PGF5003: Classical Electrodynamics I
Problem Set 3

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(Due to May 11, 2021)

Guidelines: *write down the most relevant passages in your calculations, not only the final results. Do not forget to write the mathematical expressions that you are using in order to solve the questions. We strongly recommended the use of the International System of Units.*

1 Question (1 point)

Given the following magnetic field:

$$\mathbf{B}(\mathbf{r}) = B_0\hat{\mathbf{z}} + \hat{\mathbf{z}} \times \vec{\nabla} f(\mathbf{r}) \tag{1}$$

a) *Show that this satisfies:* $\vec{\nabla} \cdot \mathbf{B} = 0$;

b) *Find the equation(s) for the field lines and show that, performing the following change of variables:* $x \to q$ *(generalized coordinate),* $y \to p$ *(conjugate momenta),* $z \to t$ *and considering* $f = -B_0 H$ *(where* H *is a Hamiltonian), this is equivalent to*

$$\dot{q} = \frac{\partial H}{\partial p} \text{ and } \dot{p} = -\frac{\partial H}{\partial q}. \tag{2}$$

Hint: *Use a scalar parameter* λ *to ensure that every differential element of magnetic field line* $d\mathbf{s}$ *is parallel to* $\mathbf{B}(\mathbf{r})$ *itself:* $d\mathbf{s} = \lambda \mathbf{B}$.

c) *What can you say about the magnetic field lines doing the analogy with the above Hamilton's equations?*

1.1 Solution

a) We can compute

$$\vec{\nabla} \cdot \mathbf{B} = \vec{\nabla} \cdot \left[B_0\hat{\mathbf{z}} + \hat{\mathbf{z}} \times \vec{\nabla} f(\mathbf{r}) \right] = 0 - \partial_y \partial_x f \hat{\mathbf{x}} + \partial_x \partial_y f \hat{\mathbf{y}} = 0 \,\square. \tag{3}$$

b) Considering the hint and the changes in question, we can write

$$
\begin{aligned}
d\mathbf{s} &= \lambda \mathbf{B} \\
dx = \lambda B_x, dy &= \lambda B_y, dz = \lambda B_z \\
\frac{dx}{dz} = \frac{B_x}{B_z} = -\frac{\partial_y f}{B_0} &\Rightarrow \frac{dq}{dt} = \frac{B_0 \partial_p H}{B_0} \Rightarrow \dot{q} = \frac{\partial H}{\partial p} \\
\frac{dy}{dz} = \frac{B_y}{B_z} = \frac{\partial_x f}{B_0} &\Rightarrow \frac{dp}{dt} = -\frac{B_0 \partial_q H}{B_0} \Rightarrow \dot{p} = \frac{\partial H}{\partial q}
\end{aligned} \tag{4}
$$

c) The magnetic field lines are similar to, following this analogy, "time-dependent" trajectories in (p, q) phase space of a "particle" with Hamiltonian $H = -f/B_0$. Since most Hamiltonian are non-integrable and produce *chaotic trajectories*, the magnetic field line configuration will be very complex!

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2 Question (1 point)

Show that the force $\mathbf{F}$ *on a magnetic dipole* $\mathbf{m}$ *exerted by an arbitrary magnetic field* $\mathbf{B}$ *is given by*

$$
\mathbf{F} = \vec{\nabla} \left(\mathbf{m} \cdot \mathbf{B} \right). \tag{5}
$$

Hint: *Given the magnetic field of a point magnetic dipole* $\mathbf{B}$ *at* $\mathbf{r}_0$

$$
\mathbf{B}(\mathbf{r}) = \mu_0 \left[\mathbf{m} \delta(\mathbf{r} - \mathbf{r}_0) - \vec{\nabla} \frac{1}{4\pi} \frac{\mathbf{m} \cdot (\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^3} \right]. \tag{6}
$$

find its current density $\mathbf{J}$.

2.1 Solution

Following the hint, the magnetic field of a point magnetic dipole $\mathbf{m}$ at $\mathbf{r}_0$ is

$$
\mathbf{B}(\mathbf{r}) = \mu_0 \left[\mathbf{m} \delta(\mathbf{r} - \mathbf{r}_0) - \vec{\nabla} \frac{1}{4\pi} \frac{\mathbf{m} \cdot (\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^3} \right]. \tag{7}
$$

This leads to the current density

$$
\mathbf{j}(\mathbf{r}) = \frac{1}{\mu_0} \vec{\nabla} \times \mathbf{B} = \vec{\nabla} \times \left[\mathbf{m} \delta(\mathbf{r} - \mathbf{r}_0) \right] = -\mathbf{m} \times \vec{\nabla} \delta(\mathbf{r} - \mathbf{r}_0). \tag{8}
$$

