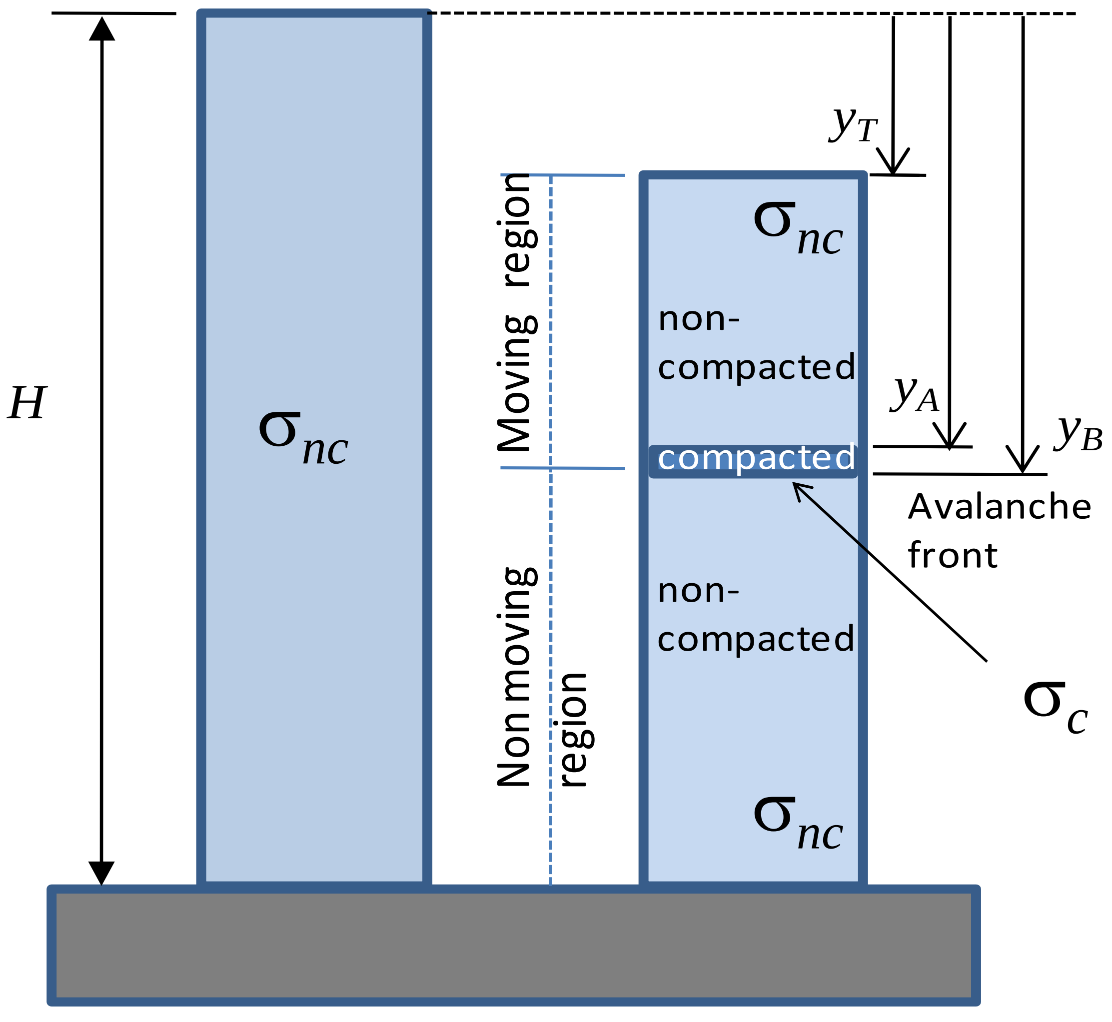
Moving region

Region at rest



Variable mass with position !!

Simple Model



Recall the Extended Lagrange Equations

Most complete case: $m_i = m_i(q_j; \dot{q}_j; t)$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = \hat{Q}_j \; ; \qquad j = 1 \qquad M$$

Mass depending explicitly on time

Extended generalized force

$$\hat{Q}_j = \sum_i \left(\mathbf{f}_i + \dot{m}_i \mathbf{w}_i\right) \cdot \frac{\partial P_i}{\partial q_j} +$$

Two new terms

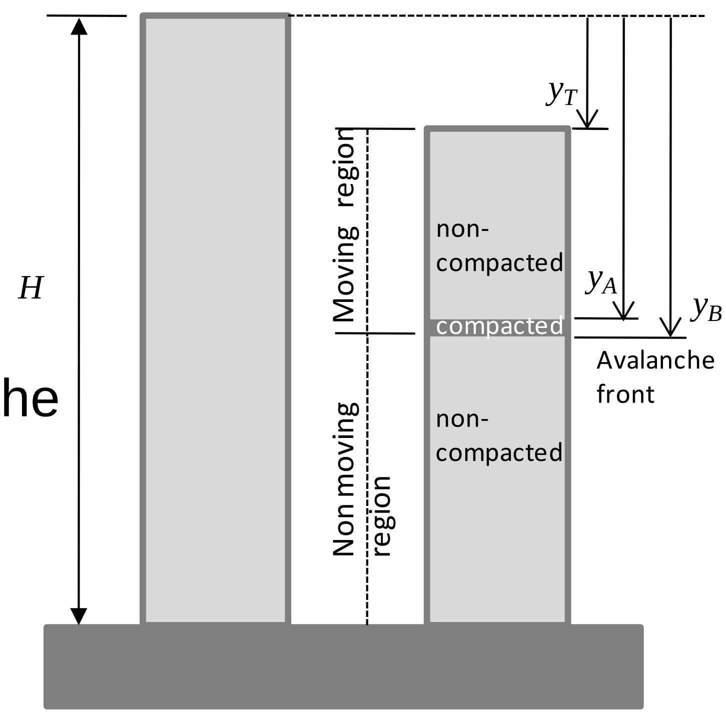
$$+\sum_i \left\{ \frac{1}{2}\frac{d}{dt}\left(\frac{\partial m_i}{\partial \dot{q}_j} (\mathbf{v}_i)^2 \right) - \frac{1}{2}\frac{\partial m_i}{\partial q_j}(\mathbf{v}_i)^2 \right\}$$

Mass depending explicitly on velocity

Mass depending explicitly on position

Simple Model

- The collapsing tower is divided in two distinct regions:
  - *the falling region;*
  - *the still region.*



- The *still* (intact) *region* transfers mass to the *falling region:*
  - the mass of the *falling region* increases;
  - the mass of the *still region* decreases.

- The *falling region* is divided in two parts:
  - The intact (*non compacted*) part;
  - The smashed (*compacted)* part.

Simplest SDOF Model

- Three Major Hypotheses:

1. ***the 'intact' upper part of the falling structure is a rigid body***, translating vertically and smashing the 'lower' part as it falls;

2. ***there is a density jump through the avalanche front***; i.e., the density of the accreted mass jumps from a 'non compacted' value to a 'compacted' value, in a continuous impacting ma[illegible]

$\sigma_{nc} = \sigma_0$

$\sigma_c > \sigma_0$

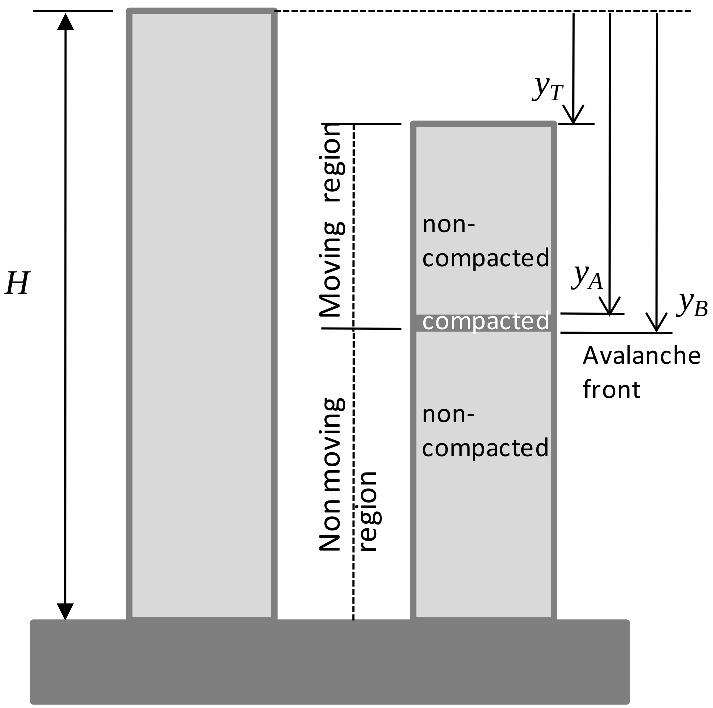
3. ***the wh[illegible] region***, composed by the 'intact' rigid falling part accreted by the instantaneously compacted part, ***translates as a rigid material system*** with mass varying explicitly with position.

Velocity jump!

Simplest SDOF Model

- Therefore:

1. both regions are material systems with varying mass;
2. A single generalized coordinate may represent the collapsing dynamics;
3. ***The varying masses may be expressed as explicit functions of the chosen coordinate*.**



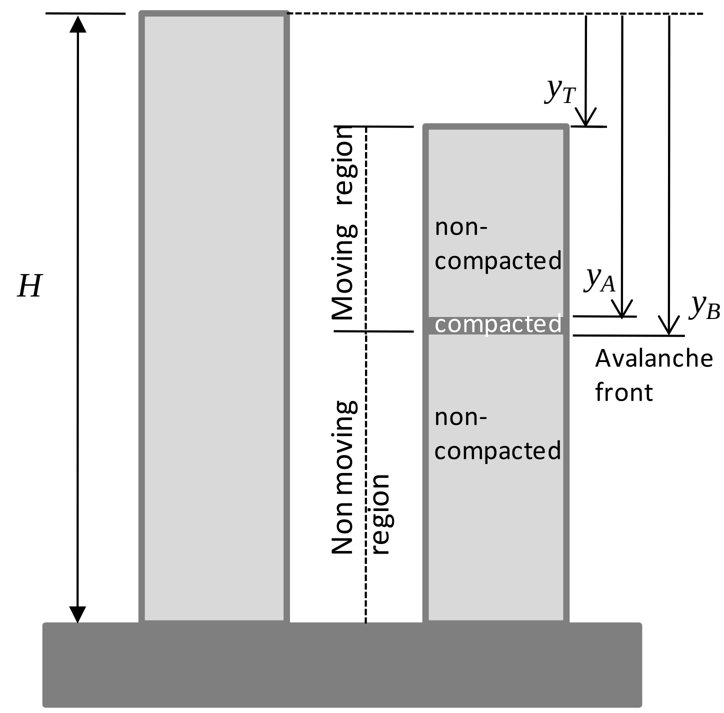
Simplest SDOF Model

Mass of the falling region

$$m_{mov} = \int_{y_T}^{y_A^-} \sigma_{nc} \mathrm{d}y + \int_{y_A}^{y_B^-} \sigma_c \mathrm{d}y = \sigma_{nc}(y_A - y_T) + \sigma_c(y_B - y_A)$$

Mass of the still region

$$m_{rest} = \int_{y_B}^{H} \sigma_{nc} \mathrm{d}y = \sigma_{nc}(H - y_B)$$

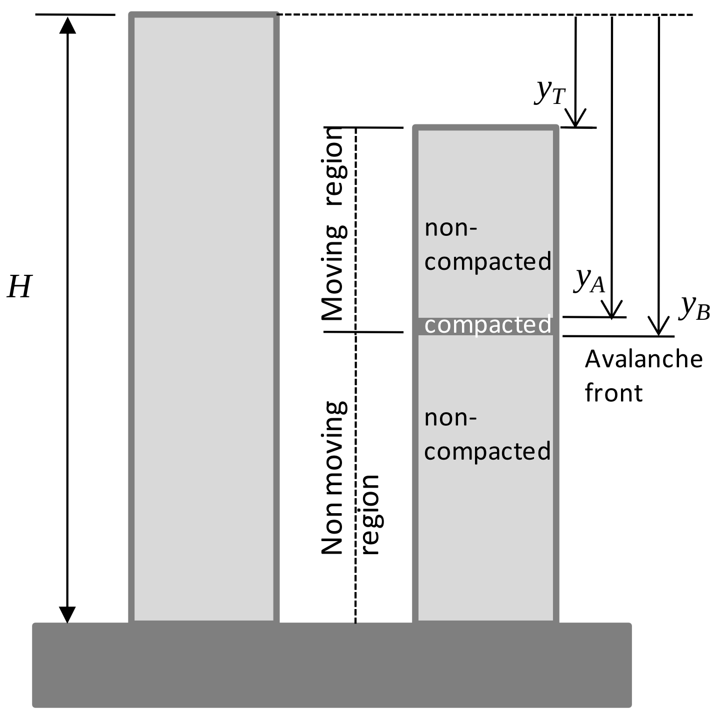


Simplest SDOF Model

Conservation of mass of the whole building

$$M = \sigma_{nc} H = m_{mov} + m_{rest}$$

$$\sigma_{nc}(y_A - y_T - y_B) + \sigma_c (y_B - y_A) = 0$$



$$\Rightarrow \quad m_{mov} = \sigma_{nc} y_B$$

Kinematic constrains: $h = (y_A - y_T) \Rightarrow$

$$\Rightarrow \quad \dot{y}_T = \dot{y}_A \quad \Rightarrow \quad \dot{y}_A = (1 - K)\dot{y}_B$$

