PGF5003: Classical Electrodynamics I
Problem Set 2

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(Due to April 27, 2021)

Guideline: *in this PS you will need to use different math relations (for instance, for Legendre Polynomials, Bessel functions of first and second kind). Use the Jackson's book or your preferred reference to look for them. Write every equation that you have used in the solution. Moreover, in every Laplacian solution, write the differential equation and general solution in each exercise.*

1 Question (1 point)

Compute the energy of a sphere uniformly charged, building the sphere from shells of charge dq brought from infinity until some radius r, in the way that the result will be a sphere of a uniform density charge distribution.

1.1 Solution

Starting from the definition of energy, we can write the element of it as

$$dW = dq'\Phi = dq'\frac{1}{4\pi\epsilon_0}\frac{q'}{r}, \tag{1}$$

where $q' = \frac{\frac{4\pi}{3}r^3 q}{\frac{4\pi}{3}R^3} = \frac{qr^3}{R^3}$ is the fraction of charge in the sphere of radius r and

$$dq' = \rho dV = \frac{q}{\frac{4}{3}\pi R^3}4\pi r^2 dr = \frac{3q}{R^3}r^2 dr. \tag{2}$$

Then, we have

$$dW = dq'\Phi = dq'\frac{1}{4\pi\epsilon_0}\frac{q'}{r} = \frac{1}{4\pi\epsilon_0}\frac{3q}{R^3}r^2 dr\frac{qr^3}{R^3}\frac{1}{r} = \frac{1}{4\pi\epsilon_0}\frac{3q^2}{R^6}r^4 dr, \tag{3}$$

and the energy is finally the integral of the above quantity from 0 to R as

$$W = \frac{1}{4\pi\epsilon_0}\frac{3q^2}{R^6}\int_0^R drr^4 = \frac{1}{4\pi\epsilon_0}\frac{3q^2}{R^6}\frac{R^5}{5} = \frac{3}{20\pi\epsilon_0}\frac{q^2}{R}. \tag{4}$$

2 Question (1 point)

Consider two cylinders (that are both coaxial and conductive), with the same length L and with radius a and b, in the way that $a < b$, $L \gg a$ and $L \gg b$. Supposing that both cylinders are uniformly charged, the inner one with $+q$ and the exterior with $-q$, find:

a) *the capacitance of this system;*

b) *the energy per unit of length (this result is in terms of the capacitance and density of charge);*

2.1 Solution

a) Using the Gauss theorem, for $a < r < b$ we have

$$\int_V dV \vec{\nabla} \cdot \mathbf{E} = \oint_{S(V)} \mathbf{E} \cdot d\mathbf{S} = 4\pi \int_V dV \rho$$
$$E 2\pi r L = \frac{q}{\epsilon_0}$$
$$\mathbf{E} = \frac{q}{2\pi\epsilon_0 L} \frac{1}{r} \hat{r}. \tag{5}$$

Then, the difference of the potential between the cylinders is given by (because $\mathbf{E} = -\vec{\nabla}\Phi$)

$$\Delta\Phi = \Phi(b) - \Phi(a) = -\int_a^b \mathbf{E} \cdot d\mathbf{l} = -\int_a^b \frac{2q}{L} \frac{dr}{r} = -\frac{q}{2\pi\epsilon_0 L} \ln\left(\frac{b}{a}\right) = \frac{q}{2\pi\epsilon_0 L} \ln\left(\frac{a}{b}\right). \tag{6}$$

By the definition of capacitance, we have

$$C = \frac{Q}{\Delta\Phi} = \frac{q}{\frac{q}{2\pi\epsilon_0 L} \ln\left(\frac{a}{b}\right)} = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{a}{b}\right)}. \tag{7}$$

b) The energy per unit of length is given by definition by (and using the capacitance per unit of length):

$$w = \frac{W}{L} = \frac{1}{2}\frac{C}{L}\Phi^2 = \frac{1}{2}\frac{C}{L}\left[\Phi(b) - \Phi(a)\right]^2 = \frac{1}{2}\frac{2\pi\epsilon_0}{\ln\left(\frac{a}{b}\right)}\left[\Phi(b) - \Phi(a)\right]^2$$
$$= \frac{\pi\epsilon_0}{\ln\left(\frac{a}{b}\right)}\left[\frac{q}{2\pi\epsilon_0 L}\ln\left(\frac{a}{b}\right)\right]^2 = \frac{q^2}{4\pi\epsilon_0 L^2}\ln\left(\frac{a}{b}\right). \tag{8}$$

3 Question (1 point)

The two-dimensional region: $\rho \geq a$, $0 \leq \phi \leq \beta$ is bounded by conducting surfaces at $\phi = 0$, $\rho = a$ and $\phi = \beta$ held at zero potential, as indicated in the figure. At large ρ, the potential is determined by some configuration of charges and/or conductors at fixed potentials.