As we want to compute the force $\mathbf{F}$, we have

$$
\begin{aligned}
\mathbf{F} &= \int d^3 r \mathbf{j} \times \mathbf{B} = \int d^3 r \mathbf{B} \times \left[\mathbf{m} \times \vec{\nabla} \delta(\mathbf{r} - \mathbf{r}_0) \right] \\
&= \int d^3 r \left[\mathbf{B} \cdot \vec{\nabla} \delta(\mathbf{r} - \mathbf{r}_0) \mathbf{m} - (\mathbf{m} \cdot \mathbf{B}) \vec{\nabla} \delta(\mathbf{r} - \mathbf{r}_0) \right] \\
&= -\int d^3 r \mathbf{m} \delta(\mathbf{r} - \mathbf{r}_0) \vec{\nabla} \cdot \mathbf{B} + \int d^3 r \delta(\mathbf{r} - \mathbf{r}_0) \vec{\nabla} (\mathbf{m} \cdot \mathbf{B}) \\
&= \vec{\nabla}_0 \left(\mathbf{m} \cdot \mathbf{B}(\mathbf{r}_0) \right) \; \square.
\end{aligned} \tag{9}
$$

where I have used $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$, from the first to second line, integrated by parts from the third to the fourth line and used that there is no magnetic charge ($\vec{\nabla} \cdot \mathbf{B} = 0$).

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3 Question (1 point)

A right-handed circular solenoid of finite length L and radius a has N turns per unit length and carries a current I. Show that the magnetic induction on the cylinder axis in the limit $NL \to \infty$ is

$$B_z = \frac{\mu_0 N I}{2} \left(\cos\theta_1 + \cos\theta_2\right) \tag{10}$$

where the angles are defined in Figure 1.

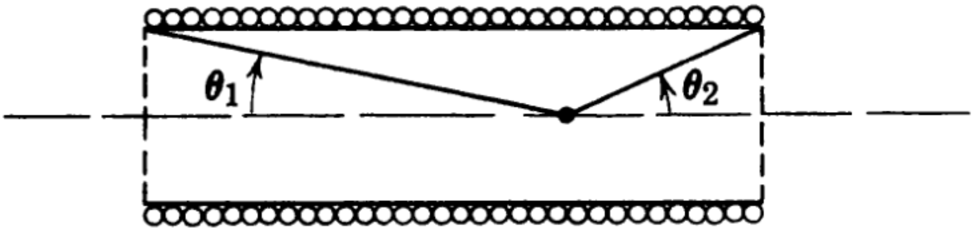

Figure 1: Figure for the question 3.

3.1 Solution

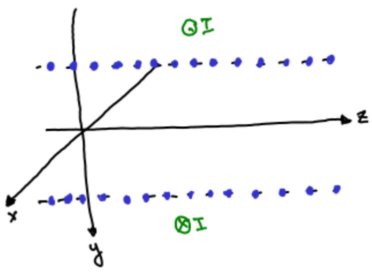

Figure 2: Visualizing the question 3.

We can compute it using **Biot-Savart**

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I d\ell \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}. \tag{11}$$

Let's explain the objects that we are going to use: considering the axis of the solenoid as z, pointing positively to the right, we could see x going out the page and y going down the page. Then, the elements are

$$d\ell = R d\theta \hat{\theta} \tag{12}$$
$$\mathbf{r} = z\hat{z} \tag{13}$$
$$\mathbf{r}' = x'\hat{x} + y'\hat{y} + z'\hat{z} = R\cos\theta\hat{x} + R\sin\theta\hat{y} + z'\hat{z} \tag{14}$$
$$\mathbf{r} - \mathbf{r}' = (z - z')\hat{z} - R(\cos\theta\hat{x} + \sin\theta\hat{y}) \tag{15}$$
$$|\mathbf{r} - \mathbf{r}'| = \sqrt{(z - z')^2 + R^2} \tag{16}$$
$$\hat{\theta} = \cos\theta\hat{y} - \sin\theta\hat{x} \tag{17}$$
$$d\ell \times (\mathbf{r} - \mathbf{r}') = R d\theta \left[(z - z')\cos\theta\hat{x} + (z - z')\sin\theta\hat{y} + R\hat{z}\right]. \tag{18}$$

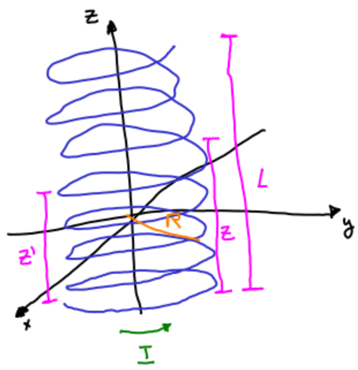

Figure 3: Solving the question 3.