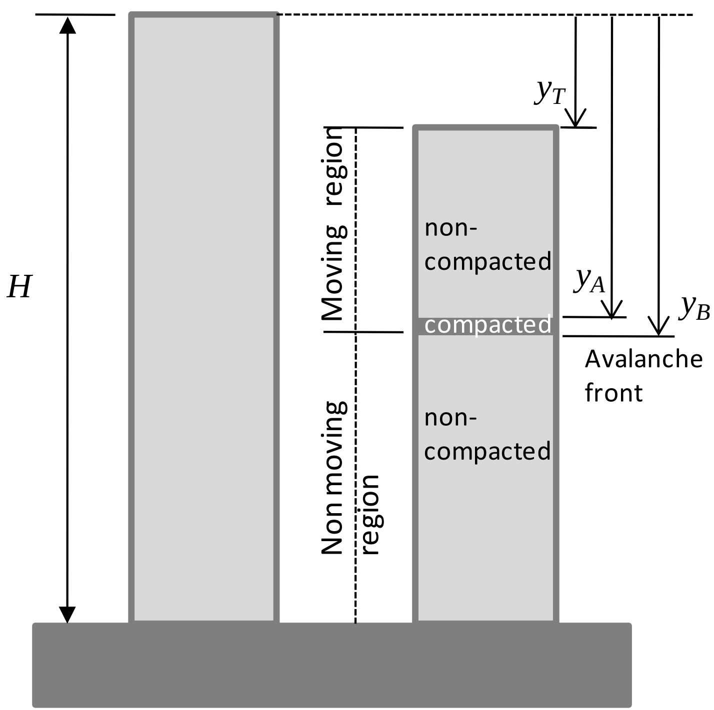
$$K = \sigma_{nc} / \sigma_c < 1$$

Compaction Factor

Simplest SDOF Model

Kinetic Energy

$$T_{mov} = \tfrac{1}{2} m_{mov} \dot{y}_A{}^2 = \tfrac{1}{2}\left[\sigma_{nc} y_B\right] \dot{y}_A{}^2 = \tfrac{1}{2}\sigma_{nc}\left[\frac{y_A - Kh}{(1-K)}\right]\dot{y}_A{}^2$$



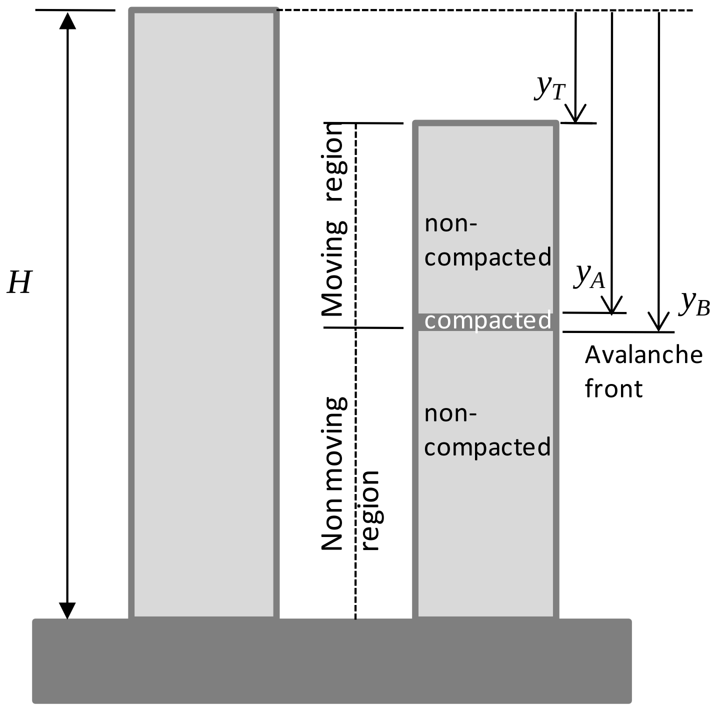
Or

$$T_{mov} = \tfrac{1}{2}\sigma_{nc}(1-K)^2 y_B \dot{y}_B{}^2$$

Simplest SDOF Model

Lagrange equation

Extended form for mass varying with position



$$\frac{d}{dt}\left(\frac{\partial T_{mov}}{\partial \dot{y}_A}\right) - \frac{\partial T_{mov}}{\partial y_A} = \hat{Q}_A$$

$$\hat{Q}_A = m_{mov} g - F - \tfrac{1}{2}\frac{\partial m_{mov}}{\partial y_A}\dot{y}_A^{\,2}$$

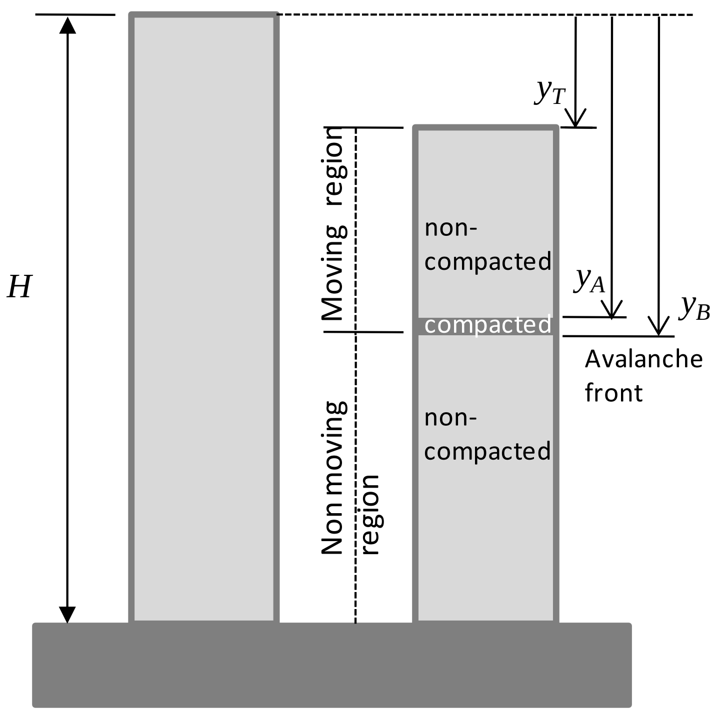
Dissipative term

Simplest SDOF Model

Actually

A Rayleigh-like function could be defined

$$R(y_A, \dot{y}_A) = \frac{1}{6}\dot{m}\,\dot{y}_A^{\;2} = \frac{1}{6}m'(y_A)\,\dot{y}_A^{\;3}$$



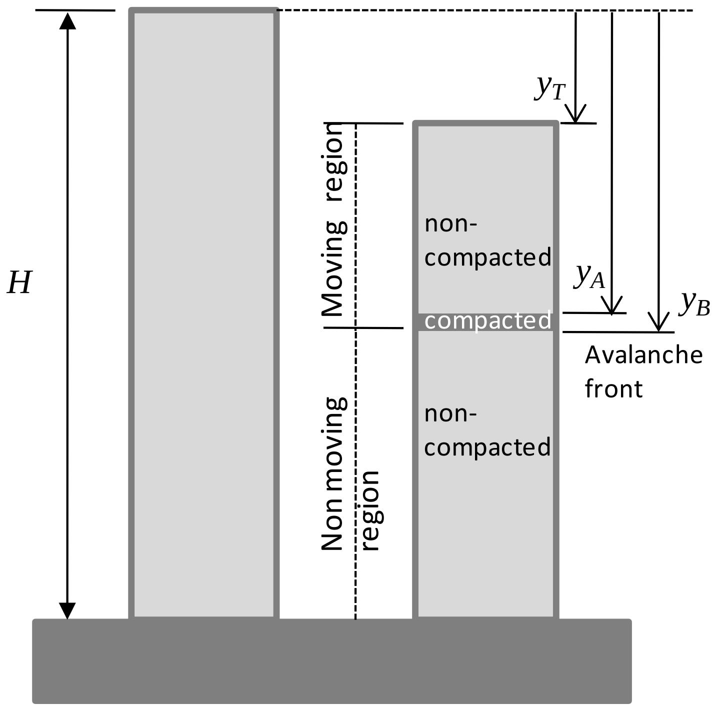
$$\frac{\partial R}{\partial \dot{y}_A} = \frac{1}{2}m'(y_A)\,\dot{y}_A^{\;2}$$

Dissipative term

Simplest SDOF Model

Lagrange equation

Extended Rayleighian form for mass varying with position



$$\frac{d}{dt}\left(\frac{\partial T_{mov}}{\partial \dot{y}_A}\right) - \frac{\partial T_{mov}}{\partial y_A} + \frac{\partial R}{\partial \dot{y}_A} = Q_A$$

$$Q_A = m_{mov} g - F$$

Dissipative term

Simplest SDOF Model

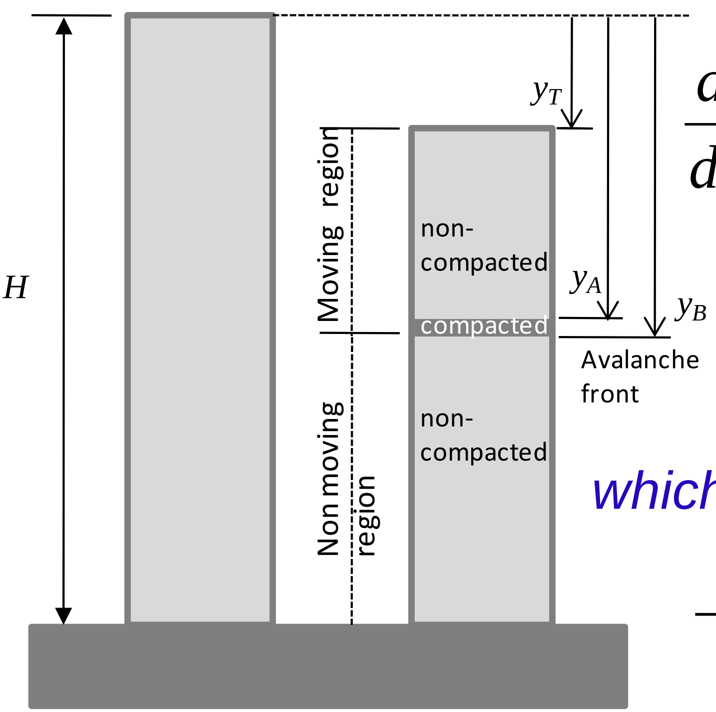
$$T_{mov} = \tfrac{1}{2} m_{mov} \dot{y}_A{}^2 = \tfrac{1}{2}\left[\sigma_{nc} y_B\right] \dot{y}_A{}^2 = \tfrac{1}{2}\sigma_{nc}\left[\frac{y_A - Kh}{(1-K)}\right]\dot{y}_A{}^2$$

Kinetic energy derivatives

$$\boxed{\dot{y}_A = (1-K)\dot{y}_B}$$

$$\frac{\partial T_{mov}}{\partial \dot{y}_A} = \sigma_{nc} y_B \dot{y}_A = (1-K)\sigma_{nc} y_B \dot{y}_B$$

$$\frac{d}{dt}\left(\frac{\partial T_{mov}}{\partial \dot{y}_A}\right) = \sigma_{nc}\frac{d}{dt}(y_B \dot{y}_A) = (1-K)\sigma_{nc}\frac{d}{dt}(y_B \dot{y}_B)$$



$$-\frac{\partial T_{mov}}{\partial y_A} = -\tfrac{1}{2}\sigma_{nc}\frac{\dot{y}_A{}^2}{1-K} = -\tfrac{1}{2}(1-K)\sigma_{nc}\dot{y}_B{}^2$$