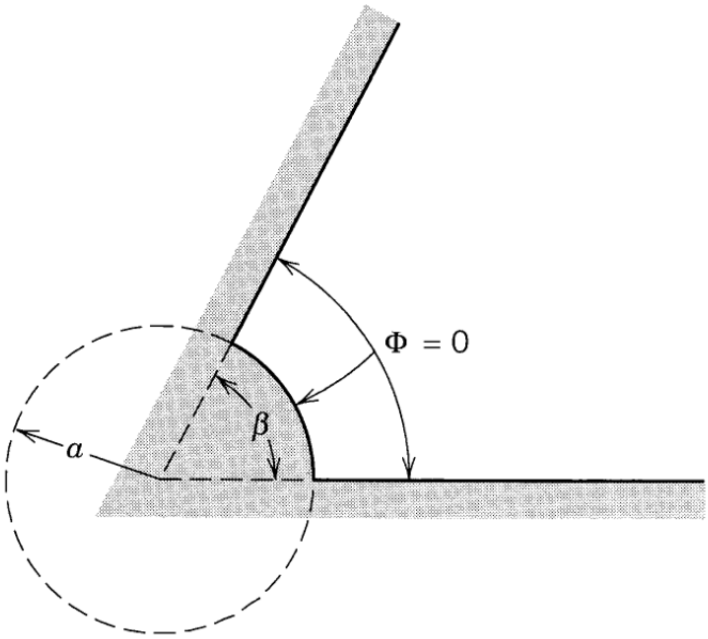

Figure 1: Figure for the question 3.

a) *Write down a solution for the potential* $\Phi(\rho, \phi)$ *that satisfies the boundary conditions for finite* ρ.

b) *Keeping only the lowest non vanishing terms, calculate the electric field components* E_ρ *and* E_ϕ *and also the surface-charge densities* $\sigma(\rho, 0)$, $\sigma(\rho, \beta)$ *and* $\sigma(a, \phi)$ *on the three boundary surfaces.*

3.1 Solution

a) In this problem we are solving the Laplace equation

$$\nabla^2 \Phi = 0 \tag{9}$$

in a 2-dimensional corners and along edges, then, the geometry suggests the use of polar coordinates (ρ, ϕ), such that the Laplace equation becomes

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} = 0. \tag{10}$$

Using separation of variables, the problem could be written as

$$\begin{aligned} \Phi(\rho, \phi) &= R(\rho)\psi(\phi) \\ \frac{\rho}{R}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) &+ \frac{1}{\psi^2}\frac{d^2\psi}{d\phi^2} = 0. \end{aligned} \tag{11}$$

Since the two terms are separately functions of ρ and ϕ, each one must be constant:

$$\begin{aligned} \frac{d^2\psi}{d\phi^2} &= -\nu^2 \psi \Rightarrow \psi(\phi) = A_\nu e^{i\nu\phi} + B_\nu e^{-i\nu\phi}, \\ \rho \frac{dR}{d\rho} + \rho^2 \frac{d^2 R}{d\rho^2} &= \nu^2 R \Rightarrow R(\rho) = \bar{a}_\nu \rho^\nu + \frac{\bar{b}_\nu}{\rho^\nu} \end{aligned} \tag{12}$$

and for $\nu = 0$ the particular solution is

$$\left\{ \begin{array}{c} R(\rho) = a_0 + b_0 \ln \rho \\ \psi(\phi) = A_0 + B_0 \phi \end{array} \right. . \tag{13}$$

Taking the boundary conditions we have

- $\Phi(\rho, 0) = 0 \Rightarrow 0 = (a_0 + b_0 \ln \rho)A_0 + (A_\nu + B_\nu) \Rightarrow B_\nu = -A_\nu$ and $A_0 = 0$;
- $\Phi(\rho, \beta) = 0$

$$\begin{aligned} (a_0 + b_0 \ln \rho)B_0\beta + A_\nu \left(e^{i\nu\beta} - e^{-i\nu\beta}\right) &= 0 \\ A_\nu \left[\cos(\nu\beta) + i\sin(\nu\beta) - \cos(\nu\beta) + i\sin(\nu\beta)\right] &= 0 \\ B_0 = 0 \text{ and } \nu &= \frac{n\pi}{\beta} \end{aligned}$$

$$\Phi(\rho, \phi) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{n\pi}{\beta}\phi\right)\left(\bar{a}\rho^{n\pi/\beta} + \frac{\bar{b}}{\rho^{n\pi/\beta}}\right). \tag{14}$$

- $\Phi(a, \phi) = 0$

$$\begin{aligned} \left(\bar{a}a^{n\pi/\beta} + \bar{b}a^{-n\pi/\beta}\right) &= 0 \\ \bar{a} &= -\bar{b}a^{-2n\pi/\beta}. \end{aligned} \tag{15}$$

Besides, as we do not have charge in $\rho = 0$, the coefficients of $\ln \rho$ is zero.

Thus, the electric potential is written as

$$\Phi(\rho, \phi) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\beta}\phi\right)\left[\left(\frac{\rho}{a}\right)^{n\pi/\beta} - \left(\frac{\rho}{a}\right)^{-n\pi/\beta}\right]. \tag{16}$$

Here, $A_n = \bar{b}_n a^{-n\pi/\beta}$ and $\bar{b}_n$ is given due the problem specification at infinity ($\rho \to \infty$).

b) The electric field is given by

$$\begin{aligned} \mathbf{E} = -\vec{\nabla}\Phi &= -\frac{\partial \Phi}{\partial \rho}\vec{\rho} - \frac{1}{\rho}\frac{\partial \Phi}{\partial \phi}\vec{\phi} \\ &= \sum_{n=0}^{\infty} A_n \left\{-\frac{n\pi}{a\beta}\left[\left(\frac{\rho}{a}\right)^{n\pi/\beta - 1} + \left(\frac{\rho}{a}\right)^{-n\pi/\beta - 1}\right]\sin\left(\frac{n\pi}{\beta}\phi\right)\right\}\hat{\rho} \\ &\quad - \frac{1}{\rho}\left\{\left[\left(\frac{\rho}{a}\right)^{n\pi/\beta} - \left(\frac{\rho}{a}\right)^{-n\pi/\beta}\right]\frac{n\pi}{\beta}\cos\left(\frac{n\pi}{\beta}\phi\right)\right\}\hat{\phi}. \end{aligned} \tag{17}$$