Putting all this together and considering that an element of length dz' contain $dI = INdz'$, we can write

$$d\mathbf{B} = \frac{\mu_0 INdz'}{4\pi}\frac{Rd\theta\,[(z-z')\cos\theta\hat{x} + (z-z')\sin\theta\hat{y} + R\hat{z}]}{[(z-z')^2 + R^2]^{3/2}}. \tag{19}$$

The magnetic field is given when we integrate this expression. Then,

$$\mathbf{B} = \int_0^{2\pi} d\theta \int_0^L dz' \frac{\mu_0 IN}{4\pi}\frac{R\,[(z-z')\cos\theta\hat{x} + (z-z')\sin\theta\hat{y} + R\hat{z}]}{[(z-z')^2 + R^2]^{3/2}}. \tag{20}$$

But, as the integrations

$$\int_0^{2\pi} d\theta \sin\theta = \int_0^{2\pi} d\theta \cos\theta = 0, \tag{21}$$

we only need to deal with the integral in z'. In this way,

$$\mathbf{B} = \frac{2\pi\mu_0 INR^2}{4\pi}\int_0^L \frac{dz'}{[(z-z')^2 + R^2]^{3/2}}\hat{z}. \tag{22}$$

We can perform this integral, changing variables

$$u = z - z', du = -dz'$$
$$z_1' = 0 \Rightarrow u_1 = z \tag{23}$$
$$z_2' = L \Rightarrow u_2 = z - L. \tag{24}$$

Thus,

$$\mathbf{B} = \frac{\mu_0 INR^2}{2}\int_{z-L}^{z} \frac{du}{[u^2 + R^2]^{3/2}}\hat{z}. \tag{25}$$

Performing now other change of variables like

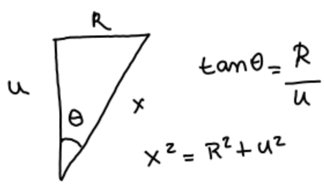

Figure 4: Changing variables.

$$x^2 = u^2 + R^2 \tag{26}$$
$$du = Rd\left(\frac{\cos\theta}{\sin\theta}\right) = -\frac{Rd\theta}{\sin^2\theta} \tag{27}$$
$$x = \frac{R}{\sin\theta} \tag{28}$$
$$u = x\cos\theta, \tag{29}$$

we have

$$\mathbf{B} = \frac{\mu_0 INR^2}{2}\left(\frac{-1}{R^2}\right)\int_{\theta=z-L}^{\theta=z} d\theta \sin\theta \hat{z} \tag{30}$$
$$= \frac{\mu_0 INR^2}{2}\frac{\cos\theta}{R^2}|_{z-L}^{L}\hat{z} = \frac{\mu_0 INR^2}{4\pi}\frac{1}{R^2}\frac{u}{\sqrt{u^2+R^2}}|_{z-L}^{L}\hat{z} \tag{31}$$
$$= \frac{\mu_0 IN}{2}\left[\frac{L}{\sqrt{L^2+R^2}} - \frac{(z-L)}{\sqrt{(z-L)^2+R^2}}\right]\hat{z}. \tag{32}$$

Here we can notice that

$$\cos\theta_1 = \frac{L}{\sqrt{L^2+R^2}} \tag{33}$$
$$\cos(\pi-\theta_2) = \frac{(z-L)}{\sqrt{(z-L)^2+R^2}} \tag{34}$$
$$\cos(\pi-\theta_2) = -\cos\theta_2. \tag{35}$$

Therefore, we finally arrive at

$$B_z = \frac{\mu_0 NI}{2}(\cos\theta_1 + \cos\theta_2)\,\square. \tag{36}$$

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4 Question (1 point)

A cylindrical conductor of radius a has a cylindrical hole of radius b cored parallel to, and centered a distance d from, the cylinder axis ($d + b < a$). The current density is uniform throughout the remaining metal of the cylinder and is parallel to the axis. Use Ampere's law and the principle of linear superposition to find the magnitude and direction of the magnetic flux density in the hole.

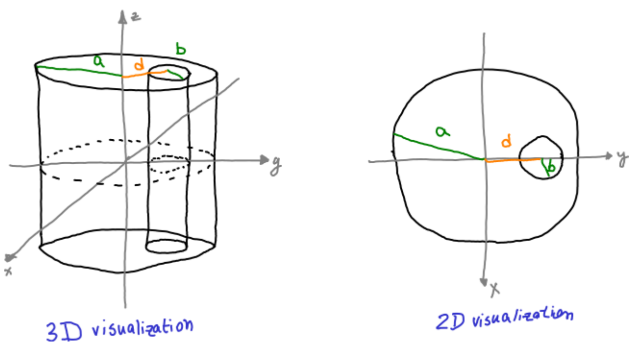

Figure 5: Figure for question 4.

4.1 Solution

First, let's say, for example, that we are looking to the system from above, i.e., we see a plane cut of the cylinder, where z comes in our direction. If a uniform current density J_0 flows in the positive z direction, for superposition you can think in a cylinder where it flows in this way everywhere plus another little cylinder (the hole) with a current density J_0 flowing in the negative z direction.