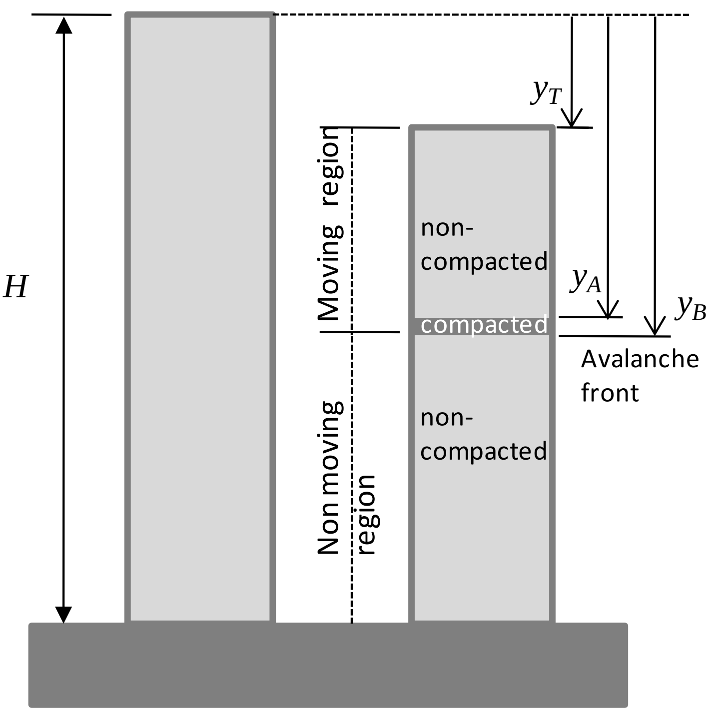
which cancels out exactly the position dependent term:

$$-\tfrac{1}{2}\frac{\partial m_{mov}}{\partial y_A}\dot{y}_A{}^2 = -\tfrac{1}{2}\frac{\sigma_{nc}}{1-K}\dot{y}_A{}^2 = -\tfrac{1}{2}(1-K)\sigma_{nc}\dot{y}_B{}^2$$

Simplest SDOF Model

Recall:

$$m_{mov} = \sigma_{nc} y_B$$



Then:

$$\frac{d}{dt}\left(\frac{\partial T_{mov}}{\partial \dot{y}_A}\right) - \frac{\partial T_{mov}}{\partial y_A} = \hat{Q}_A$$

$$\hat{Q}_A = m_{mov} g - F - \tfrac{1}{2}\frac{\partial m_{mov}}{\partial y_A}\dot{y}_A{}^2$$

$$(1-K)\sigma_{nc}\frac{d}{dt}(y_B \dot{y}_B) = \sigma_{nc} g y_B - F$$

$$\ddot{y}_B = \frac{g}{1-K} - \frac{\dot{y}_B{}^2}{y_B} - \frac{1}{(1-K)}\frac{F}{\sigma_{nc} y_B}$$

Simplest SDOF Model

All this leads to

Extended Lagrange Equation

$$\ddot{y}^* = -\frac{\dot{y}^{*2}}{y^*} + \frac{1}{1-K} - \frac{\Phi}{(1-K)y^*}$$

Usual Lagrange Equation

$$\ddot{y}^* = -\frac{1}{2}\frac{\dot{y}^{*2}}{y^*} + \frac{1}{1-K} - \frac{\Phi}{(1-K)y^*}$$

Compaction factor

$$K = \sigma_{nc} / \sigma_c$$

Avalanche front

$$y^* = \frac{y_B}{H}$$

Tower height

Resistive load

$$\Phi = F / P$$

Tower weight

Aparent Paradox

Neither Bažant & Verdure, or Seffen were conclusive on **which one should be the proper equation!**

Extended Lagrange Equation

$$\ddot{y}^* = -\frac{\dot{y}^{*2}}{y^*} + \frac{1}{1-K} - \frac{\Phi}{(1-K)y^*}$$

Proper Eq.!

Usual Lagrange Equation

$$\ddot{y}^* = -\frac{1}{2}\frac{\dot{y}^{*2}}{y^*} + \frac{1}{1-K} - \frac{\Phi}{(1-K)y^*}$$

Non -proper Eq.!

Crash down duration (tower 1): **11s**

Crush down duration (tower 1): **9,8s**

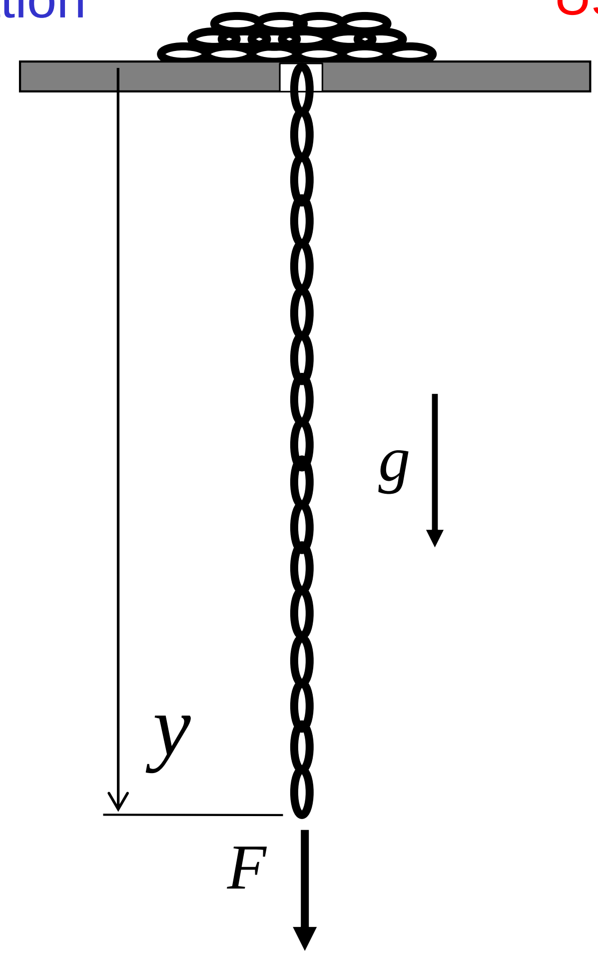
Similarity with falling chains

$F=0;\;\; K=0$

Cayley's

Extended Lagrange Equation

Usual Lagrange Equation



$$\ddot{y} = g - \frac{\dot{y}^2}{y}$$

Proper Eq.!

$$\ddot{y} = g - \frac{1}{2}\frac{\dot{y}^2}{y}$$

Non -proper Eq.!

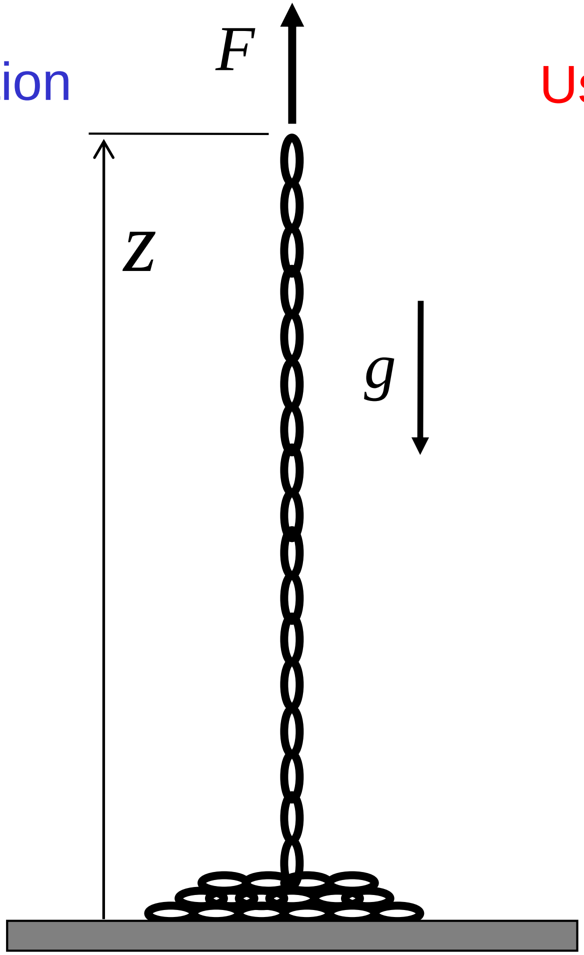
Similarity with falling chains

$F=0; \; K=0$

Extended Lagrange Equation

$$\ddot{z} = -g - \frac{\dot{z}^2}{z}$$

Proper Eq.!



Buquoy's

Usual Lagrange Equation

$$\ddot{z} = -g - \frac{1}{2}\frac{\dot{z}^2}{z}$$

Non -proper Eq.!

Simplest SDOF Model

Case Study: the WTC Towers

$H = 407\,\mathrm{m}$ $\quad P = 3.073\,\mathrm{GN}$ $\quad \sigma_{nc} = 770 \times 10^3\,\mathrm{t/m}$

$0 < \Phi = F / P < 0.21$

$K = \sigma_{nc} / \sigma_c = 0.2$

Bažant, Z. P., Verdure, M., 2007, "Mechanics of progressive collapse: learning from World Trade Center and building demolitions". Journal of Engineering Mechanics, v. 133, n. 3, pp. 308-19.

Seffen, K. A., 2008, "Progressive collapse of the World Trade Center: simple analysis". Journal of Engineering Mechanics, v. 134, n. 2, p. 125-32.

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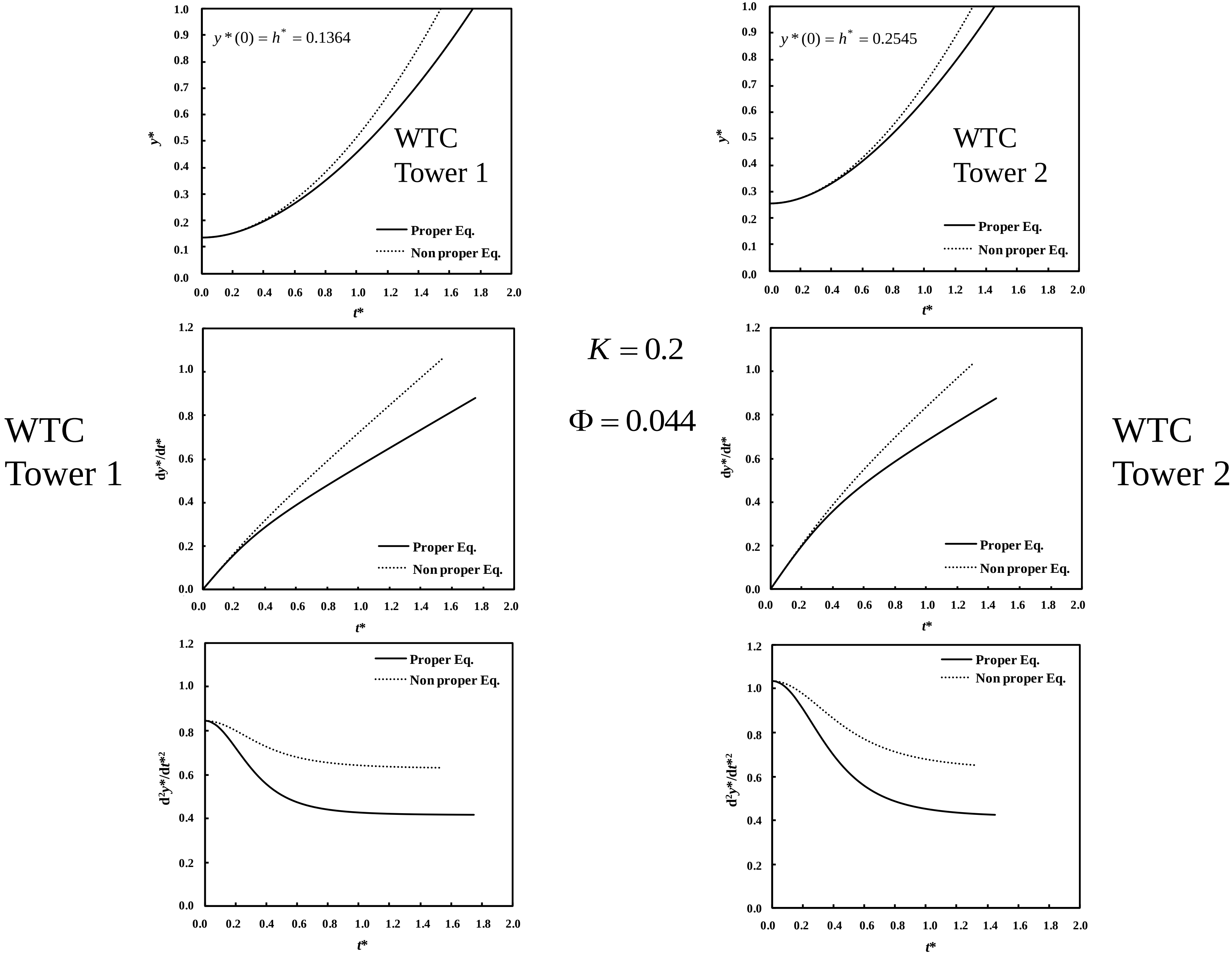
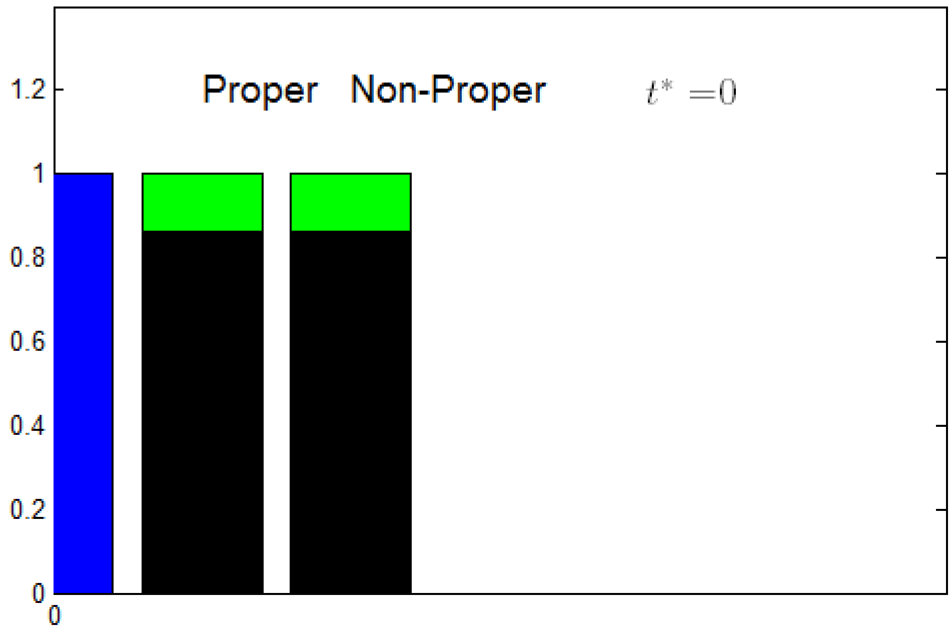


