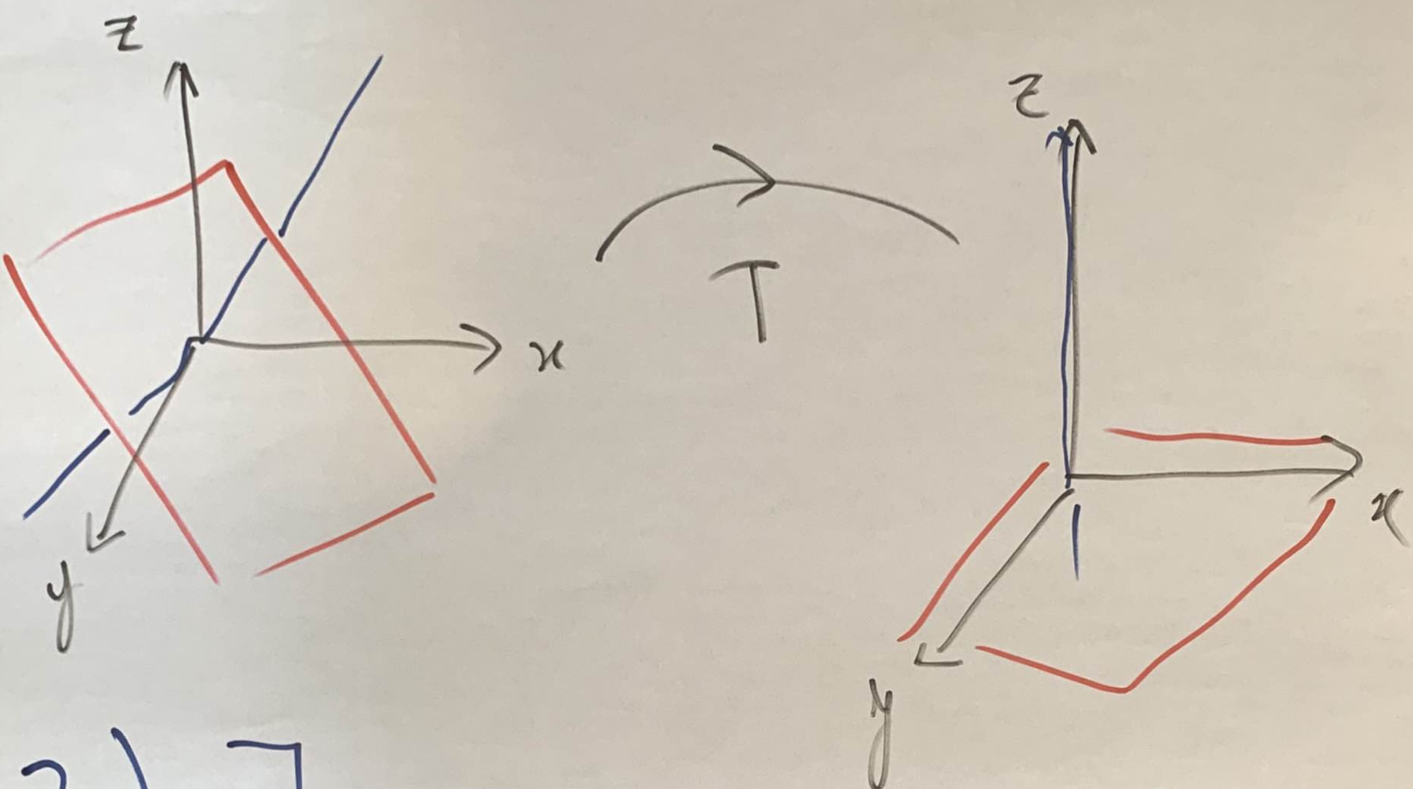
MONITORIA MAT 4 10/05



2) $\exists$ subespaços V, W de $\mathbb{R}^n$ tais que:

(i) $\mathbb{R}^n = V + W$ $(\forall x \in \mathbb{R}^n, \exists v \in V, w \in W$ t.q. $x = v + w)$

(ii) $V \cap W = \{0\}$

$\dim V = m$, $\dim W = r$

(a) $\nexists$ $n = m + r$

$\mathcal{B}_V = \{v_1, \dots, v_m\}$ base de V

$B_W = \{w_1, \dots, w_r\}$ base de W

$\vdash \mathcal{B} = \mathcal{B}_V \cup \mathcal{B}_W = \{v_1, \dots, v_m, w_1, \dots, w_r\}$ é base de $\mathbb{R}^n$?