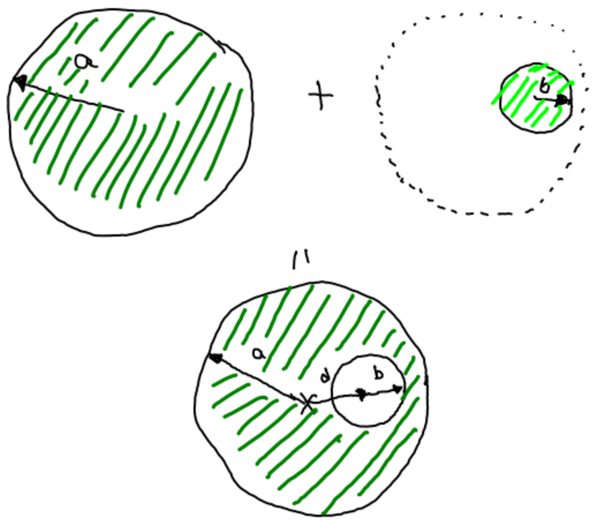

Figure 6: Visualizing question 4.

Due the **Ampere's Law**

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S d\mathbf{S} \cdot \mathbf{J}, \tag{37}$$

if we take the Amperian circuit is at some radius r inside the cylinder with no hole we can write

$$\begin{aligned} B 2\pi r &= \mu_0 J_0 \pi r^2 \\ \mathbf{B}_{cylinder} &= \frac{\mu_0 J_0 r}{2} \hat{\phi} = \frac{\mu_0 J_0}{2} \left(-y\hat{x} + x\hat{y} \right). \end{aligned} \tag{38}$$

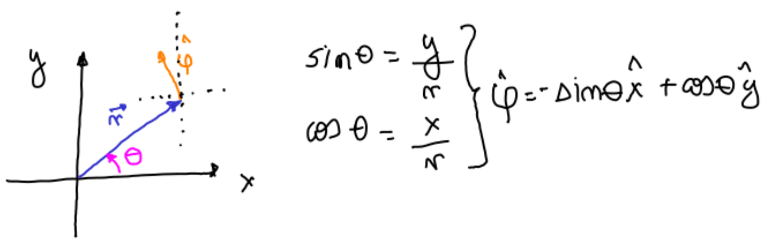

Figure 7: Change from $\hat{\phi}$ to (x, y).

Using the same idea, but now for the hole, we have the same expression for its field, considering that the current if flowing in the opposite direction. However, considering that the hole is with its center on $(x, y) = (d, 0)$, it is

$$\mathbf{B}_{hole} = -\frac{\mu_0 J_0}{2} \left[-y\hat{x} + (x - d)\hat{y} \right]. \tag{39}$$

By superposition, the total magnetic field is the sum of both ones

$$\mathbf{B} = \mathbf{B}_{cylinder} + \mathbf{B}_{hole} \tag{40}$$
$$= \frac{\mu_0 J_0}{2}(-y\hat{x} + x\hat{y}) - \frac{\mu_0 J_0}{2}\left[-y\hat{x} + (x-d)\hat{y}\right]$$
$$= \frac{\mu_0 J d}{2}\hat{y}. \tag{41}$$

If considering according to the figure it is $(x, y) = (0, d)$, it is

$$\mathbf{B}_{hole} = -\frac{\mu_0 J_0}{2}\left[-(y-d)\hat{x} + x\hat{y}\right]. \tag{42}$$

By superposition, the total magnetic field is the sum of both ones

$$\mathbf{B} = \mathbf{B}_{cylinder} + \mathbf{B}_{hole} \tag{43}$$
$$= \frac{\mu_0 J_0}{2}(-y\hat{x} + x\hat{y}) - \frac{\mu_0 J_0}{2}\left[-(y-d)\hat{x} + x\hat{y}\right]$$
$$= -\frac{\mu_0 J d}{2}\hat{x}. \tag{44}$$

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5 Question (2 points)

A small loop # 1 of wire (radius a) is held at a distance z above the center of a large loop # 2 (radius b), as shown in Figure 2. The planes of the two loops are parallel and perpendicular to the common axis.