Keeping only the lowest non-vanishing terms means to consider $n = 1$. Then:

$$E_\rho = -A_1 \left\{\frac{\pi}{a\beta}\left[\left(\frac{\rho}{a}\right)^{\pi/\beta - 1} + \left(\frac{\rho}{a}\right)^{-\pi/\beta - 1}\right]\sin\left(\frac{\pi}{\beta}\phi\right)\right\} \tag{18}$$

$$E_\phi = -A_1 \frac{1}{\rho}\left\{\left[\left(\frac{\rho}{a}\right)^{\pi/\beta} - \left(\frac{\rho}{a}\right)^{-\pi/\beta}\right]\frac{\pi}{\beta}\cos\left(\frac{\pi}{\beta}\phi\right)\right\} \tag{19}$$

Moreover, the surface charge density is given by

$$\sigma = \epsilon_0 \mathbf{E}\cdot\mathbf{n}|_S$$

$$\sigma(\rho, 0) = \epsilon_0 \mathbf{E}\cdot\hat{\phi}|_{\phi=0} = -\frac{A_1\pi\epsilon_0}{4\beta\rho}\left\{\left[\left(\frac{\rho}{a}\right)^{\pi/\beta} - \left(\frac{\rho}{a}\right)^{-\pi/\beta}\right]\right\} \tag{20}$$

$$\sigma(\rho, \beta) = \epsilon_0 \mathbf{E}\cdot(-\hat{\phi})|_{\phi=\beta} = -\frac{A_1\pi\epsilon_0}{4\beta\rho}\left\{\left[\left(\frac{\rho}{a}\right)^{\pi/\beta} - \left(\frac{\rho}{a}\right)^{-\pi/\beta}\right]\right\} \tag{21}$$

$$\sigma(a, \phi) = \epsilon_0 \mathbf{E}\cdot\hat{\rho}|_{\rho=a} = -A_1\left\{\frac{2\pi\epsilon_0}{a\beta}\sin\left(\frac{\pi}{\beta}\phi\right)\right\}. \tag{22}$$

4 Question (1 point)

Consider a cube with conductive faces and side a. If the face on $z = a$ has constant potential Φ_0 and the other faces have zero potential, what is the electric potential inside the cube?

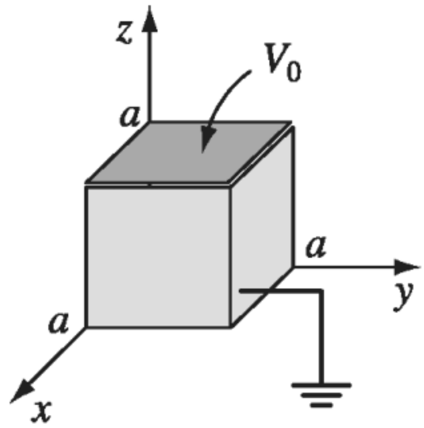

Figure 2: Figure for the question 4.

4.1 Solution

We are solving the Poisson equation

$$\nabla^2 \Phi = 0 \tag{23}$$

in Cartesian coordinates

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Phi = 0. \tag{24}$$

Proposing the following separation of variables, we get

$$\begin{aligned} \Phi(x, y, z) &= X(x)Y(y)Z(z), \\ \frac{d^2X}{dx^2} &= -\alpha^2 X(x) \\ \frac{d^2Y}{dy^2} &= -\beta^2 Y(y) \\ \frac{d^2Z}{dz^2} &= \gamma^2 Y(y), \end{aligned}$$

since $\alpha^2 + \beta^2 = \gamma^2$. The general solutions follows as

$$\begin{aligned} X(x) &= Ae^{i\alpha x} + Be^{-i\alpha x} \\ Y(y) &= Ce^{i\beta y} + De^{-i\beta y} \\ X(x) &= Ee^{\gamma z} + Fe^{-\gamma z}. \end{aligned}$$

Imposing the boundary conditions:

- $\Phi(0, y, z) = 0 \Rightarrow 0 = A + B \Rightarrow B = -A$;
- $\Phi(a, y, z) = 0$

$$\begin{aligned} A\left(e^{i\alpha a} - e^{-i\alpha a}\right) &= 0 \\ A\left[\cos(\alpha a) + i\sin(\alpha a) - \cos(\alpha a) + i\sin(\alpha a)\right] &= 0 \\ \alpha &= \frac{n\pi}{a} \\ X(x) &= \sum_{n=0}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right) \end{aligned} \tag{25}$$

  for $n \in [1, 2, \dots)$;

- $\Phi(x, 0, z) = 0 \Rightarrow 0 = C + D \Rightarrow D = -C$;
- $\Phi(x, a, z) = 0$

$$\begin{aligned} C\left(e^{i\beta a} - e^{-i\beta a}\right) &= 0 \\ C\left[\cos(\beta a) + i\sin(\beta a) - \cos(\beta a) + i\sin(\beta a)\right] &= 0 \\ \beta &= \frac{m\pi}{a} \\ Y(y) &= \sum_{m=0}^{\infty} C_n \sin\left(\frac{m\pi}{a}y\right) \end{aligned} \tag{26}$$

  for $m \in [1, 2, \dots)$;

- Because $\gamma^2 = \alpha^2 + \beta^2$, we have:

$$\gamma = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2} = \frac{\pi}{a}\sqrt{m^2 + n^2}; \tag{27}$$

- $\Phi(x, y, 0) = 0 \Rightarrow 0 = E + F \Rightarrow F = -E$:

$$\begin{aligned} Z(z) &= E\left(e^{\gamma a} - e^{-\gamma z}\right) \\ &= 2E\sinh(\gamma z). \end{aligned} \tag{28}$$

- $\Phi(x, y, a) = \Phi_0$:

  Up to now, the general equation becomes

$$\begin{aligned} \Phi(x, y, z)|_{z=a} &= \sum_{m,n} A_n C_m E_{n,m} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}y\right)\sinh\left[\left(\frac{\pi}{a}\sqrt{m^2+n^2}\right)z\right]|_{z=a} \\ &= \sum_{m,n} K_{n,m} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}y\right)\sinh\left[\left(\frac{\pi}{a}\sqrt{m^2+n^2}\right)z\right]|_{z=a} \\ &= \sum_{m,n} K_{n,m} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}y\right)\sinh\left[\left(\frac{\pi}{a}\sqrt{m^2+n^2}\right)a\right] = \Phi_0. \end{aligned} \tag{29}$$

  Integrating in both sides as follows

$$\begin{aligned} &\int_0^a dx \sin\left(\frac{n'\pi}{a}x\right)\int_0^a dy \sin\left(\frac{m'\pi}{a}y\right)\Phi_0 = \\ &\quad \sum_{m,n} K_{n,m}\sinh\left[\left(\frac{\pi}{a}\sqrt{m^2+n^2}\right)a\right] \\ &\qquad \int_0^a dx \sin\left(\frac{n'\pi}{a}x\right)\int_0^a dy \sin\left(\frac{m'\pi}{a}y\right)\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}y\right), \end{aligned} \tag{30}$$

due to orthogonality, $n' = n$, $m' = m$. Because $\int_0^a dx \sin^2\left(\frac{n\pi}{a}x\right) = \frac{a}{2}$ and $\int_0^a dx \sin\left(\frac{n\pi}{a}x\right) = \frac{a(1-\cos(\pi n))}{\pi n} = \frac{2a}{\pi n}$, for odd numbers n:

$$K_{n,m} = \frac{16\Phi_0}{nm\pi^2} \frac{1}{\sinh\left[\left(\frac{\pi}{a}\sqrt{m^2+n^2}\right)a\right]}. \tag{31}$$

Finally, we get the electric potential inside the cube as

$$\Phi(x,y,z) = \sum_{m,n=0}^{\infty} \frac{16\Phi_0}{nm\pi^2} \frac{\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}y\right)}{\sinh\left[\left(\frac{\pi}{a}\sqrt{m^2+n^2}\right)a\right]} \sinh\left[\left(\frac{\pi}{a}\sqrt{m^2+n^2}\right)z\right], \tag{32}$$

valid for m and n odd numbers.

5 Question (2 points)

A spherical surface of radius R has charge uniformly distributed over its surface with a density $Q/4\pi R^2$, except for a spherical cap at the north pole, defined by the cone $\theta = \alpha$.

a) *Show that the potential inside the spherical surface can be expressed as*

$$\Phi = \frac{Q}{2}\sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)} \left[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)\right] \frac{r^\ell}{R^{\ell+1}} P_\ell(\cos\theta), \tag{33}$$

where, for $\ell = 0$, $P_{\ell-1}(\cos\alpha) = -1$. What is the electric potential outside?

b) *Find the magnitude and the direction of the electric field at the origin.*

5.1 Solution

a) We are going to solve the Laplace equation

$$\nabla^2\Phi = 0. \tag{34}$$

Because we have a spherical geometry, let's do it in spherical coordinates:

$$\Phi(r,\theta,\phi) = R(r)\Theta(\theta)\psi(\phi)$$

$$\begin{aligned}
\frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin\theta}\left(\frac{\partial^2}{\partial\phi^2}\right)\right]\Phi &= 0\\
\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\psi}\frac{d^2\psi}{d\phi^2} &= 0\\
\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) &= m^2\\
\frac{1}{\psi}\frac{d^2\psi}{d\phi^2} &= -m^2
\end{aligned}$$

But, as we have azymutal symmetry, $m = 0$, i.e., the dependence in ϕ is zero. Then, we can write

$$\begin{aligned}
\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \lambda R &= 0,\\
\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \lambda\Theta &= 0
\end{aligned}$$

and the general solution is given using Legendre polynomials $P_\ell(\cos\theta)$ as

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta). \tag{35}$$

To solve this problem, we need to impose the boundary conditions, in the splitted solution: $\Phi = \begin{cases}\Phi_{in}, r < R\\ \Phi_{out}, r > R\end{cases}$, then

- $\Phi_{ins} = \Phi_{out}|_{r=R} \Rightarrow$ **Dirichlet**;
- $\left(\frac{\partial \Phi_{out}}{\partial r} - \frac{\partial \Phi_{ins}}{\partial r}\right)|_{r=R} = -4\pi\sigma \Rightarrow$ **Neumann**

  Where we had to remember that:

$$\sigma = \frac{-1}{4\pi}\frac{\partial \Phi}{\partial r} = \begin{cases} \frac{Q}{4\pi R^2}, \theta > \alpha \\ 0, \theta < \alpha \end{cases} \tag{36}$$

In this way we get

- $B_\ell = 0$ inside (in order to not diverge the potential in $r = 0$):

$$\Phi_{ins} = \sum_{\ell}^{\infty} A_\ell r^\ell P_\ell(\cos\theta);$$

- $A_\ell = 0$ outside (in order to not diverge the potential when $r \to \infty$):

$$\Phi_{out} = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) + \bar{B} \tag{37}$$

  and $\bar{B}$ is due electric potential for $r \to \infty$. Diving deeper in this limit we can write

$$\begin{aligned} \Phi_{out}(r \to \infty) = \frac{Q_{total}}{r} &= \left[\sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) + \bar{B}\right]_{r\to\infty} \\ &= \frac{B_0}{r} + \bar{B} \Rightarrow \bar{B} = 0 \Leftrightarrow B_0 = Q_{total}, \end{aligned} \tag{38}$$

  in other words, there is no other charge in the problem! Just notice that we have used $P_0(\cos\theta) = 1$. But, what is Q_{total}? It is given by the integral

$$\begin{aligned} Q_{total} &= \int dS\sigma = \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi R^2 \sigma \\ &= \frac{Q}{2}\int_\alpha^\pi d\theta \sin\theta = \frac{Q}{2}\left(1 + \cos\alpha\right). \end{aligned} \tag{39}$$