$\vdash$ <u>$\mathcal{B}$ gera $\mathbb{R}^n$</u>: Seja $x \in \mathbb{R}^n$ arbitrário

Por (i), existem $v \in V, w \in W$ t.q.

$x = v + w$

$$\begin{cases} v \in V, \mathcal{B}_V \text{ é base de } V \Rightarrow v = \sum_{i=0}^{m} \alpha_i v_i \\ w \in W, \mathcal{B}_W \text{ é base de } W \Rightarrow w = \sum_{j=0}^{r} \beta_j w_j \end{cases}$$

$\hookrightarrow x = v + w = \sum_{i=0}^{m} \alpha_i v_i + \sum_{j=0}^{r} \beta_j w_j$

$\hookrightarrow x \in \mathcal{L}(\mathcal{B})$. ✓

- $\mathcal{B}$ é L.I.:

$$0 = \sum_{i=1}^{m} \alpha_i v_i + \sum_{j=1}^{r} \beta_j w_j \Rightarrow \alpha_i = \beta_j = 0$$

O que queremos

$$0 = \sum_{i=1}^{m} \alpha_i v_i + \sum_{j=1}^{r} \beta_j w_j \Rightarrow$$

$$y = \underbrace{\sum_{i=1}^{m} \alpha_i v_i}_{\in V = L(\mathcal{B}_V)} = \underbrace{\sum_{j=1}^{r} (-\beta_j) w_j}_{\in W = L(\mathcal{B}_W)} \in V \cap W$$

Por (ii), $V \cap W = \{0\} \Rightarrow$

$$\begin{cases} \sum_{i=1}^{m} \alpha_i v_i = 0 \overset{\mathcal{B}_V \text{ é L.I.}}{\Rightarrow} \alpha_i = 0 \\ \sum_{j=1}^{r} (-\beta_j) w_j = 0 \overset{\mathcal{B}_W \text{ é L.I.}}{\Rightarrow} -\beta_j = 0 \Rightarrow \beta_j = 0 \end{cases}$$

Concluímos que $\mathcal{B} = \{v_1, \dots, v_m, w_1, \dots, w_r\}$ é base de $\mathbb{R}^n \Rightarrow \underline{n = m + n}$ $\square$

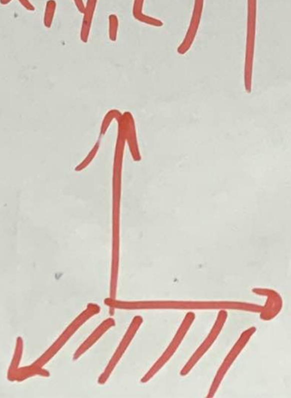
(b) $\exists\, T : \mathbb{R}^n \to \mathbb{R}^n$ isomorfismo linear (transf. linear bijetora) t.q.

$$T(V) = \{(x_1, \dots, x_n) \mid x_{m+1} = \dots = x_n = 0\} \cong \mathbb{R}^m$$

$$T(W) = \{(x_1, \dots, x_n) \mid x_1 = \dots = x_m = 0\} \cong \mathbb{R}^n$$

Obs.: $n = 3$, $m = 2$, $r = 1$

$$\{(x, y, z) \mid z = 0\} = \{(x, y, 0)\}$$



$$\{(x, y, z) \mid x = y = 0\} = \{(0, 0, z)\}$$

Obs.: Se $x = \sum_{i=1}^{m} \alpha_i(x) v_i + \sum_{j=1}^{r} \beta_j(x) w_j$

então:

(i) $x \in V \iff \beta_j(x) = 0 \ \forall j$

(ii) $x \in W \iff \alpha_i(x) = 0 \ \forall i$

Isso motiva a seguinte def.:

$T(x) = (\alpha_1(x), \dots, \alpha_m(x), \beta_1(x), \dots, \beta_r(x))$

$\hookrightarrow T(V) = \mathbb{R}^m, \ T(W) = \mathbb{R}^n$

$\vdash$ T é linear bijetora

$\vdash$ T é linear

Exercício

I - T é bijetora

T é bijetora $\Longleftrightarrow$ $\dim \ker T = 0$

TNI

$\dim \mathbb{R}^n = \dim \mathbb{R}^n$

$\ker T = \{x \mid \alpha_1(x) = \cdots = \alpha_m(x) = \beta_1(x) = \cdots$
$= \beta_r(x) = 0\} = \{0\}$ ✓ ✓

$$x = \sum_{i=1}^{m} \alpha_i(x) v_i + \sum_{j=1}^{r} \beta_j(x) w_j$$

$\alpha_i \to 0$, $\beta_j \to 0$

$\square$

Ex. 15)

V, W esp. vet. de dim. $< \infty$

$T: V \to W$ é linear

(i) $\dim \ker T \leq \dim V$ (sub.)

(ii) $\dim T(V) \leq \dim W$ (sub.)

(iii) $\dim \ker(T) + \dim T(V) = \dim V$ (TNI)

(iv) $\dim T(V) \leq \dim V$

Caso 1: $\dim V = \dim W$

T inj. $\iff \dim \ker T \underset{(iii)}{=} \dim V - \dim T(V) \underset{c1}{=} \dim W - \dim T(V) = 0 \iff$

$\dim W = \dim T(V) \iff T(V) = W$

$\iff T$ é sobrej.

Caso 2: $\dim V < \dim W$

T é sobrej. $\Rightarrow \dim T(V) = \dim W$

$\Rightarrow \dim \ker T = \dim V - \dim T(V) \overset{(iii)}{=}$

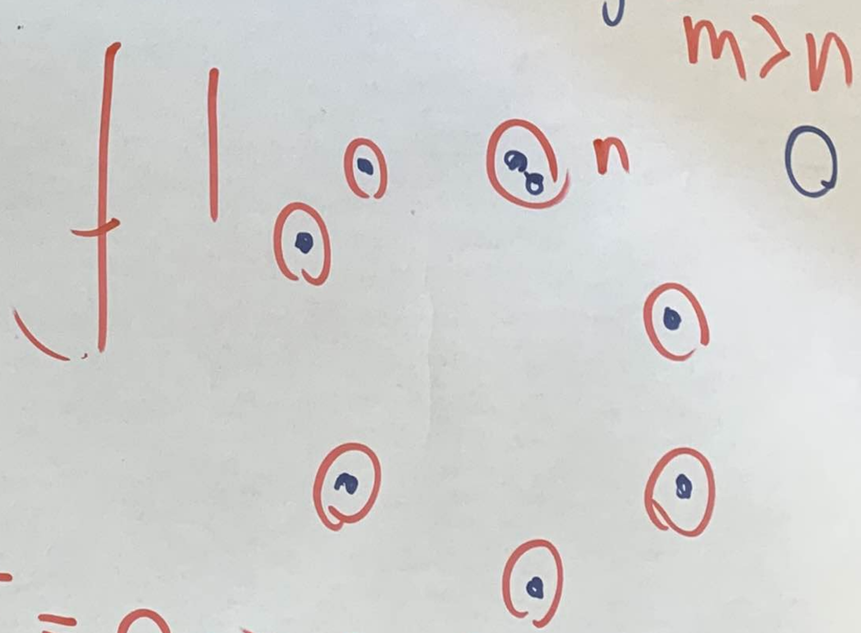
$= \dim V - \dim W \underset{C2}{<} 0 \Rightarrow$

Conclusão: T não pode ser sobrej.

