PGF5003: Classical Electrodynamics I
Problem Set 3

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(Due to May 11, 2021)

Guidelines: *write down the most relevant passages in your calculations, not only the final results. Do not forget to write the mathematical expressions that you are using in order to solve the questions. We strongly recommended the use of the International System of Units.*

1 Question (1 point)

Given the following magnetic field:

$$\mathbf{B}(\mathbf{r}) = B_0\hat{\mathbf{z}} + \hat{\mathbf{z}} \times \vec{\nabla} f(\mathbf{r}) \tag{1}$$

a) *Show that this satisfies:* $\vec{\nabla} \cdot \mathbf{B} = 0$;

b) *Find the equation(s) for the field lines and show that, performing the following change of variables:* $x \to q$ *(generalized coordinate),* $y \to p$ *(conjugate momenta),* $z \to t$ *and considering* $f = -B_0 H$ *(where* H *is a Hamiltonian), this is equivalent to*

$$\dot{q} = \frac{\partial H}{\partial p} \text{ and } \dot{p} = -\frac{\partial H}{\partial q}. \tag{2}$$

Hint: *Use a scalar parameter* λ *to ensure that every differential element of magnetic field line* $d\mathbf{s}$ *is parallel to* $\mathbf{B}(\mathbf{r})$ *itself:* $d\mathbf{s} = \lambda\mathbf{B}$.

c) *What can you say about the magnetic field lines doing the analogy with the above Hamilton's equations?*

2 Question (1 point)

Show that the force $\mathbf{F}$ *on a magnetic dipole* $\mathbf{m}$ *exerted by an arbitrary magnetic field* $\mathbf{B}$ *is given by*

$$\mathbf{F} = \vec{\nabla}(\mathbf{m} \cdot \mathbf{B}). \tag{3}$$

Hint: *Given the magnetic field of a point magnetic dipole* $\mathbf{B}$ *at* $\mathbf{r}_0$

$$\mathbf{B}(\mathbf{r}) = \mu_0 \left[\mathbf{m}\delta(\mathbf{r}-\mathbf{r}_0) - \vec{\nabla}\frac{1}{4\pi}\frac{\mathbf{m}\cdot(\mathbf{r}-\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}_0|^3}\right]. \tag{4}$$

find its current density $\mathbf{J}$.

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3 Question (1 point)

A right-handed circular solenoid of finite length L *and radius* a *has* N *turns per unit length and carries a current* I*. Show that the magnetic induction on the cylinder axis in the limit* $NL \to \infty$ *is*

$$B_z = \frac{\mu_0 N I}{2}(\cos\theta_1 + \cos\theta_2) \quad (5)$$

where the angles are defined in Figure 1.

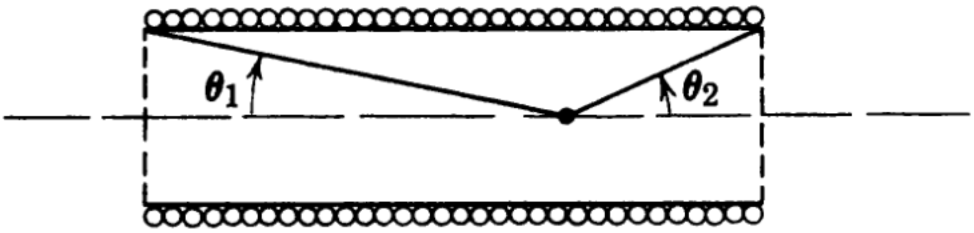

Figure 1: Figure for the question 3.

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4 Question (1 point)

A cylindrical conductor of radius a *has a cylindrical hole of radius* b *cored parallel to, and centered a distance* d *from, the cylinder axis* ($d + b < a$)*. The current density is uniform throughout the remaining metal of the cylinder and is parallel to the axis. Use Ampere's law and the principle of linear superposition to find the magnitude and direction of the magnetic flux density in the hole.*

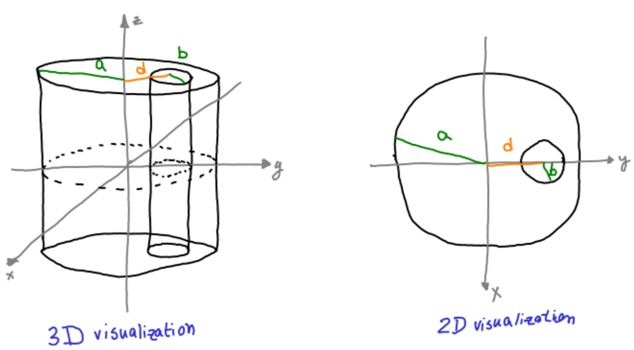

Figure 2: Figure for question 4.

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5 Question (2 points)

A small loop # 1 of wire (radius a) is held at a distance z above the center of a large loop # 2 (radius b), as shown in Figure 2. The planes of the two loops are parallel and perpendicular to the common axis.

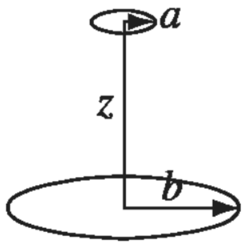

Figure 3: Figure for the question 5.

a) *First, find the magnetic field of a loop in its axis (here consider it as z, as indicated in Figure). Second, suppose current I flows in the big loop. Find the flux through the little loop.* **Hint:** *the little loop is so small that you may consider the field of the big loop to be essentially constant.*

b) *Suppose current I flows in the little loop. Find the flux through the big loop.* **Hint:** *The little loop is so small that you may treat it as a magnetic dipole. Consider its magnetic field as*

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right). \tag{6}$$

c) *Find the mutual inductance and confirm that $M_{12} = M_{21}$.*

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6 Question (2 point)

A sphere of linear magnetic material (with permeability μ) is placed in an otherwise uniform magnetic field $\mathbf{B}_0$. Find the new field inside the sphere. **Hint:** *solve this problem using separation of variables.*

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7 Question (2 points)

A current distribution $\mathbf{J}(\mathbf{x})$ exists in a medium of unit relative permeability adjacent to a semi-infinite slab of material having relative permeability μ and filling the half-space, $z < 0$.

a) *Show that for $z > 0$ the magnetic induction can be calculated by replacing the medium of permeability μ by an image current distribution $J^\star$, with components*

$$\mathbf{J}^\star = \frac{(\mu-1)}{(\mu+1)}J_x(x,y,-z)\hat{x} + \frac{(\mu-1)}{(\mu+1)}J_y(x,y,-z)\hat{y} - \frac{(\mu-1)}{(\mu+1)}J_z(x,y,-z)\hat{z} \tag{7}$$

b) *Show that for $z < 0$ the magnetic induction appears to be due to a current distribution $\frac{2\mu}{(\mu+1)}\mathbf{J}$ in a medium of unit relative permeability.*