  Finally, we can write the solution in the outside as

$$\Phi_{out} = \frac{Q}{2}\frac{(1+\cos\alpha)}{r} + \sum_{\ell=1}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta). \tag{40}$$

Just now, we can impose the boundary conditions properly:

- $\Phi_{ins} = \Phi_{out}|_{r=R} \Rightarrow$ **Dirichlet**:

$$\begin{gathered} \sum_{\ell}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = \frac{Q}{2}\frac{(1+\cos\alpha)}{R} + \sum_{\ell=1}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta) \\ A_0 = \frac{Q}{2}\frac{(1+\cos\alpha)}{R} \\ \sum_{\ell=1}^{\infty} P_\ell(\cos\theta)\left[A_\ell R^\ell - \frac{B_\ell}{R^{\ell+1}}\right] = 0 \Rightarrow B_\ell = A_\ell R^{2\ell+1} \end{gathered} \tag{41}$$

and, from this condition we get

$$\begin{aligned}
\Phi_{ins} &= \frac{Q}{2}\frac{(1+\cos\alpha)}{R} + \sum_{\ell=1}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) \\
\Phi_{out} &= \frac{Q}{2}\frac{(1+\cos\alpha)}{r} + \sum_{\ell=1}^{\infty} \frac{A_\ell R^{2\ell+1}}{r^{\ell+1}} P_\ell(\cos\theta)
\end{aligned} \tag{42}$$

- $\left(\frac{\partial \Phi_{out}}{\partial r} - \frac{\partial \Phi_{ins}}{\partial r}\right)|_{r=R} = -4\pi\sigma = \frac{-Q}{R^2} \Rightarrow$ **Neumann**

$$\begin{aligned}
\frac{\partial \Phi_{out}}{\partial r} &= -\frac{Q}{2}\frac{(1+\cos\alpha)}{r^2} - \sum_{\ell=1}^{\infty} \frac{A_\ell R^{2\ell+1}}{r^{\ell+2}} P_\ell(\cos\theta) \\
\frac{\partial \Phi_{ins}}{\partial r} &= \sum_{\ell=1}^{\infty} A_\ell \ell r^{\ell-1} P_\ell(\cos\theta) \\
-4\pi\sigma &= \left[-\frac{Q}{2}\frac{(1+\cos\alpha)}{R^2} - \sum_{\ell=1}^{\infty} \frac{A_\ell R^{2\ell+1}}{R^{\ell+2}} P_\ell(\cos\theta) - \sum_{\ell=1}^{\infty} A_\ell \ell R^{\ell-1} P_\ell(\cos\theta)\right] \\
4\pi\sigma - \frac{Q}{2}\frac{(1+\cos\alpha)}{R^2} &= \sum_{\ell=1}^{\infty} A_\ell R^{\ell-1} P_\ell(\cos\theta)\,(2\ell+1) \\
\int_0^\pi d\theta \sin\theta P_{\ell'}(\cos\theta)\left[4\pi\sigma - \frac{Q}{2}\frac{(1+\cos\alpha)}{R^2}\right] &= \\
& \sum_{\ell=1}^{\infty} A_\ell R^{\ell-1}\,(2\ell+1) \int_0^\pi d\theta \sin\theta P_{\ell'}(\cos\theta) P_\ell(\cos\theta) \\
\int_0^\pi d\theta \sin\theta P_{\ell'}(\cos\theta)\left[4\pi\sigma - \frac{Q}{2}\frac{(1+\cos\alpha)}{R^2}\right] &= \sum_{\ell=1}^{\infty} A_\ell R^{\ell-1}\,(2\ell+1)\,\frac{2\delta_{\ell,\ell'}}{2\ell+1} \\
\int_0^\pi d\theta \sin\theta P_{\ell}(\cos\theta)\left[4\pi\sigma - \frac{Q}{2}\frac{(1+\cos\alpha)}{R^2}\right] &= 2A_\ell R^{\ell-1} \\
A_\ell =& \frac{R^{-\ell+1}}{2}\left[4\pi\sigma \int_\alpha^\pi d\theta \sin\theta P_\ell(\cos\theta) - \frac{Q}{2}\frac{(1+\cos\alpha)}{R^2}\int_0^\pi d\theta \sin\theta P_\ell(\cos\theta)\right]
\end{aligned} \tag{43}$$

To solve this integral, we can use the relation

$$P_\ell(x) = \frac{d}{dx}\left[\frac{P_{\ell+1}(x) - P_{\ell-1}(x)}{(2\ell+1)}\right] \tag{44}$$

changing the variables. Then,

$$\begin{aligned}
A_\ell &= \frac{R^{-\ell+1}}{2}\left\{4\pi\frac{Q}{4\pi R^2}\left[\frac{P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)}{(2\ell+1)}\right]\right. \\
&\qquad \left. -\frac{Q}{2}\frac{(1+\cos\alpha)}{R^2}\left[\frac{P_{\ell+1}(-1) - P_{\ell-1}(-1)}{(2\ell+1)}\right]\right\} \\
&= \frac{R^{-\ell+1}}{2}\left\{4\pi\frac{Q}{4\pi R^2}\left[\frac{P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)}{(2\ell+1)}\right]\right. \\
&\qquad \left. -\frac{Q}{2}\frac{(1+\cos\alpha)}{R^2}\left[\frac{-1+1}{(2\ell+1)}\right]\right\} \\
&= \frac{QR^{-\ell-1}}{2(2\ell+1)}\left[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)\right].
\end{aligned} \tag{45}$$