Caso 3: $\dim V > \dim W$

$\vdash$ T não pode ser inj.

(Exo.)

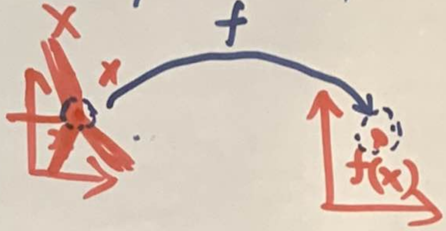


Ideia:

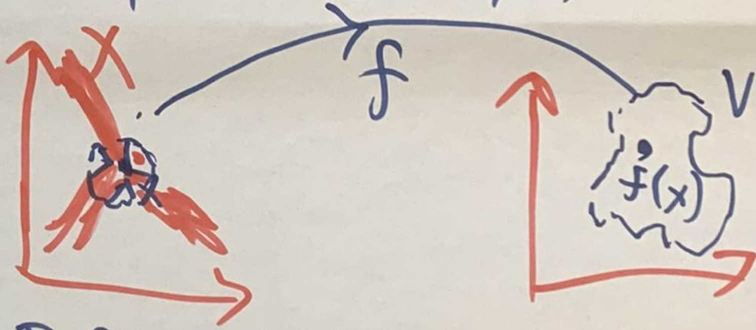
$\dim \ker T = 0 \Rightarrow \dim T(V) > \dim W$

Ex. 5) $f: X \subset \mathbb{R}^p \to \mathbb{R}^q$

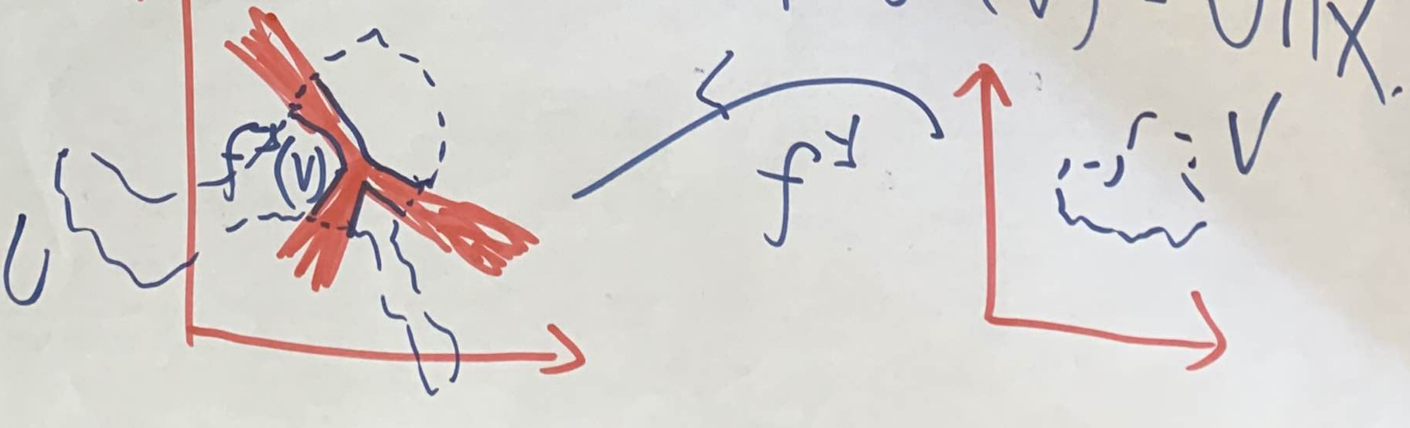
(i) f é contínua: $\forall x \in X \ \forall \varepsilon > 0$ $\exists \delta > 0$ t.q., se $y \in X$ e $\|x - y\| < \delta$, então $\|f(x) - f(y)\| < \varepsilon$;



(ii) $\forall x \in X \ \forall V \subset \mathbb{R}^q$ viz. ab. de $f(x)$ $\exists \delta > 0$ t.q. $f(B(x,\delta) \cap X) \subset$ $\subset V$;



(iii) $\forall V \subset \mathbb{R}^q$ aberto, existe $U \subset \mathbb{R}^p$ ab. t.q. $f^{-1}(V) = U \cap X$.