WTC
Tower 1
WTC
Tower 2
$K = 0.2$
$\Phi = 0.044$
$y^*(0) = h^* = 0.1364$
$y^*(0) = h^* = 0.2545$
y*
dy*/dt*
d²y*/dt*²
t*
Proper Eq.
Non proper Eq.

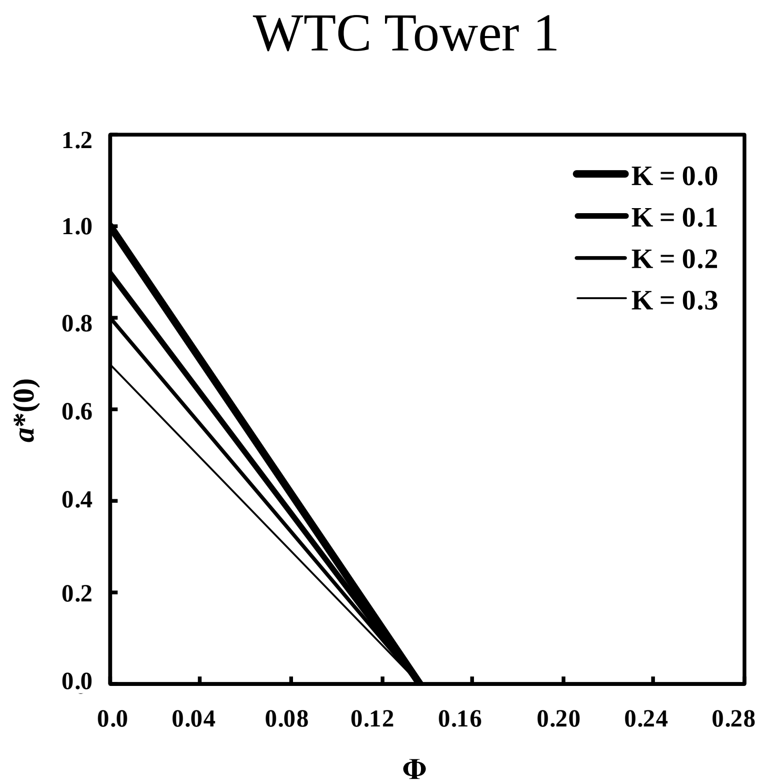
Table 1. ‘Crush-down’ time. WTC: towers 1 and 2. Comparing the results from the proper and non proper ones equations.

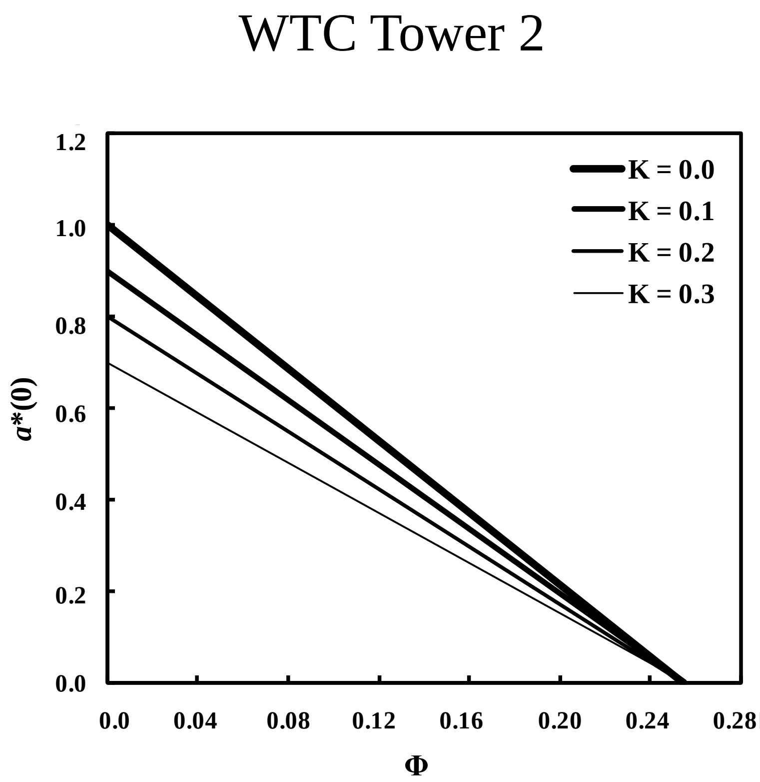
		K=0.2; Φ=0.044		K=0.2; Φ=0.0	
Tower	Equation	t_C^*	t_C (s)	t_C^*	t_C (s)
1	Eq. (102) - proper	1.75	11.3	1.59	10.2
1	Eq. (103) - non-proper	1.55	10.0	1.39	9.0
2	Eq. (102) - proper	1.45	9.3	1.36	8.8
2	Eq. (103) - non-proper	1.32	8.5	1.23	7.9

Pesce, C.P., Casetta, L., Santos, F.M., 2012, “The equation of motion governing the dynamics of vertically collapsing buildings”,
http://dx.doi.org/10.1061/(ASCE)EM.1943-7889.0000453



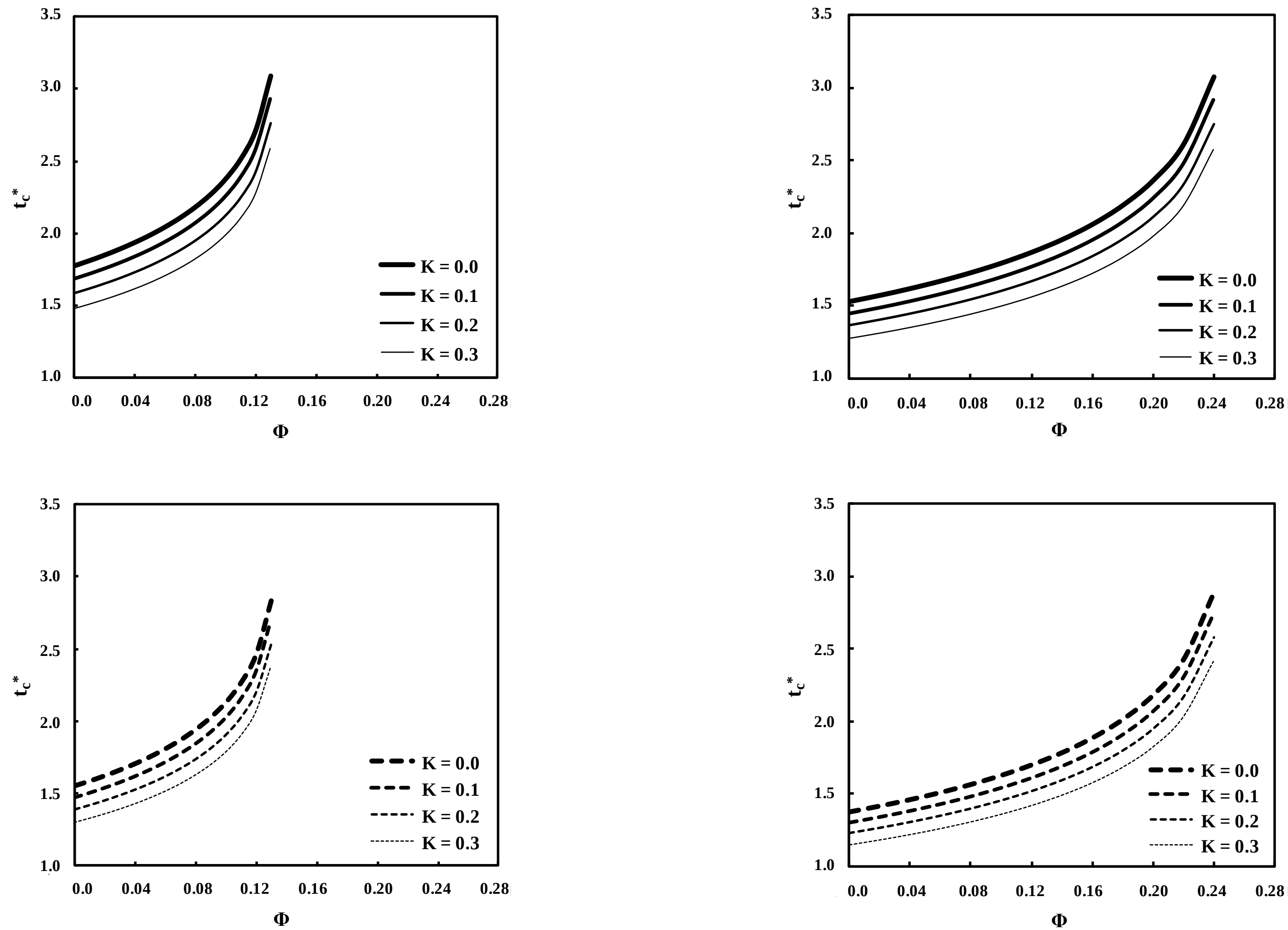
Proper
Non-Proper
$t^* = 0$
1.2
1
0.8
0.6
0.4
0.2
0
0





Initial acceleration of the avalanche front, as function of the resistive force, Φ, having as parameter the compaction factor K.

WTC Tower 1 | WTC Tower 2



Non dimensional ‘crush-down’ time, as function of the resistive force, Φ.

CONCLUSIONS

Problems of variable mass systems in Engineering Mechanics are rather classical and very well explored in the technical literature, since von Buquoy’s work, 1812-1815, Cayley, 1857, and Meshchersky’s, 1897.

However, its subtlety sometimes reserve trappings to students and even to scholars. As a matter of fact, much work is still being carried out on the subject, as testimonies the excellent and recent review by Irschik and Holl (2004).

Nevertheless, from time to time, misinterpretations are found on the correct application of Newton’s second law or concerning the Lagrangian Equation to this kind of systems

Sometimes, motivated by nonlinear dynamics applications, aroused from engineering problems, other times by theoretical issues, see, e.g. Mušicki (2005), variable mass system dynamics is still a state-of-the-art matter, both, grounding the rational formulation of open systems dynamics or directly linked to technical applications.