Therefore, the electric fields (inside and outside) becomes

$$\begin{aligned}\Phi_{ins} &= \frac{Q}{2}\frac{(1+\cos\alpha)}{R} + \frac{Q}{2}\sum_{\ell=1}^{\infty}\frac{R^{-\ell-1}}{(2\ell+1)}r^{\ell}\left[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)\right]P_{\ell}(\cos\theta)\\ &= \frac{Q}{2}\sum_{\ell=0}^{\infty}\frac{1}{(2\ell+1)}\frac{r^{\ell}}{R^{\ell+1}}\left[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)\right]P_{\ell}(\cos\theta), \qquad (46)\\ \Phi_{out} &= \frac{Q}{2}\frac{(1+\cos\alpha)}{r} + \sum_{\ell=1}^{\infty}\frac{R^{\ell}}{r^{\ell+1}}P_{\ell}(\cos\theta)\frac{Q}{2(2\ell+1)}\left[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)\right]\\ &= \frac{Q}{2}\sum_{\ell=1}^{\infty}\frac{1}{(2\ell+1)}\frac{R^{\ell}}{r^{\ell+1}}\left[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)\right]P_{\ell}(\cos\theta). \qquad (47)\end{aligned}$$

b) The electric field in the origin is given by

$$\begin{aligned}\mathbf{E}(r=0) = -\vec{\nabla}\Phi|_{r=0} &= -\left(\frac{\partial\Phi_{ins}}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\Phi_{ins}}{\partial\theta}\hat{\theta}\right)_{r=0}\\ &= -\Bigg\{\frac{Q}{2}\sum_{\ell=1}^{\infty}\frac{1}{(2\ell+1)}\frac{\ell r^{\ell-1}}{R^{\ell+1}}\left[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)\right]P_{\ell}(\cos\theta)\hat{r}\\ &\quad + \frac{1}{r}\frac{Q}{2}\sum_{\ell=1}^{\infty}\frac{1}{(2\ell+1)}\frac{r^{\ell}}{R^{\ell+1}}\left[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)\right]\left[\frac{\ell\cos\theta P_{\ell}(\cos\theta) - \ell P_{\ell-1}(\cos\theta)}{\sin\theta}\right]\hat{\theta}\Bigg\}_{r=0}\\ &= -\Bigg\{\frac{Q}{2}\sum_{\ell=1}^{\infty}\frac{1}{(2\ell+1)}\frac{\ell r^{\ell-1}}{R^{\ell+1}}\left[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)\right]P_{\ell}(\cos\theta)\hat{r}\\ &\quad + \frac{Q}{2}\sum_{\ell=1}^{\infty}\frac{1}{(2\ell+1)}\frac{\ell r^{\ell-1}}{R^{\ell+1}}\left[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)\right]\left[\frac{\cos\theta P_{\ell}(\cos\theta) - P_{\ell-1}(\cos\theta)}{\sin\theta}\right]\hat{\theta}\Bigg\}_{r=0}.\end{aligned}$$

Here, we can say that the terms $\ell = 2$ and higher are all zero at the origin ($r = 0$), resting just $\ell = 1$. Then,

$$\begin{aligned}\mathbf{E}(r=0) &= -\Bigg\{\frac{Q}{2}\frac{1}{3}\frac{1}{R^2}\left[P_2(\cos\alpha) - P_0(\cos\alpha)\right]P_1(\cos\theta)\hat{r}\\ &\qquad + \frac{Q}{2}\frac{1}{3}\frac{1}{R^2}\left[P_2(\cos\alpha) - P_0(\cos\alpha)\right]\left[\frac{\cos\theta P_1(\cos\theta) - P_0(\cos\theta)}{\sin\theta}\right]\hat{\theta}\Bigg\}\\ &= -\frac{Q}{6R^2}\left[P_2(\cos\alpha) - P_0(\cos\alpha)\right]\left\{P_1(\cos\theta)\hat{r} + \left[\frac{\cos\theta P_1(\cos\theta) - P_0(\cos\theta)}{\sin\theta}\right]\hat{\theta}\right\}\\ &= -\frac{Q}{6R^2}\left[P_2(\cos\alpha) - P_0(\cos\alpha)\right]\left\{P_1(\cos\theta)\hat{r} + \left[\frac{\cos\theta P_1(\cos\theta) - P_0(\cos\theta)}{\sin\theta}\right]\hat{\theta}\right\}\\ &= -\frac{Q}{6R^2}\left[\frac{1}{2}\left(3\cos\theta^2 - 1\right) - 1\right]\left\{\cos\theta\hat{r} + \left[\frac{\cos^2\theta - 1}{\sin\theta}\right]\hat{\theta}\right\}\\ &= -\frac{Q}{6R^2}\left(-\frac{3}{2}\sin^2\alpha\right)\left[\cos\theta\hat{r} - \sin\theta\hat{\theta}\right] = \frac{Q}{4R^2}\sin^2\alpha\hat{z} \qquad (48)\end{aligned}$$

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6 Question (2 points)

Consider a half-infinite cylinder grounded, with a electric potential ϕ_0 at its closed end.

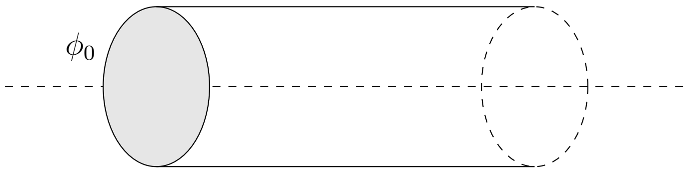

Figure 3: Semi-infinite cylinder.