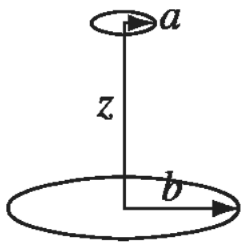

Figure 8: Figure for the question 5.

a) *First, find the magnetic field of a loop in its axis (here consider it as z, as indicated in Figure). Second, suppose current I flows in the big loop. Find the flux through the little loop.* **Hint:** *the little loop is so small that you may consider the field of the big loop to be essentially constant.*

b) *Suppose current I flows in the little loop. Find the flux through the big loop.* **Hint:** *The little loop is so small that you may treat it as a magnetic dipole. Consider its magnetic field as*

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right). \tag{45}$$

c) *Find the mutual inductance and confirm that $M_{12} = M_{21}$.*

5.1 Solution

a)

i) We can deduce the magnetic field using **Biot-Savart law**

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{Id\mathbf{s}\times\hat{r}}{r^2}. \tag{46}$$

As $d\mathbf{s} \perp \hat{r}$ (because $\hat{r}$ points to a point in $\hat{z}$), $|\mathbf{s}\times\hat{r}| = ds$, then

$$dB = \frac{\mu_0}{4\pi}\frac{Ids}{(z^2+R^2)}, \tag{47}$$

where R is the radius of the loop. We only have components in z because the other cancel. Then, we only needs to integrate in z: $dB_z = dB\cos\theta$, where $\cos\theta = R/\sqrt{z^2+R^2}$, in the way that

$$\begin{aligned} B_z &= \oint dB\cos\theta = \oint \frac{\mu_0}{4\pi}\frac{Ids}{(z^2+R^2)}\frac{R}{\sqrt{z^2+R^2}} \\ &= \frac{\mu_0 IR}{4\pi(z^2+R^2)^{3/2}}\int_0^{2\pi} d\phi R = \frac{\mu_0 IR^2}{2(z^2+R^2)^{3/2}}. \end{aligned} \tag{48}$$

Which is a general magnetic field for a loop on its axis.

ii) The flux through loop # 1 is given by

$$\Phi_1 = \oint_S \mathbf{B}_2 d\mathbf{S}_1. \tag{49}$$

According to the hint, the field of the larger loop is constant in the region of the smaller loop, then, using the magnetic field computed before for $R = b$ gives

$$\Phi_1 = \oint_S \mathbf{B}_2 d\mathbf{S}_1 = B_2\pi a^2 = \frac{\mu_0\pi I a^2 b^2}{2(z^2+b^2)^{3/2}}. \tag{50}$$

b) The flux through loop # 2 is given by

$$\Phi_2 = \oint_S \mathbf{B}_1 d\mathbf{S}_2. \tag{51}$$

We can use the hint remembering that $m = I\pi a^2$. Moreover, this integral may be taken over any surface that is bounded by the loop # 2. Choosing the spherical "cap" centered at the little loop in the way that $d\mathbf{S}_2 = r^2\sin\theta d\theta d\phi\hat{r}$, we have

$$\begin{aligned} \Phi_2 &= \oint_S \mathbf{B}_1 d\mathbf{S}_2 \\ &= \int_0^{\theta'} d\theta\sin\theta\int_0^{2\pi} d\phi r^2\hat{r}\cdot\frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\hat{r}+\sin\theta\hat{\theta}\right) \\ &= 2\pi\int_0^{\theta'} d\theta\sin\theta\frac{\mu_0 m}{4\pi r}(2\cos\theta) \\ &= \frac{\mu_0 I\pi a^2}{r}\int_0^{\theta'} d\theta\sin\theta\cos\theta. \end{aligned} \tag{52}$$

Writing r and θ' in terms of the geometry of the problem

$$\begin{aligned} r &= \sqrt{b^2+z^2} \\ \sin\theta' &= \frac{b}{r} = \frac{b}{\sqrt{b^2+z^2}}, \end{aligned} \tag{53}$$

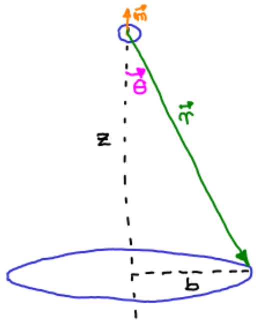

Figure 9: Solving question 5.

we end up with

$$\Phi_2 = \frac{\mu_0 I \pi a^2}{\sqrt{b^2+z^2}} \int_0^{\theta'} d\theta \sin\theta\cos\theta = \frac{\mu_0 I \pi a^2}{\sqrt{b^2+z^2}} \left(\frac{-\cos^2\theta}{2}\right)_0^{\theta'}$$
$$= \frac{\mu_0 I \pi a^2}{\sqrt{b^2+z^2}} \frac{\sin^2\theta'}{2} = \frac{\mu_0 I \pi a^2}{2\sqrt{b^2+z^2}} \left(\frac{b}{\sqrt{b^2+z^2}}\right)^2$$
$$= \frac{\mu_0 I \pi a^2 b^2}{2(b^2+z^2)^{3/2}}. \tag{54}$$

c) The mutual inductance of each loop is given by

$$\Phi_1 = M_{12} I_2 \text{ and } \Phi_2 = M_{21} I_1. \tag{55}$$

As $I_1 = I_2 = I$, we have

$$M_{12} = \frac{\Phi_1}{I} = \frac{\mu_0 \pi a^2 b^2}{2(z^2+b^2)^{3/2}} \tag{56}$$
$$M_{21} = \frac{\Phi_2}{I} = \frac{\mu_0 \pi a^2 b^2}{2(z^2+b^2)^{3/2}}, \tag{57}$$

which are clearly the same!