Its importance goes beyond applications on engineering, extending from solids and fluids dynamics to complex flows of mixtures, fluid flows in porous media, or even reaching quite distinct problems in theoretical physics.

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"Be extremely careful when dealing with variable mass systems!!"

Thank you!

Acknowledgments:

CNPq

FAPESP

Especial thanks

Dr. Leonardo Casetta

Appendix I

Derivation of the Extended Lagrange Equation for General Variable Mass Systems

2003, Pesce, C. P.

"The Application of Lagrange Equations to Mechanical Systems with Mass Explicitly Dependent on Position", Journal of Applied Mechanics, Vol. 70, pp. 751-756.

Extended Lagrange Equations
Derivation via D'Alembert Principle and PVW

$$\sum_i \left(\frac{d\mathbf{p}_i}{dt} - \mathbf{F}_i \right) \cdot \delta P_i = \mathbf{0} \qquad \mathbf{F}_i = \mathbf{f}_i + \mathbf{h}_i$$

$\mathbf{f}_i$ Active forces

$\mathbf{h}_i = \dot{m}_i \mathbf{v}_{oi}$ Reactive forces

Velocity of expelled (gained) mass measured in an inertial frame.

or

$$\sum_i \left(m_i \frac{d\mathbf{v}_i}{dt} - (\mathbf{f}_i + \Phi_i) \right) \cdot \delta P_i = \mathbf{0}$$

Relative velocity of expelled (gained) mass

$$\Phi_i = \dot{m}_i \left(\mathbf{v}_{oi} - \mathbf{v}_i \right) = \mathbf{h}_i - \dot{m}_i \mathbf{v}_i$$

Meschersky's Force

Lagrange Equations: via Principle of Virtual Work

$$\delta P_i = \sum_j \frac{\partial P_i}{\partial q_j} \cdot \delta q_j$$

$$\mathbf{v}_i = \mathbf{v}_i(q_j; \dot{q}_j; t); \; j = 1, \ldots, M$$

$$\frac{\partial \mathbf{v}_i}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial P_i}{\partial q_j}\right)$$

Kinematic relations

$$\frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} = \frac{\partial P_i}{\partial q_j}$$

$$\frac{d\mathbf{v}_i}{dt} \cdot \frac{\partial P_i}{\partial q_j} = \frac{d}{dt}\left(\frac{1}{2}\frac{\partial \mathbf{v}_i{}^2}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}\left(\frac{1}{2}\mathbf{v}_\mathrm{i}{}^2\right)$$

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial P_i}{\partial q_j} = \sum_i (\mathbf{f}_i + \mathbf{h}_i) \cdot \frac{\partial P_i}{\partial q_j}$$

Generalized forces

Simplest case: constant mass

$$m_i \frac{d\mathbf{v}_i}{dt} \cdot \frac{\partial P_i}{\partial q_j} = \frac{d}{dt}\left(\frac{1}{2}\frac{\partial m_i \mathbf{v}_i{}^2}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}\left(\frac{1}{2} m_i \mathbf{v}_\mathrm{i}{}^2\right) = \frac{d}{dt}\left(\frac{\partial T_i}{\partial \dot{q}_j}\right) - \frac{\partial T_i}{\partial q_j}$$

$$\frac{dm_i}{dt} = 0 \quad \Rightarrow \quad \frac{d\mathbf{p}_i}{dt} = m_i \frac{d\mathbf{v}_i}{dt}$$

$$Q_j = \sum_i \mathbf{f}_i \cdot \frac{\partial P_i}{\partial q_j}$$

$$\Rightarrow \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j \; ; \qquad j = 1, \ldots, M$$

Case: m(t)

PVW: $$\sum_i \left(\frac{d\mathbf{p}_i}{dt} - (\mathbf{f}_i + \mathbf{h}_i) \right) \cdot \delta P_i = \sum_j \sum_i \left(m_i \frac{d\mathbf{v}_i}{dt} + \frac{dm_i}{dt}\mathbf{v}_i - (\mathbf{f}_i + \mathbf{h}_i) \right) \cdot \frac{\partial P_i}{\partial q_j} \delta q_j = \mathbf{0}$$

Integration by parts, first term:

$$m_i \frac{d\mathbf{v}_i}{dt} \cdot \frac{\partial P_i}{\partial q_j} = \frac{d}{dt}\left(\frac{1}{2} m_i \frac{\partial \mathbf{v}_i^{\,2}}{\partial \dot{q}_j} \right) - \frac{dm_i}{dt}\left(\frac{1}{2} \frac{\partial \mathbf{v}_i^{\,2}}{\partial \dot{q}_j} \right) - \frac{\partial}{\partial q_j}\left(\frac{1}{2} m_i \mathbf{v}_i^{\,2} \right) =$$

$$= \frac{d}{dt}\left(\frac{1}{2} \frac{\partial m_i \mathbf{v}_i^{\,2}}{\partial \dot{q}_j} \right) - \frac{dm_i}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial}{\partial m_i}\left(\frac{1}{2} m_i \mathbf{v}_i^{\,2} \right)\right)\right) - \frac{\partial}{\partial q_j}\left(\frac{1}{2} m_i \mathbf{v}_i^{\,2} \right) =$$

$$= \frac{d}{dt}\left(\frac{\partial T_i}{\partial \dot{q}_j} \right) - \frac{dm_i}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial T_i}{\partial m_i} \right)\right) - \frac{\partial T_i}{\partial q_j}$$

$$\frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} = \frac{\partial P_i}{\partial q_j}$$

Cancel each other

second term:

$$\frac{dm_i}{dt}\mathbf{v}_i \cdot \frac{\partial P_i}{\partial q_j} = \frac{dm_i}{dt}\mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} = \frac{1}{2}\frac{dm_i}{dt}\frac{\partial \mathbf{v}_i^{\,2}}{\partial \dot{q}_j} = \frac{dm_i}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial T_i}{\partial m_i} \right)\right)$$

Most general form

Case: m(t)
(continued)

Leading to ***the same usual form***

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j \; ; \qquad j = 1, \ldots, M$$

with

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial P_i}{\partial q_j} = \sum_i \left(\mathbf{f}_i + \mathbf{h}_i\right) \cdot \frac{\partial P_i}{\partial q_j}$$

where

$$\mathbf{h}_i = \dot{m}_i \mathbf{v}_{oi}$$

Most complete case: $m_i = m_i(q_j;\dot{q}_j;t)$

Integration by parts, first term:

$$m_i \frac{d\mathbf{v}_i}{dt}\cdot\frac{\partial P_i}{\partial q_j} =$$

$$\frac{d}{dt}\left(\frac{1}{2}m_i\frac{\partial \mathbf{v}_i^{\,2}}{\partial \dot{q}_j}\right)-\frac{1}{2}\frac{dm_i}{dt}\left(\frac{\partial \mathbf{v}_i^{\,2}}{\partial \dot{q}_j}\right)-\frac{\partial}{\partial q_j}\left(\frac{1}{2}m_i\mathbf{v}_i^{\,2}\right)+\frac{1}{2}\frac{\partial m_i}{\partial q_j}\left(\mathbf{v}_i^{\,2}\right)=$$

$$=\frac{d}{dt}\left(\frac{1}{2}\frac{\partial m_i\mathbf{v}_i^{\,2}}{\partial \dot{q}_j}\right)-\frac{1}{2}\frac{d}{dt}\left(\frac{\partial m_i}{\partial \dot{q}_j}\mathbf{v}_i^{\,2}\right)-\frac{1}{2}\frac{dm_i}{dt}\left(\frac{\partial \mathbf{v}_i^{\,2}}{\partial \dot{q}_j}\right)-\frac{\partial}{\partial q_j}\left(\frac{1}{2}m_i\mathbf{v}_i^{\,2}\right)+\frac{1}{2}\frac{\partial m_i}{\partial q_j}\left(\mathbf{v}_i^{\,2}\right)$$

$$=\frac{d}{dt}\left(\frac{\partial T_i}{\partial \dot{q}_j}\right)-\frac{1}{2}\frac{d}{dt}\left(\frac{\partial m_i}{\partial \dot{q}_j}\mathbf{v}_i^{\,2}\right)-\frac{dm_i}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial T_i}{\partial m_i}\right)\right)-\frac{\partial T_i}{\partial q_j}+\frac{1}{2}\frac{\partial m_i}{\partial q_j}\left(\mathbf{v}_i^{\,2}\right)$$

Two new terms

Cancel each other, as before

second term:

$$\frac{dm_i}{dt}\mathbf{v}_i\cdot\frac{\partial P_i}{\partial q_j}=\frac{dm_i}{dt}\mathbf{v}_i\cdot\frac{\partial \mathbf{v}_i}{\partial \dot{q}_j}=\frac{1}{2}\frac{dm_i}{dt}\frac{\partial \mathbf{v}_i^{\,2}}{\partial \dot{q}_j}=\frac{dm_i}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial T_i}{\partial m_i}\right)\right)$$

Most complete case: $m_i = m_i(q_j; \dot{q}_j; t)$

Leading to the Extended Form of the Lagrange Equations:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = \hat{Q}_j \; ; \qquad j = 1, \ldots, M$$

Mass depending explicitly on time

Extended generalized force

$$\hat{Q}_j = \sum_i \left(\mathbf{f}_i + \dot{m}_i \mathbf{v}_{oi}\right) \cdot \frac{\partial P_i}{\partial q_j} +$$

Two new terms

$$+ \sum_i \left\{ \frac{1}{2}\frac{d}{dt}\left(\frac{\partial m_i}{\partial \dot{q}_j} (\mathbf{v}_i)^2 \right) - \frac{1}{2}\frac{\partial m_i}{\partial q_j} (\mathbf{v}_i)^2 \right\}$$

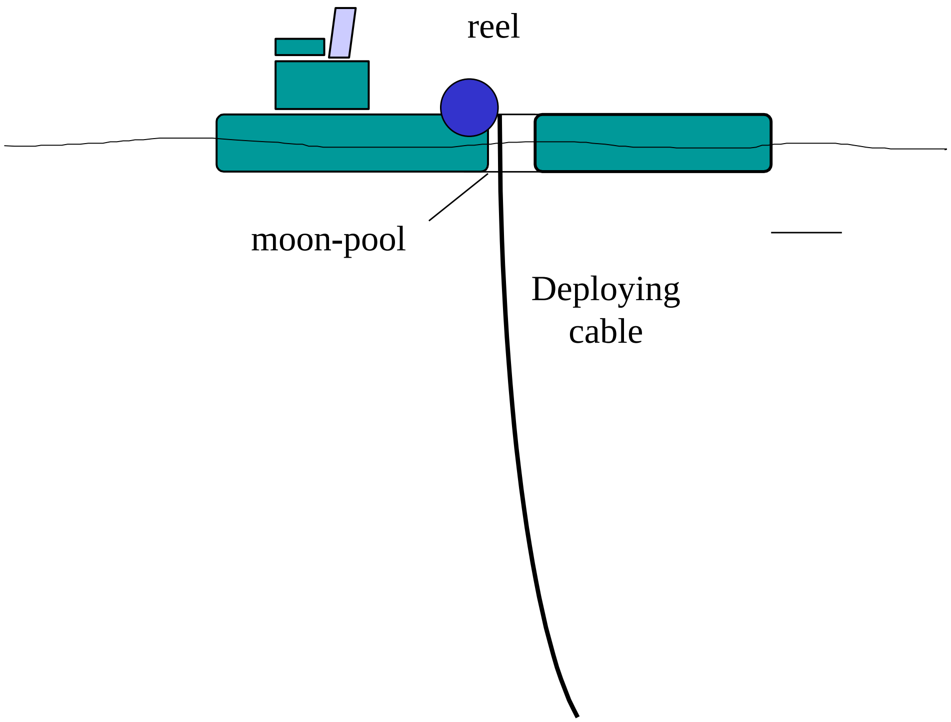
Mass depending explicitly on velocity

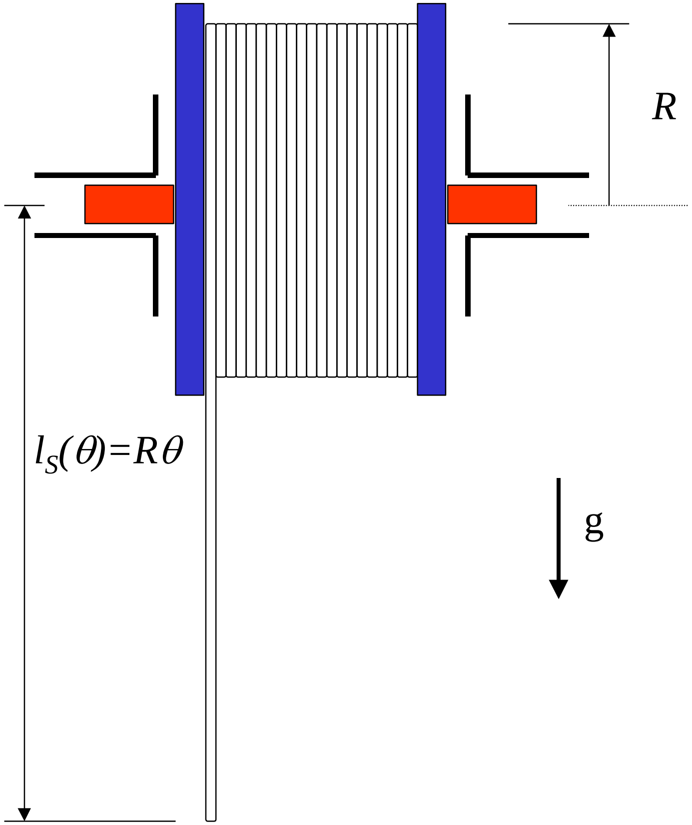
Mass depending explicitly on position

Appendix II

Ocean Engineering Problems

Ocean Engineering problems





Cable Deployment from a Reel

Cable Deployment from a Reel

System

Wound cable+reel:

Variable mass

$$\tau(\theta) = \mu R\theta\big((1-\beta)g - R\ddot{\theta}\big)$$ Hanging traction

$$\dot{m}_R(\theta) = -\mu R\dot{\theta}$$ Mass rate

$$T_1 = 1/2\,(I_R)\dot{\theta}^2 = 1/2\,(I_O + \mu R^2(L - R\theta))\dot{\theta}^2$$

$$\frac{d}{dt}\left(\frac{\partial T_1}{\partial\dot{\theta}}\right) - \frac{\partial T_1}{\partial\theta} = Q_\theta$$

$$Q_\theta = \big(\tau(\theta) + \dot{m}_R(\theta)R\dot{\theta}\big)R$$

INCORRECT

$$(I_O + mR^2)\ddot{\theta} + \frac{1}{2}\mu R^3\dot{\theta}^2 - (1-\beta)\mu g R^2\theta = 0$$

Cable Deployment from a Reel

System

$$\tau(\theta) = \mu R\theta\left((1-\beta)g - R\ddot{\theta}\right)$$ Hanging traction

Wound cable+reel:

$$\dot{m}_R(\theta) = -\mu R\dot{\theta}$$ Mass rate

Variable mass

$$T_1 = 1/2\,(I_R)\dot{\theta}^2 = 1/2\,(I_O + \mu R^2(L - R\theta))\dot{\theta}^2$$

$$\frac{d}{dt}\left(\frac{\partial T_1}{\partial\dot{\theta}}\right) - \frac{\partial T_1}{\partial\theta} = \hat{Q}_\theta \qquad \hat{Q}_\theta = \left(\tau(\theta) + \dot{m}_R(\theta)R\dot{\theta}\right)R - \frac{1}{2}\frac{dm_R}{d\theta}R^2\dot{\theta}^2$$

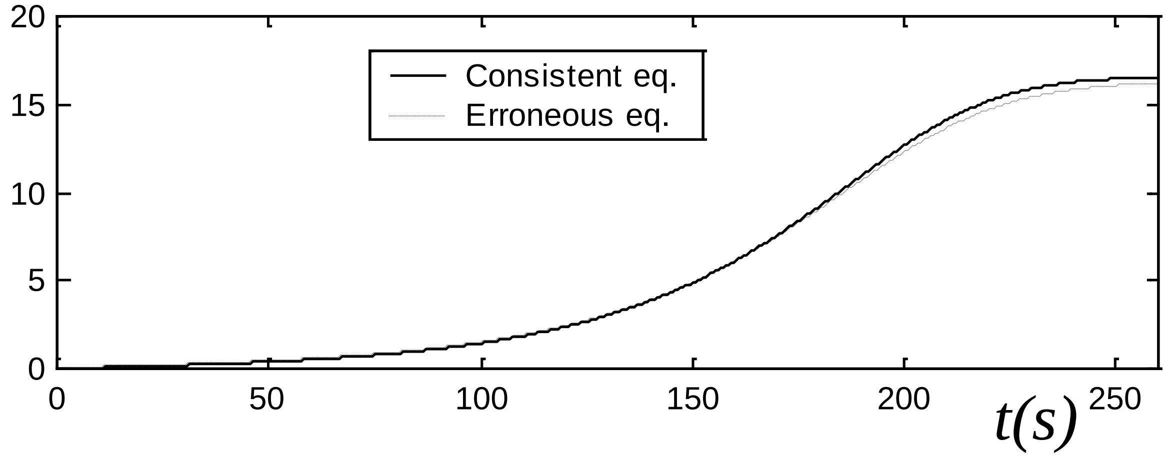
CORRECT

$$(I_O + mR^2)\ddot{\theta} - (1-\beta)\mu gR^2\theta = 0$$

Cable Deployment from a Reel: ***typical analysis***

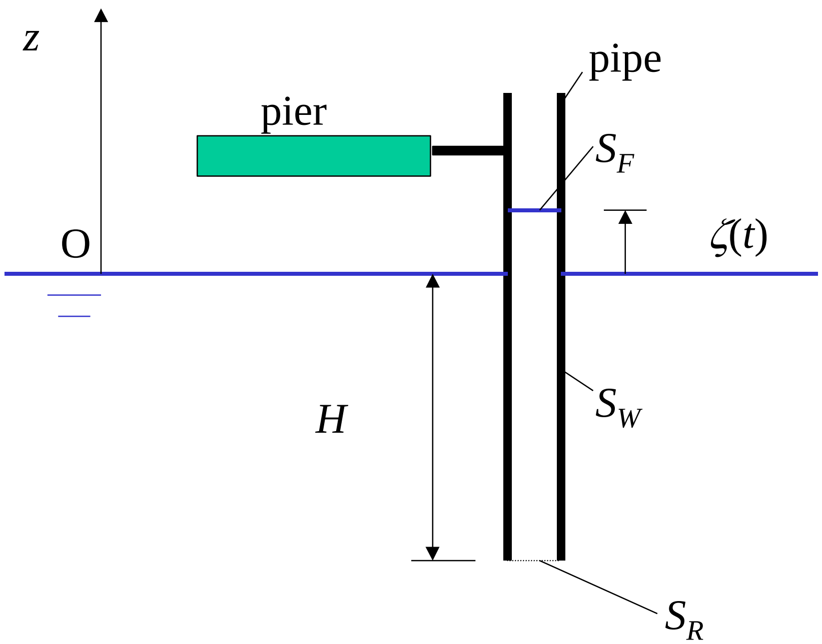
I.C.: $l_S(\theta(0)) = 10\text{ m}; \quad \dot{\theta}(0) = 0$

$\dot{\theta}(t)(rad/s)$



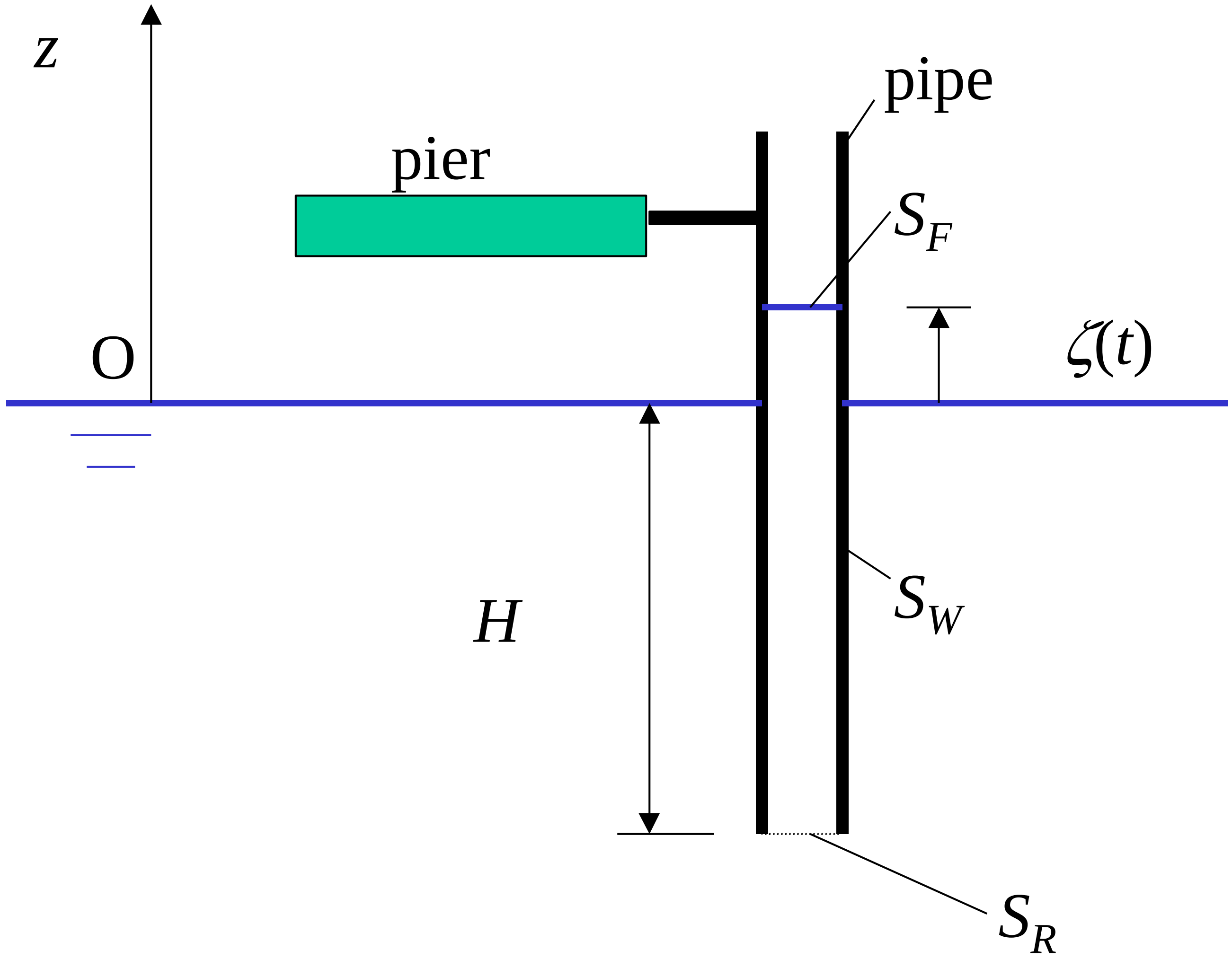
$$F_f = -1/2\,C_f\,\rho D(R\dot{\theta})^2 l_s(\theta) = -1/2\,C_f\,\rho D R^3 \theta \dot{\theta}^2$$

Ocean Engineering problems



Oscillating water column in open pipes

Oscillating water column in the moon-pool of a mono-column platform



Oscillating water column in open pipes

Oscillating water column in open pipes

$$T = \frac{1}{2}\rho A(\zeta + H)\dot{\zeta}^2$$

$$F = f + \dot{m}v_o = \left(F_S + F_D\right) + \dot{m}v_o =$$

$$= \left(-\rho A g\zeta - \frac{1}{2}\rho A\dot{\zeta}^2\right) + \left(\rho A\dot{\zeta}^2\right) = -\rho A g\zeta + \frac{1}{2}\rho A\dot{\zeta}^2$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\zeta}} - \frac{\partial T}{\partial \zeta} = F \quad \Rightarrow \quad \ddot{\zeta} + \frac{\dot{\zeta}^2}{(\zeta + H)} + g\frac{\zeta}{(\zeta + H)} = 0$$

INCORRECT

Oscillating water column in open pipes

$$T = \frac{1}{2}\rho A(\zeta + H)\dot{\zeta}^2$$

$$\hat{F} = f + \dot{m}v_o - \frac{1}{2}\sum_i \frac{\partial m_i}{\partial \zeta}\mathbf{v}_i^{\,2} = \left(F_S + F_D\right) + \dot{m}v_o - \frac{1}{2}\frac{\partial m}{\partial \zeta}\dot{\zeta}^2 =$$

$$= \left(-\rho A g\zeta - \frac{1}{2}\rho A\dot{\zeta}^2\right) + \left(\rho A\dot{\zeta}^2\right) - \left(\frac{1}{2}\rho A\dot{\zeta}^2\right) = -\rho A g\zeta$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\zeta}} - \frac{\partial T}{\partial \zeta} = \hat{F}$$

$$\ddot{\zeta} + \frac{1}{2}\frac{\dot{\zeta}^2}{\left(\zeta + H\right)} + g\frac{\zeta}{\left(\zeta + H\right)} = 0$$