A $\Longrightarrow$ B

~~"todo ovo é amarelo"~~ 2

"não existe ovo"

1 "Se Sócrates era grego, então

"Se Sócrates era homem, então Sócrates morreu" 3

4 Sócrates não está vivo"

Sócrates é grego $\Longrightarrow$ Sócrates

era homem

todos os gregos são homens

$\Downarrow$

Sócrates morreu $\Longrightarrow$ Sócrates não está vivo

C

TODOS QUE MORREM NÃO ESTÃO VIVOS

$(i) \Leftrightarrow (ii) \Longleftrightarrow (iii)$

$$\begin{tikzcd} (i) \arrow[r, Rightarrow] & (ii) \arrow[r, Rightarrow] & (iii) \arrow[ll, Rightarrow, bend left] \end{tikzcd}$$

[$(i) \Rightarrow (ii)$] $x \in X$, V viz. ab. de $f(x)$ arbitrárias

$f(x) \in V$, V ab. $\Rightarrow \exists \varepsilon > 0$ t.q.

$B(f(x), \varepsilon) \subset V \Rightarrow \forall x, \forall \varepsilon$

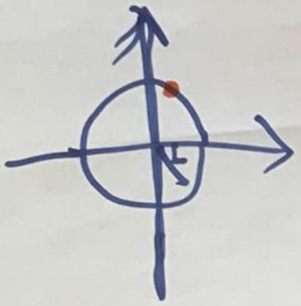
~~$\Rightarrow \exists \delta > 0$ t.q.~~ $\Rightarrow \exists \delta > 0$ t.q.

$\forall y \in X$, $\underbrace{\|y - x\| < \delta}_{y \in B(x,\delta) \cap X} \Rightarrow \underbrace{\|f(x) - f(y)\| < \varepsilon}_{f(y) \in B(f(x), \varepsilon)}$

$y \in B(x,\delta)\cap X \Rightarrow f(y) \in B(f(x),$
$\varepsilon) \subset V$

$\hookrightarrow f(B(x,\delta)\cap X) \subset V$ □

$\left(f(t) = \frac{1}{t} : \underbrace{\mathbb{R}-\{0\}}_{X} \to \mathbb{R} \right.$



$f: S^1 \to [0,2\pi)$

$(x_1,x_2) \mapsto \theta$ t.q.

$x_1 = \cos\theta$

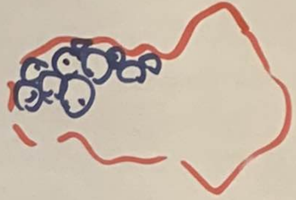
$x_2 = \operatorname{sen}\theta$

[(ii) $\Rightarrow$ (iii)] $V \subset \mathbb{R}^q$ aberto qualquer

$\forall x \in f^{-1}(V)$:

por (ii), $\exists\, \delta_x > 0$ t.q.

$f(B(x,\delta_x)\cap X) \subset V \Rightarrow$

$B(x,\delta_x)\cap X \subset f^{-1}(V)$ 

$U = \bigcup_{x\in f^{-1}(V)} B(x,\delta_x)$ é aberto e:

(a) $U\cap X = \left[\bigcup_{x\in f^{-1}(V)} B(x,\delta_x)\right]\cap X =$

$= \bigcup_{x\in f^{-1}(V)} \underbrace{B(x,\delta_x)\cap X}_{\subset f^{-1}(V)} \subset f^{-1}(V)$

(b) $\forall x\in f^{-1}(V)$, $x \in B(x,\delta_x)\cap X$

$\subset \bigcup_{y\in f^{-1}(V)} B(y,\delta_y)\cap X = U\cap X$

$\hookrightarrow f^{-1}(V)\subset U\cap X$