Show that the electric potential in the inner parts of the cylinder is given by:

$$\phi(\rho, \theta, z) = 2\phi_0 \sum_k \frac{e^{-kz}}{ka} \frac{J_0(k\rho)}{J_1(ka)}, \tag{49}$$

where $J_0(ka) = 0$ and, of course, J_i is the Bessel function of first kind and order i.

6.1 Solution

Here we are solving the Poisson equation

$$\nabla^2 \Phi = 0 \tag{50}$$

in spherical coordinates

$$0 = \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \right) + \frac{\partial}{\partial \phi} \left(\frac{1}{\rho} \frac{\partial}{\partial \phi} \right) + \frac{\partial}{\partial z} \left(\frac{1}{\rho} \frac{\partial}{\partial z} \right) \right].$$

Proposing the following separation of variables, we get

$$\begin{aligned}
\Phi(\rho, \phi, z) &= R(\rho)\psi(\phi)Z(z), \\
\frac{d^2 Z}{dz^2} &= k^2 Z(z) \\
\frac{d^2 \psi}{d\phi^2} &= -\nu^2 \psi(z) \\
\frac{d^2 R}{d\rho^2} &= -\frac{1}{\rho}\frac{dR}{d\rho} - \left(k - \frac{\nu^2}{\rho^2} \right) R.
\end{aligned}$$

The general solution follow as

$$\Phi(\rho, \phi, z) = \sum_{k,\nu} \left[A J_\nu(k\rho) + B N_\nu(k\rho) \right] \left[C e^{i\nu\phi} + D e^{-i\nu\phi} \right] \left[E e^{kz} + F e^{-kz} \right], \tag{51}$$

where $J_\nu(k\rho)$, $N_\nu(k\rho)$ are, respectively, the Bessel functions of first and second kind (the last one could be called as *Neumann function* too). We can say some things about the conditions (not boundary, but in general now) that we can impose in this problem:

- $B = 0$ because $N_\nu(k\rho) \propto \rho^{-k}$ and potential is finite at $\rho = 0$;
- $E = 0$ because we need that e^{kz} to not diverge on infinity;
- $\nu = 0$ because the problem has azimuthal symmetry;

then, the general solution follow as

$$\Phi(\rho, \phi, z) = \sum_k \bar{A} J_0(k\rho) e^{-kz}, \tag{52}$$

where, clearly, $\bar{A} = AF$.

Now, imposing the boundary conditions:

- $\Phi(\rho = a) = 0 \Rightarrow$ due to the cylinder being grounded

$$\Phi(\rho = a) = \sum_k \bar{A} J_0(ka) e^{-kz} = 0 \Rightarrow J_0(ka) = 0 \Rightarrow k \to k_m = \frac{x_m}{a} \tag{53}$$

  and $x_m = [2.405, 5.520, 8.654, \dots)$ are the roots from Bessel functions of first kind, which implies that k assume discrete values.

- $\Phi(z = 0) = \Phi_0$

$$\Phi(\rho = a) = \sum_k \bar{A} J_0(ka) = \Phi_0 \tag{54}$$

  Here we need to use some machinery from mathematics. We can, for example, integrate both sides of the above equation as follows

$$\int_0^a d\rho\rho J_\alpha(k'\rho) \sum_k \bar{A} J_0(ka) = \int_0^a d\rho\rho J_\alpha(k'\rho)\Phi_0$$
$$\sum_k \bar{A} \frac{a^2}{2} \left[J_1(ka)\right]^2 \delta_{k,k'} = \Phi_0 J_1(ka)$$
$$\bar{A} = \frac{2\Phi_0}{a^2 J_1(ka)},$$

  where we need to use the equation (3.95) from Jackson's book

$$\int_0^a d\rho\rho J_\nu\left(x_{\nu n'}\frac{\rho}{a}\right) J_\nu\left(x_{\nu n}\frac{\rho}{a}\right) = \frac{a^2}{2}\left[J_{\nu+1}(x_{\nu n})\right]^2 \delta_{n,n'} \tag{55}$$

  and the integral expression

$$\int_0^a d\rho\rho J_0(k\rho) = \frac{aJ_0(ka)}{k}. \tag{56}$$

Finally, we got the potential as

$$\phi(\rho, \theta, z) = 2\phi_0 \sum_k \frac{e^{-kz}}{ka} \frac{J_0(k\rho)}{J_1(ka)} \square. \tag{57}$$

7 Question (2 points)

Two concentric conducting spheres of inner and outer radii a and b, respectively, carry charges $\pm Q$. The empty space between the spheres is half-filled by a hemispherical shell of dielectric (of dielectric constant ϵ), as shown in the Figure.

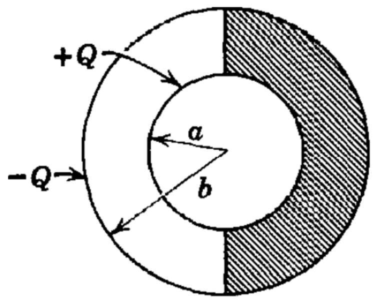

Figure 4: Figure for the question 7.

a) *Find the electric field everywhere between the spheres.*

b) *Calculate the surface-charge distribution on the inner sphere.*

7.1 Solution

a) We can solve this problem splitting it on two different regions:

- empty (e);
- filled (f).