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6 Question (2 point)

A sphere of linear magnetic material (with permeability μ) is placed in an otherwise uniform magnetic field $\mathbf{B}_0$. Find the new field inside the sphere. **Hint:** *solve this problem using separation of variables.*

6.1 Solution

As we do not have any mention to a free current in the sphere, we can write

$$\mathbf{H} = -\vec{\nabla}\Phi$$
$$\nabla^2\Phi = 0, \tag{58}$$

for the field $\mathbf{H}$ and scalar potential Φ. Because we have a spherical geometry, let's do it in spherical coordinates:

$$\Phi(r,\theta,\phi) = R(r)\Theta(\theta)\psi(\phi)$$

$$\begin{aligned}
\frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin\theta}\left(\frac{\partial^2}{\partial\phi^2}\right)\right]\Phi = 0\\
\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\psi}\frac{d^2\psi}{d\phi^2} = 0\\
\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = m^2\\
\frac{1}{\psi}\frac{d^2\psi}{d\phi^2} = -m^2
\end{aligned}$$

But, as we have azymutal symmetry, $m = 0$, i.e., the dependence in ϕ is zero. Then, we can write

$$\begin{aligned}
\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \lambda R = 0,\\
\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \lambda\Theta = 0
\end{aligned}$$

and the general solution is given using Legendre polynomials $P_\ell(\cos\theta)$ as

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{C_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta). \tag{59}$$

Inside the sphere, to not diverge the solution, we have

$$\Phi_{ins}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta). \tag{60}$$

Before write the general expression for the potential outside, let's remember that, for large r we want to have the uniform magnetic field $\mathbf{B}_0$. In order to have this we can follow

$$\mathbf{B}_0 = B_0\hat{z} \Rightarrow \mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \frac{B_0}{\mu_0}\hat{z} \tag{61}$$

and hence, the potential stays

$$\Phi(r\to\infty) = \frac{-B_0}{\mu_0}z = -\frac{B_0 r\cos\theta}{\mu_0}. \tag{62}$$

Just having this in mind we can write the potential outside (to not diverge) as

$$\begin{aligned}
\Phi_{out}(r,\theta) &= \Phi(r\to\infty) + \sum_{\ell=0}^{\infty}\frac{C_\ell}{r^{\ell+1}}P_\ell(\cos\theta)\\
&= -\frac{B_0 r\cos\theta}{\mu_0} + \sum_{\ell=0}^{\infty}\frac{C_\ell}{r^{\ell+1}}P_\ell(\cos\theta).
\end{aligned} \tag{63}$$

Imposing now the boundary conditions:

- $\Phi_{ins}(r = R) = \Phi_{out}(r = R)$

$$\begin{aligned} \Phi_{ins}(r = R) &= \Phi_{out}(r = R) \\ \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) &= -\frac{B_0 R \cos\theta}{\mu_0} + \sum_{\ell=0}^{\infty} \frac{C_\ell}{R^{\ell+1}} P_\ell(\cos\theta). \end{aligned} \tag{64}$$

Here we see that $\cos\theta = P_1(\cos\theta)$, which gives

$$\begin{aligned} A_1 R &= -\frac{B_0}{\mu_0} R + \frac{C_1}{R^2}, \ell = 1 \\ A_\ell R^\ell &= \frac{C_\ell}{R^{\ell+1}}, \ell \neq 1. \end{aligned} \tag{65}$$

- $\mu \partial_r \Phi_{ins}(r = R) = \mu_0 \partial_r \Phi_{out}(r = R)$

$$\begin{aligned} \mu_0 \partial_r \Phi_{out}(r = R) &= \mu_0 \left[-\frac{B_0}{\mu_0} \cos\theta - \sum_{\ell=0}^{\infty} \frac{(\ell+1) C_\ell}{R^{\ell+2}} P_\ell(\cos\theta) \right] \\ &= -B_0 \cos\theta - \sum_{\ell=0}^{\infty} \frac{\mu_0 (\ell+1) C_\ell}{R^{\ell+2}} P_\ell(\cos\theta) \\ \mu \partial_r \Phi_{ins}(r = R) &= \sum_{\ell=0}^{\infty} \mu \ell A_\ell R^{\ell-1} P_\ell(\cos\theta) \end{aligned} \tag{66}$$

Once again this gives

$$\begin{aligned} -B_0 - \frac{2\mu_0 C_1}{R^3} &= \mu A_1, \ell = 1 \\ -\frac{\mu_0 (\ell+1) C_\ell}{R^{\ell+2}} &= \mu \ell A_\ell R^{\ell-1}, \ell \neq 1. \end{aligned} \tag{67}$$