CORRECT

Oscillating water column in open pipes

$$\omega = \sqrt{g/H}$$

Normalizing

$$\eta(t) = \zeta(t)/H$$

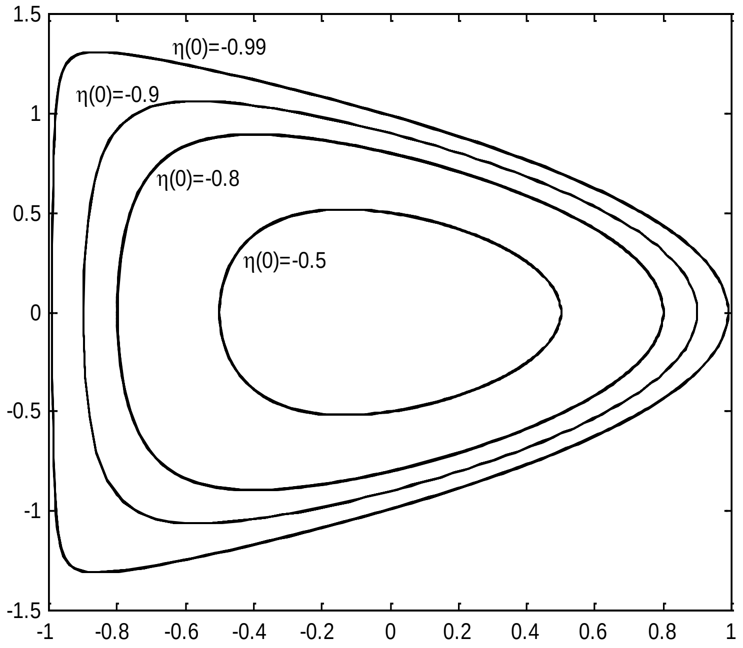
$$-1 < \eta$$

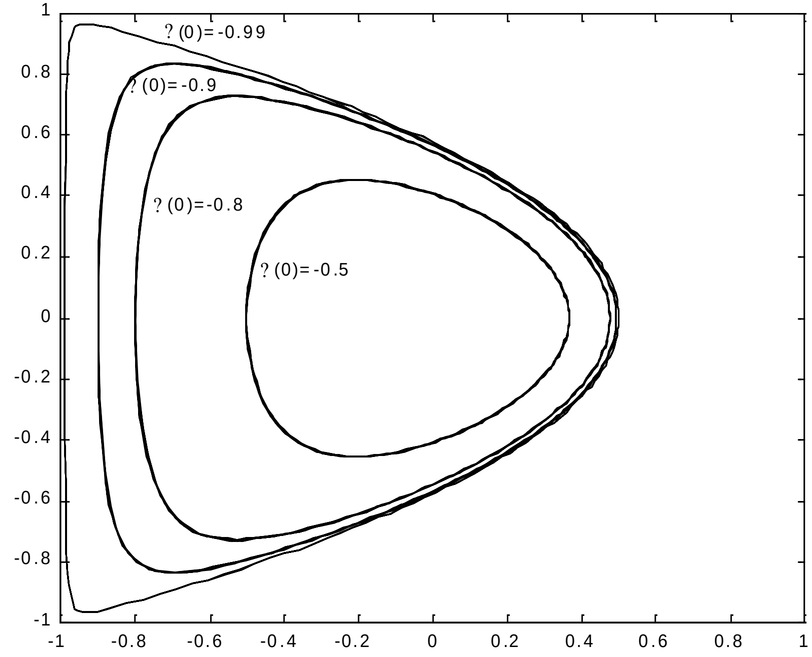
$$\ddot{\eta} + \frac{1}{2}\frac{\dot{\eta}^2}{(\eta+1)} + \frac{\eta}{(\eta+1)} = 0$$

Small oscillations:

$$\ddot{\eta} + \eta = 0$$

Oscillating water column in open pipes





CORRECT

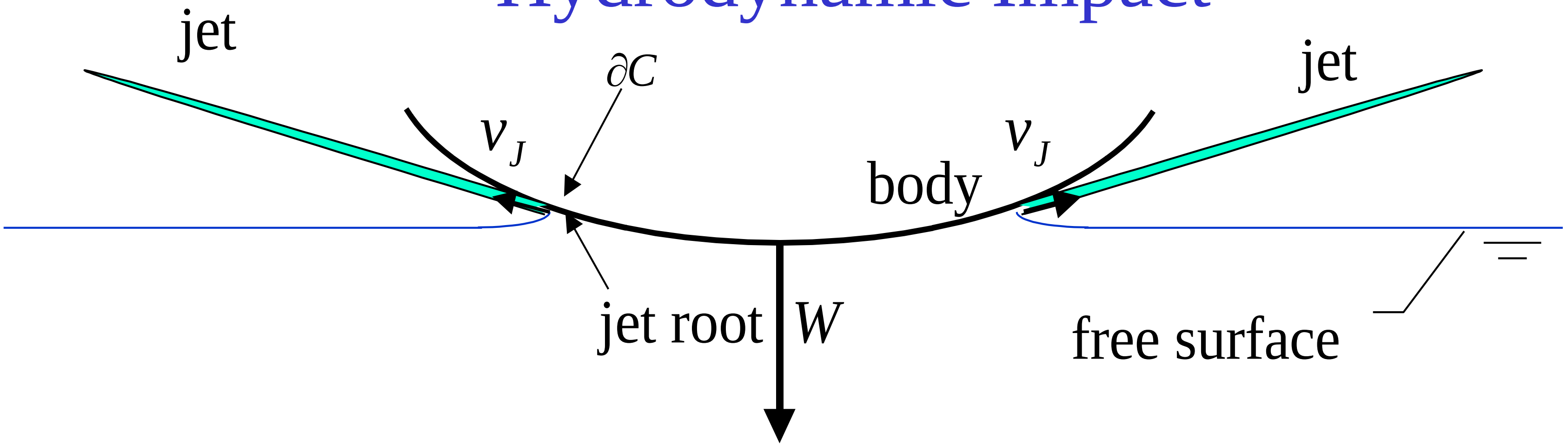
Phase portraits

INCORRECT

I.C.: $\eta(0) = -0.99..-0.5; \dot{\eta}(0) = 0$

Ocean Engineering problems

Hydrodynamic Impact



$$T = \frac{1}{2} M_{zz} W^2$$

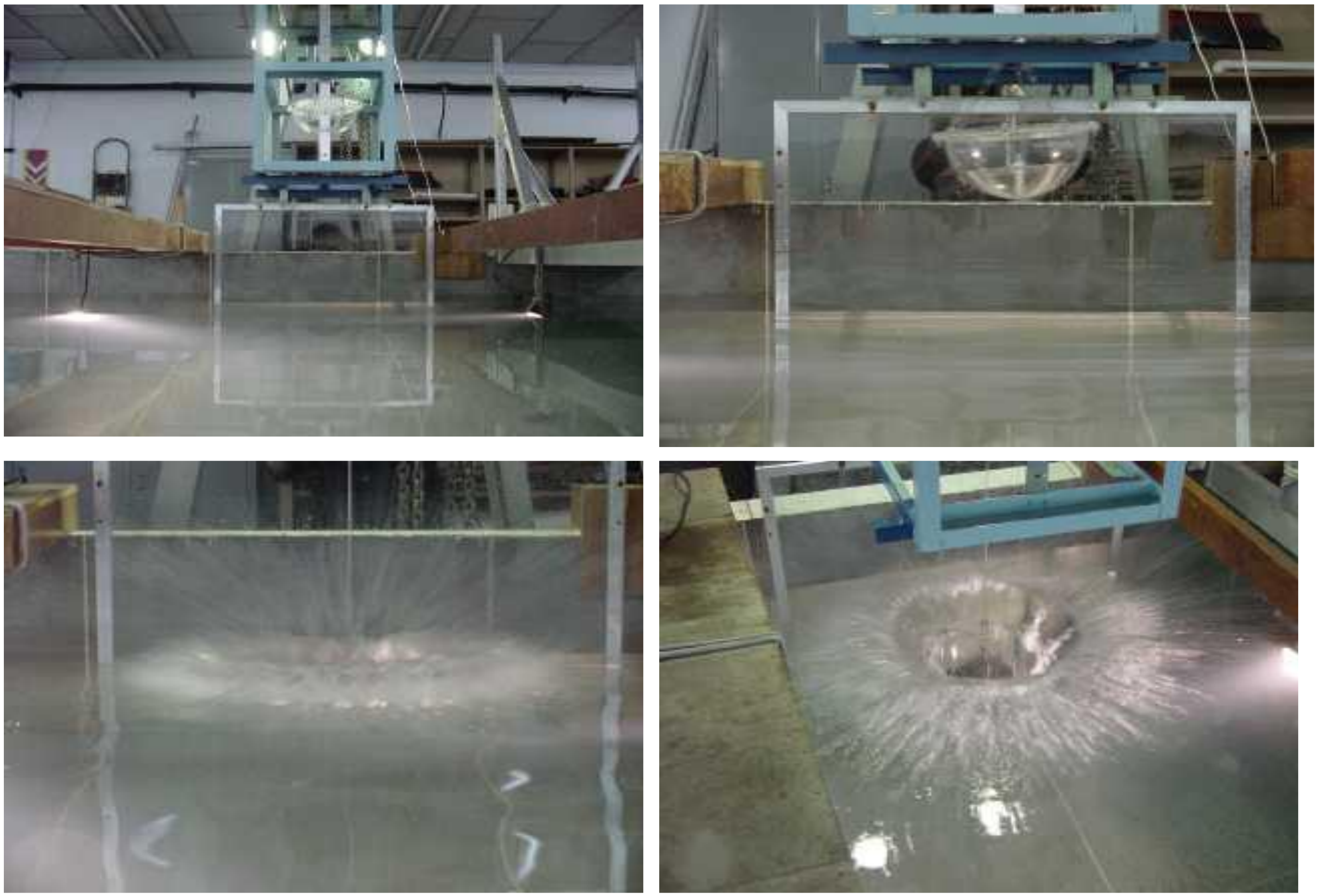
$$M_{zz} = M_{zz}(\zeta)$$

$$\zeta = \int_{0^+}^{t} W dt$$

Added mass dependent on position

Buoyancy and Gravitational Forces are neglectable at the very instant of impact

Hydrodynamic Impact



Sphere

Hydrodynamic Impact

Force applied on the body via non-extended form

$$F_z = -\frac{d}{dt}\left(\frac{\partial T}{\partial W}\right) + \frac{\partial T}{\partial \zeta}$$

↓

INCORRECT

$$F_z = -\frac{1}{2}\frac{dM_{zz}}{dt}W - M_{zz}\frac{dW}{dt}$$

Hydrodynamic Impact

Second order

Force applied on the bulk of the liquid

$$-F_z^{\,B} = -\frac{d}{dt}\left(\frac{\partial T}{\partial W}\right) + \frac{\partial T}{\partial \zeta} - \frac{1}{2}\frac{dM_{zz}}{d\zeta}W^2 - 2\dot{m}v_J \sin\alpha$$

Force applied on the body via extended form

$$F_z = -\frac{d}{dt}\left(\frac{\partial T}{\partial W}\right) + \frac{\partial T}{\partial \zeta} - \frac{1}{2}\frac{dM_{zz}}{d\zeta}W^2$$

Extra term

CORRECT

$$F_z = -\frac{d}{dt}\left(M_{zz}W\right) + \frac{1}{2}W^2\frac{dM_{zz}}{d\zeta} - \frac{1}{2}\frac{dM_{zz}}{d\zeta}W^2 = -\frac{d}{dt}\left(M_{zz}W\right)$$

Hydrodynamic Impact

Sphere of radius *R* and mass *m*

Non-dimensional time

normalizing

Mass ratio

$$\mathrm{t} = W_0 t / R \qquad \beta = m / m_D$$

$$\eta = \zeta / R \qquad \dot{\eta} = \frac{d\eta}{d\mathrm{t}} = \frac{1}{W_0}\frac{d\zeta}{dt} \qquad \ddot{\eta} = \frac{d^2\eta}{d\mathrm{t}^2} = \frac{R}{W_0{}^2}\frac{d^2\zeta}{dt^2}$$