Let's start with the easy side: the **empty**. In this region, as we do not have any charges (it is empty), we can write the Poisson equation

$$\vec{\nabla}\Phi_e = 0, \tag{58}$$

which has the solution

$$\Phi_e = \sum_{\ell=0}^{\infty} \left(A_{ell} r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta). \tag{59}$$

Considering the the potential on the surface of the first conductor ($r = a$) is a constant value, for instance C, we can impose our first boundary condition and find out that

$$\begin{aligned}
\Phi_e(r = a) &= \sum_{\ell=0}^{\infty} \cancelto{0}{\left(A_{ell} a^\ell + \frac{B_\ell}{a^{\ell+1}} \right)} P_\ell(\cos\theta) = C \\
C &= A_0 + \frac{B_0}{a} \\
B_\ell &= -A_\ell a^{2\ell+1}, \ell > 0,
\end{aligned}$$

and we use the fact that the terms assigned as zero have this value because the constant is a constant, so it is independent of polar variable θ.

Considering now the other conductor ($r = b$), that needs to have a constant potential ($\bar{C}$) as well, we have

$$\begin{aligned}\Phi_e(r=b) &= \sum_{\ell=0}^{\infty}\left(A_{ell}b^{\ell} - \frac{A_\ell a^{2\ell+1}}{b^{\ell+1}}\right)^{\nearrow 0} P_\ell(\cos\theta) = \bar{C} \\ \bar{C} &= A_0 - \frac{B_0}{b} \\ b^{2\ell+1} &= a^{2\ell+1} \Rightarrow A_\ell = B_\ell = 0, \ell > 0. \end{aligned} \tag{60}$$

Because this relation hold for all values of the independent polar variable and because the Legendre polynomials are orthogonal, all coefficients must equate independently. Then, A_ℓ must be zero for all $\ell > 1$. Thus, the potential in the empty region is given by

$$\Phi_e(r,\theta,\phi) = A_0 + \frac{B_0}{r}, \tag{61}$$

and the electric field follow as

$$\mathbf{E}_e = -\vec{\nabla}\Phi_e = \frac{B_0}{r^2}\hat{r}. \tag{62}$$

But this is the solution only for the empty region. The **filled** region, again, there is no charge (at least in the region between the shells). Then, one more time we can solve the same equations, finding

$$\begin{aligned}\Phi_f &= \bar{A}_0 + \frac{\bar{B}_0}{r}\hat{r} \\ \mathbf{E}_f &= -\vec{\nabla}\Phi_f = \frac{\bar{B}_0}{r^2}\hat{r}. \end{aligned} \tag{63}$$

Having the splitted solution we only need to apply the boundary condition between the dielectric and empty regions:

$$\begin{aligned}(\mathbf{D}_f - \mathbf{D}_e)\cdot\hat{n} = 0 \Rightarrow (\epsilon_0\mathbf{E}_e = \epsilon\mathbf{E}_f)_\theta\,, \\ (\mathbf{E}_f - \mathbf{E}_e)\cdot\hat{t} = 0 \Rightarrow (\mathbf{E}_e = \mathbf{E}_f)_r\,. \end{aligned}$$

Taking the radial part, we conclude that

$$\frac{B_0}{r^2} = \frac{\bar{B}_0}{r^2} \Rightarrow B_0 = \bar{B}_0 \tag{64}$$

and that the electrical field ($\mathbf{E}$) and the displacement fields ($\mathbf{D}_e$ and $\mathbf{D}_f$) are

$$\mathbf{E} = \frac{B_0}{r^2}\hat{r}, \mathbf{D}_e = \epsilon_0\frac{B_0}{r^2}\hat{r} \text{ and } \mathbf{D}_f = \epsilon\frac{B_0}{r^2}\hat{r}. \tag{65}$$

But what is the value of B_0? The answer to this question is given thinking about the total charge that we could have access, for example, $+Q$, in the inner shell. Then, due the fact that $\vec{\nabla}\cdot\mathbf{D} = \rho$, the Gauss' law in the region between the shells give it as following

$$\begin{aligned}Q &= \oint_{S(a<r<b)} d\mathbf{S}\cdot\mathbf{D} \\ Q &= \int_0^{2\pi} d\phi r^2\left[\int_0^{\pi/2} d\theta\sin\theta\epsilon_0\frac{B_0}{r^2} + \int_{\pi/2}^{\pi} d\theta\sin\theta\epsilon\frac{B_0}{r^2}\right] \\ &= 2\pi B_0(\epsilon + \epsilon_0) \Rightarrow B_0 = \frac{Q}{2\pi(\epsilon+\epsilon_0)}. \end{aligned} \tag{66}$$

Finally,

$$\mathbf{E} = \frac{Q}{2\pi(\epsilon + \epsilon_0)}\frac{1}{r^2}\hat{r}, \mathbf{D}_e = \frac{\epsilon_0 Q}{2\pi(\epsilon + \epsilon_0)}\frac{1}{r^2}\hat{r} \text{ and } \mathbf{D}_f = \frac{\epsilon Q}{2\pi(\epsilon + \epsilon_0)}\frac{1}{r^2}\hat{r}. \tag{67}$$

b) The surface-charge distribution in the inner sphere could be computed using

$$\sigma = (\mathbf{D}_2 - \mathbf{D}_1) \cdot \hat{n}|_S$$

and, because inside a conductor there is no fields $\mathbf{D}_1 = 0$. Then,

$$\begin{aligned} \sigma_e &= \mathbf{D}_e \cdot \hat{r}|_{r=a} = \frac{\epsilon_0 Q}{2\pi(\epsilon + \epsilon_0)}\frac{1}{a^2}, \\ \sigma_f &= \mathbf{D}_f \cdot \hat{r}|_{r=a} = \frac{\epsilon Q}{2\pi(\epsilon + \epsilon_0)}\frac{1}{a^2}. \end{aligned} \tag{68}$$