Now we need to match the results from both boundary conditions:

- $\ell \neq 1$:

$$\begin{aligned} C_\ell &= A_\ell R^{2\ell+1} \\ R^{\ell-1} A_\ell \left[\mu \ell + \mu_0 (\ell+1) \right] &= 0 \\ A_\ell &= 0. \end{aligned} \tag{68}$$

- $\ell = 1$:

$$\begin{aligned} A_1 R &= -\frac{B_0 R}{\mu_0} + \frac{B_1}{R^2} \\ B_0 + 2\mu_0 \frac{B_1}{R^3} + \mu A_1 &= 0 \\ A_1 &= \frac{-3B_0}{(2\mu_0 + \mu)} \end{aligned} \tag{69}$$

$$C_1 = \frac{R^3 B_0 (\mu - \mu_0)}{\mu_0 (2\mu_0 + \mu)}. \tag{70}$$

This result leads to the potential inside the sphere as

$$\Phi_{ins}(r,\theta) = \frac{-3B_0}{(2\mu_0+\mu)} r\cos\theta = \frac{-3B_0 z}{(2\mu_0+\mu)}. \tag{71}$$

Because $\mathbf{H}_{ins} = -\vec{\nabla}\Phi_{ins}$, we get

$$\mathbf{H}_{ins} = \frac{3B_0}{(2\mu_0+\mu)}\hat{z} = \frac{3\mathbf{B}_0}{(2\mu_0+\mu)} \tag{72}$$

and therefore

$$\mathbf{B}_{ins} = \mu\mathbf{H}_{ins} = \frac{3\mu\mathbf{B}_0}{(2\mu_0+\mu)}. \tag{73}$$

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7 Question (2 points)

A current distribution $\mathbf{J}(\mathbf{x})$ exists in a medium of unit relative permeability adjacent to a semi-infinite slab of material having relative permeability μ and filling the half-space, $z<0$.

a) *Show that for $z>0$ the magnetic induction can be calculated by replacing the medium of permeability μ by an image current distribution $J^\star$, with components*

$$\mathbf{J}^\star = \frac{(\mu-1)}{(\mu+1)}J_x(x,y,-z)\hat{x} + \frac{(\mu-1)}{(\mu+1)}J_y(x,y,-z)\hat{y} - \frac{(\mu-1)}{(\mu+1)}J_z(x,y,-z)\hat{z} \tag{74}$$

b) *Show that for $z<0$ the magnetic induction appears to be due to a current distribution $\frac{2\mu}{(\mu+1)}\mathbf{J}$ in a medium of unit relative permeability.*

7.1 Solution

a) Using the suggested **method of images**, we replace the effects of the actual currents on the material interface with an image current deep within the magnetic material. The real current distribution J is known but its image needs to be determined. Because the "mirror surface" is a flat plane, we can assume that each piece i of current in J at (x,y,z) will be mirrored by a piece i of current in $J^\star$ at the opposite z, i.e., $(x,y,-z)$:

$$J_i^\star(x,y,z) = J_i(x,y,-z). \tag{75}$$

As the magnetic field is given by

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int dV' \frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{76}$$

we can write the magnetic field in all space as

$$\begin{aligned}\mathbf{B}_{z>0}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int dV' \frac{[\mathbf{J}(\mathbf{r}')+\mathbf{J}^\star(\mathbf{r}')]\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \\ \mathbf{B}_{z>0}(\mathbf{r}) &= \frac{\mu_0\mu}{4\pi}\int dV' \frac{\alpha\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\end{aligned} \tag{77}$$

where I designate by $\alpha\mathbf{J}(\mathbf{r}')$ the current bellow the plane because to match the boundary conditions.

These conditions are then:

$$\vec{\nabla}\cdot\mathbf{B}=0\Rightarrow(\mathbf{B}_2-\mathbf{B}_1)\cdot\hat{n}|_{z=0}=0 \tag{78}$$

$$\vec{\nabla}\times\mathbf{H}=0\Rightarrow\hat{n}\times(\mathbf{H}_2-\mathbf{H}_1)|_{z=0}=0. \tag{79}$$

Using the first one we have

$$\begin{aligned}
\mathbf{B}_{z>0}(\mathbf{r})\cdot\hat{z}|_{z=0} &= \mathbf{B}_{z<0}(\mathbf{r})\cdot\hat{z}|_{z=0}\\
\frac{\mu_0}{4\pi}\int dV'\frac{[\mathbf{J}(\mathbf{r}')+\mathbf{J}^\star(\mathbf{r}')]\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\cdot\hat{z}|_{z=0} &= \frac{\mu_0\mu}{4\pi}\int dV'\frac{\alpha\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\cdot\hat{z}|_{z=0}\\
[\mathbf{J}(\mathbf{r}')+\mathbf{J}^\star(\mathbf{r}')]\times(\mathbf{r}-\mathbf{r}')\cdot\hat{z}|_{z=0} &= \alpha\mu\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')\cdot\hat{z}|_{z=0}.
\end{aligned} \tag{80}$$