INCORRECT

Asymptotics, similitude solutions and Non-Extended Lagrange Equations:

$$\ddot{\eta} + \frac{1}{2}\frac{\frac{9\sqrt{3}}{2\pi}\eta^{1/2}\dot{\eta}^2}{\beta + \frac{3\sqrt{3}}{\pi}\eta^{3/2}} = 0$$

Hydrodynamic Impact

Sphere of radius *R* and mass *m*

normalizing

$$\mathrm{t} = W_0 t / R \qquad \beta = m / m_D$$

$$\eta = \zeta / R \qquad \dot{\eta} = \frac{d\eta}{d\mathrm{t}} = \frac{1}{W_0}\frac{d\zeta}{dt} \qquad \ddot{\eta} = \frac{d^2\eta}{d\mathrm{t}^2} = \frac{R}{W_0^{\,2}}\frac{d^2\zeta}{dt^2}$$

CORRECT

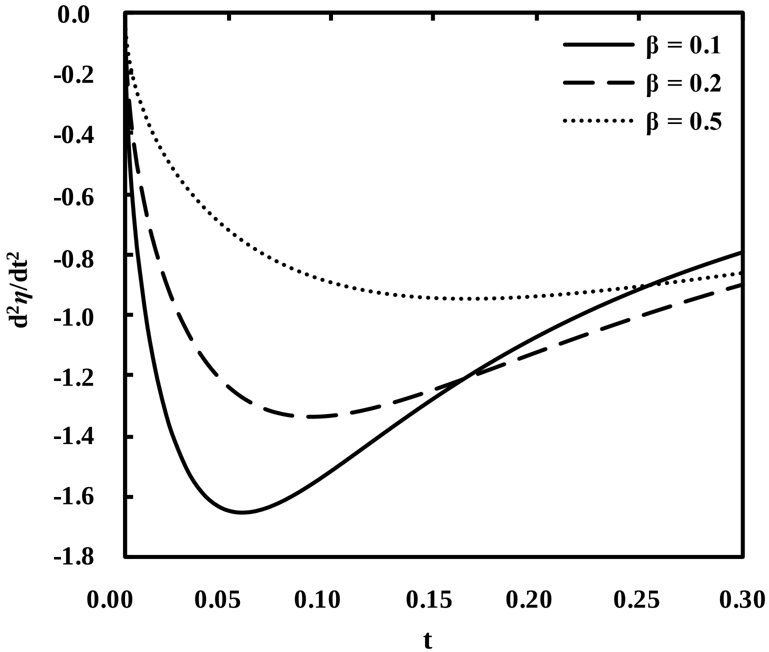
Asymptotics, similitude solutions and ***Extended*** Lagrange Equations:

$$\ddot{\eta} + \frac{\dfrac{9\sqrt{3}}{2\pi}\eta^{1/2}\dot{\eta}^2}{\beta + \dfrac{3\sqrt{3}}{\pi}\eta^{3/2}} = 0$$

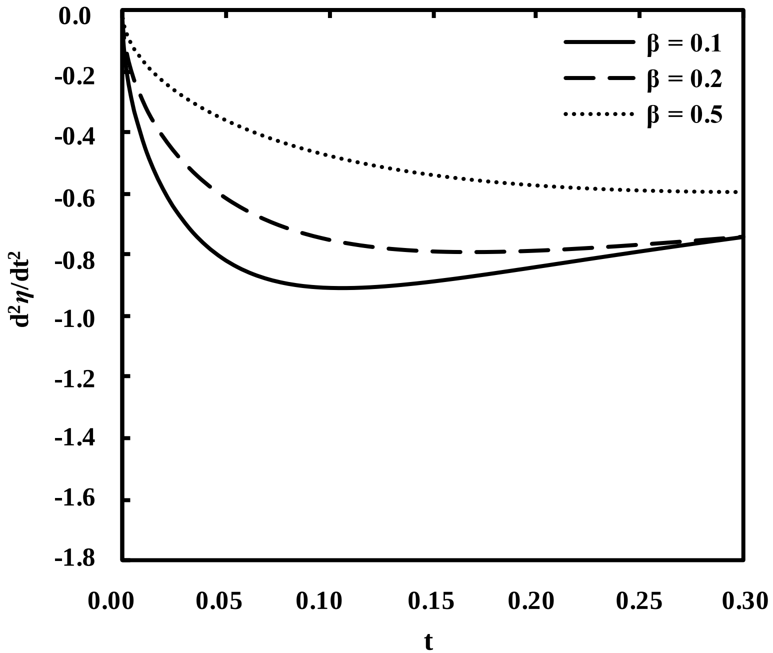
Hydrodynamic Impact

Sphere of radius *R* and mass *m*

$$\ddot{\eta} = \frac{d^2\eta}{d\mathrm{t}^2} = \frac{R}{W_0^{\ 2}} \frac{d^2\zeta}{dt^2}$$



$\mathrm{t} = W_0 t / R$



$\mathrm{t} = W_0 t / R$

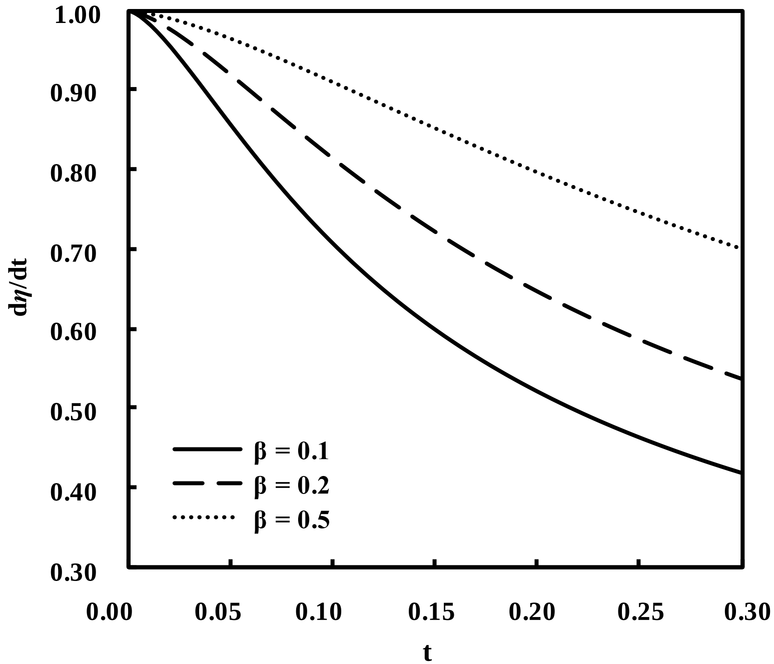
CORRECT ***INCORRECT***

$$\beta = m / m_D = 3m / \left(4\rho\pi R^3\right): \text{ specific mass}$$

Hydrodynamic Impact

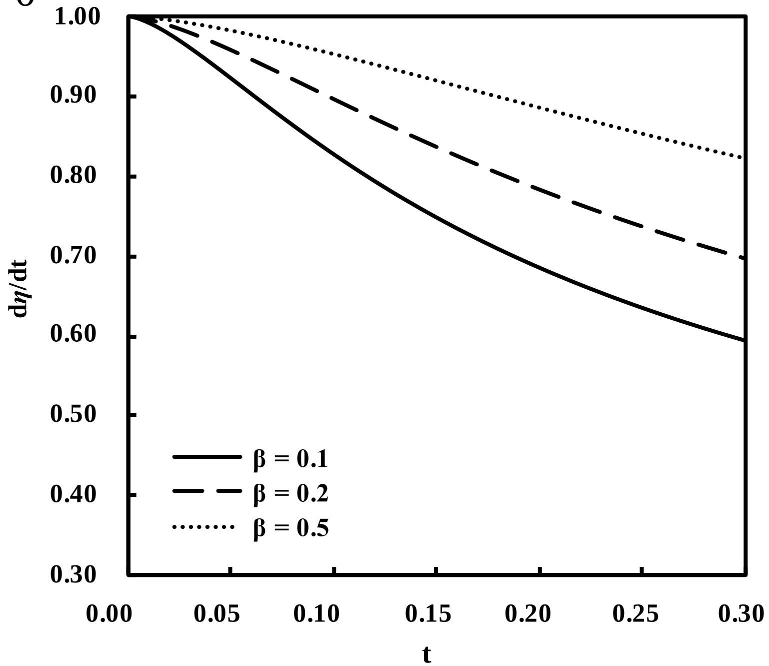
Sphere of radius *R* and mass *m*

$$\dot{\eta} = \frac{d\eta}{d\mathrm{t}} = \frac{1}{W_0}\frac{d\zeta}{dt}$$



$\mathrm{t} = W_0 t / R$

CORRECT



$\mathrm{t} = W_0 t / R$

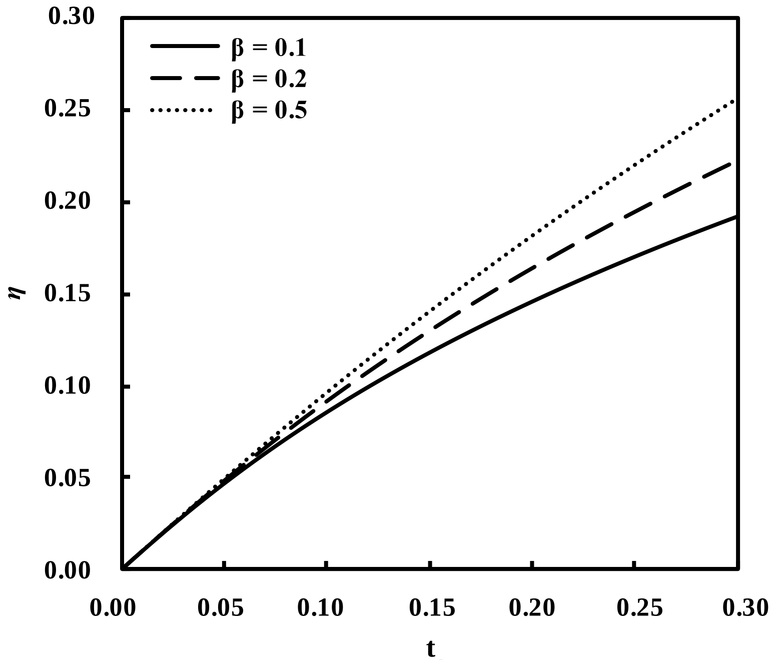
INCORRECT

$$\beta = m / m_D = 3m / \left(4\rho\pi R^3\right): \text{ specific mass}$$

Hydrodynamic Impact

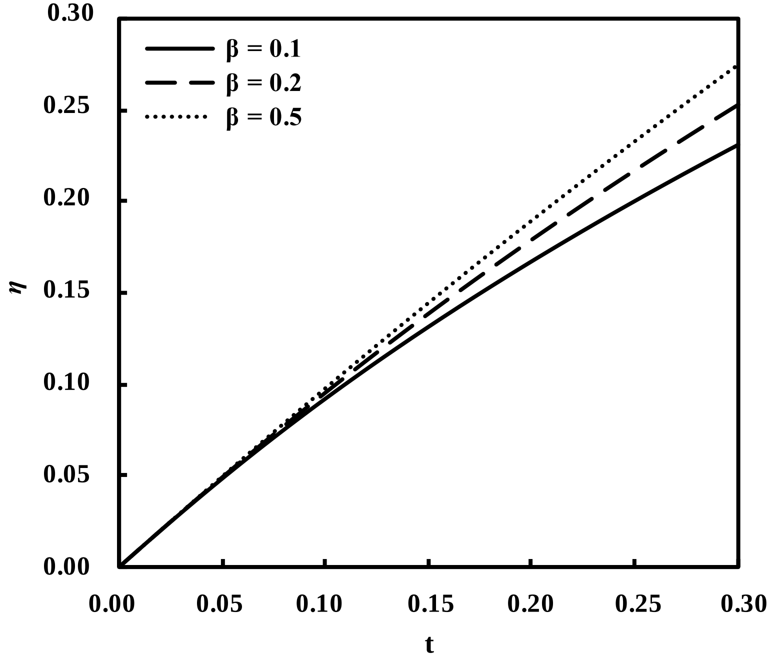
Sphere of radius *R* and mass *m*

$$\eta = \zeta / R$$



$\mathrm{t} = W_0 t / R$

CORRECT



$\mathrm{t} = W_0 t / R$

INCORRECT

$$\beta = m / m_D = 3m / \left(4\rho\pi R^3\right):\ \text{specific mass}$$