The above computation is much easier if we break it on transverse and z components. Besides, I will use the identity: $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$:

$$\begin{aligned}
[(J_z\hat{z}+\mathbf{J}_t)+(J_z^\star\hat{z}+\mathbf{J}_t^\star)]\times(\mathbf{r}-\mathbf{r}')\cdot\hat{z}|_{z=0} &= \alpha\mu(J_z\hat{z}+\mathbf{J}_t)\times(\mathbf{r}-\mathbf{r}')\cdot\hat{z}|_{z=0}\\
\{\hat{z}\times[(J_z\hat{z}+\mathbf{J}_t)+(J_z^\star\hat{z}+\mathbf{J}_t^\star)]\}\cdot(\mathbf{r}-\mathbf{r}')|_{z=0} &= \mu\alpha\{\hat{z}\times[(J_z\hat{z}+\mathbf{J}_t)]\cdot(\mathbf{r}-\mathbf{r}')\}|_{z=0}\\
\{[\hat{z}\times\mathbf{J}_t+\hat{z}\times\mathbf{J}_t^\star]\cdot(\mathbf{r}-\mathbf{r}')\}|_{z=0} &= \mu\alpha\{[\hat{z}\times\mathbf{J}_t]\cdot(\mathbf{r}-\mathbf{r}')\}|_{z=0}\\
\left\{\left[J_x\hat{y}-J_y\hat{x}+J_x^\star\hat{y}-J_y^\star\hat{x}\right]\cdot(\mathbf{r}-\mathbf{r}')\right\}|_{z=0} &= \mu\alpha\{[J_x\hat{y}-J_y\hat{x}]\cdot(\mathbf{r}-\mathbf{r}')\}|_{z=0}\\
\left[(-J_y-J_y^\star)(x-x')+(J_x+J_x^\star)(y-y')\right] &= \mu\alpha\left[-J_y(x-x')+J_x(y-y')\right]\\
J_y+J_y^\star &= \mu\alpha J_y\\
J_x+J_x^\star &= \mu\alpha J_x\\
J_x^\star &= (\mu\alpha-1)\,J_x, &&(81)\\
J_y^\star &= (\mu\alpha-1)\,J_y. &&(82)
\end{aligned}$$

Applying the second boundary condition

$$\begin{aligned}
\hat{n}\times\mathbf{H}_2=\hat{n}\times\mathbf{H}_1\Rightarrow\frac{1}{\mu_0}\hat{z}\times\mathbf{B}_{z>0}|_{z=0} &= \frac{1}{\mu_0\mu}\hat{z}\times\mathbf{B}_{z<0}|_{z=0}\\
\frac{1}{\mu_0}\mu_0\hat{z}\times[\mathbf{J}(\mathbf{r}')+\mathbf{J}^\star(\mathbf{r}')]\times(\mathbf{r}-\mathbf{r}')|_{z=0} &= \frac{1}{\mu_0\mu}\mu_0\mu\hat{z}\times[\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')]|_{z=0}.
\end{aligned} \tag{83}$$

Again, I am going to use a identity, which is $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

$$\begin{aligned}
(z-z')[\mathbf{J}(\mathbf{r}')+\mathbf{J}^\star(\mathbf{r}')]-(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')\,(J_z+J_z^\star) &= \alpha\left[\mathbf{J}(\mathbf{r}')(z-z')-(\mathbf{r}-\mathbf{r}')J_z\right]\\
J_x^\star &= (1-\alpha)\,J_x, &&(84)\\
J_y^\star &= (1-\alpha)\,J_y, &&(85)\\
J_z^\star &= (\alpha-1)\,J_z. &&(86)
\end{aligned}$$

And using the previous results with this one we finally get that: $\alpha=\frac{2}{(\mu+1)}$.

Finally, we deduce that

$$\mathbf{J}^\star=\frac{(\mu-1)}{(\mu+1)}J_x(x,y,-z)\hat{x}+\frac{(\mu-1)}{(\mu+1)}J_y(x,y,-z)\hat{y}-\frac{(\mu-1)}{(\mu+1)}J_z(x,y,-z)\hat{z} \tag{87}$$

b) We showed that

$$\mathbf{B}_{z>0}(\mathbf{r})=\frac{\mu_0\mu}{4\pi}\int dV'\frac{\alpha\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}=\frac{\mu_0\mu}{4\pi}\frac{2}{(\mu+1)}\int dV'\frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{88}$$

Then, we can just pick up the terms and see that we have an effective current, in a medium with a unit permeability, given by

$$\mathbf{J}_{eff}=\frac{2\mu}{(\mu+1)}\mathbf{J}. \tag{89}$$