Electrodynamics

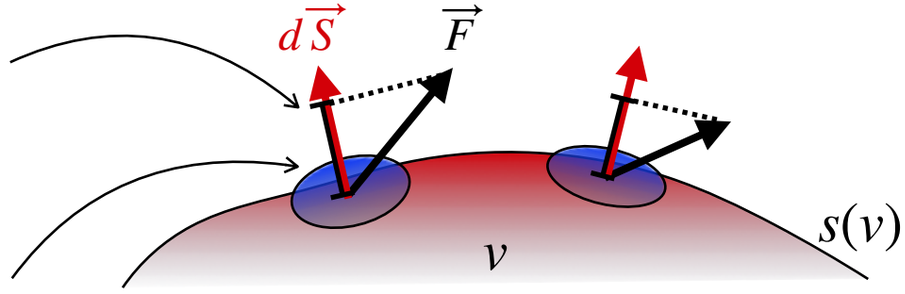
- ⚡Green's functions
- ⚡Boundary Conditions
- ⚡Variational method
- ⚡Laplace Equation
- ⚡Relaxation techniques

Helmholtz Theorem

- The Helmholtz theorem tells us that all we need in order to fully determine a field in any given volume is to provide its **divergence** and **curl**:

$$\vec{\nabla} \cdot \vec{F} = S_{\text{div}} \qquad \text{and} \qquad \vec{\nabla} \times \vec{F} = \vec{S}_{\text{rot}} \quad ,$$

- The field is then determined uniquely once we provide the **boundary conditions** on the surface that envelops this volume:



$$\vec{F} = -\vec{\nabla}\Psi + \vec{\nabla} \times \vec{R} \quad , \quad \text{with}$$

$$\Psi = \frac{1}{4\pi}\int_v d^3x' \, \frac{S_{\text{div}}(\vec{x}')}{|\vec{x} - \vec{x}'|} - \frac{1}{4\pi}\oint_{s(v)} d\vec{s}' \cdot \frac{\vec{F}(\vec{x}')}{|\vec{x} - \vec{x}'|} \qquad \Longleftarrow \qquad \vec{F}_{\perp}$$

$$\vec{R} = \frac{1}{4\pi}\int_v d^3x' \, \frac{\vec{S}_{\text{rot}}(\vec{x}')}{|\vec{x} - \vec{x}'|} - \frac{1}{4\pi}\oint_{s(v)} d\vec{s}' \times \frac{\vec{F}(\vec{x}')}{|\vec{x} - \vec{x}'|} \qquad \Longleftarrow \qquad \vec{F}_{||}$$

Green's Theorem

- Now let's write a result which is very similar to the Helmholtz theorem, but that is **more powerful**. Recall **Green's theorem**:

$$\int_V d^3x \left(f\,\nabla^2 g - g\,\nabla^2 f\right) = \oint_{S(V)} d\vec{S}\cdot\left(f\,\vec{\nabla}\,g - g\,\vec{\nabla}f\right)$$

- Now, we will make a very particular choice. First, let's use $x \to x'$ as the integration variable, and for the functions f and g we pick:

$$f \to \phi(\vec{x}') \quad , \quad \text{and}$$

$$g \to G(\vec{x}', \vec{x})$$

- The theorem now reads as follows:

$$\int d^3x' \left[\phi(\vec{x}')\,\nabla'^2 G(\vec{x}', \vec{x}) - G(\vec{x}', \vec{x})\,\nabla'^2\phi(\vec{x}')\right] = \oint d\vec{S}'\cdot\left[\phi(\vec{x}')\,\vec{\nabla}'G(\vec{x}', \vec{x}) - G(\vec{x}', \vec{x})\,\vec{\nabla}'\phi(\vec{x}')\right]$$

- But remember that $\nabla^2 G(\vec{x}, \vec{x}') = \delta(\vec{x} - \vec{x}')$, or $\nabla'^2 G(\vec{x}', \vec{x}) = \delta(\vec{x}' - \vec{x})$, and $\nabla'^2\phi(\vec{x}') = -\rho(\vec{x}')/\epsilon_0$, therefore:

$$\int d^3x' \left[\phi(\vec{x}')\,\delta(\vec{x}' - \vec{x}) + G(\vec{x}', \vec{x})\frac{\rho(\vec{x}')}{\epsilon_0}\right] = \oint d\vec{S}'\cdot\left[\phi(\vec{x}')\,\vec{\nabla}'G(\vec{x}', \vec{x}) - G(\vec{x}', \vec{x})\,\vec{\nabla}'\phi(\vec{x}')\right]$$

$$\Rightarrow \quad \phi(\vec{x}) = -\int d^3x'\,G(\vec{x}', \vec{x})\frac{\rho(\vec{x}')}{\epsilon_0} + \oint d\vec{S}'\cdot\left[\underbrace{\phi(\vec{x}')\,\vec{\nabla}'G(\vec{x}', \vec{x})}_{\text{Dirichlet}} - \underbrace{G(\vec{x}', \vec{x})\,\vec{\nabla}'\phi(\vec{x}')}_{\text{Neumann}}\right]$$

Green's Theorem

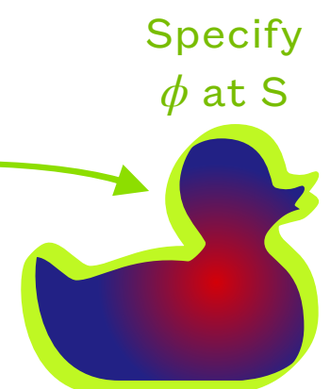
ATTENTION:
Jackson has a "funny" definition for the sign and normalization of the Green function, which differs from mine. Just beware of this difference in notation!

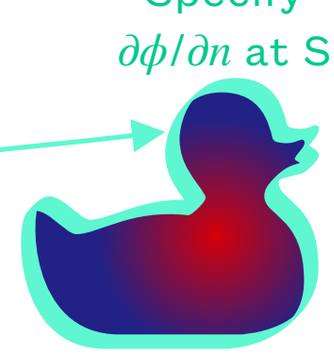
- This means that the field is determined by:

$$\Rightarrow \quad \phi(\vec{x}) = -\int_V d^3x'\, G(\vec{x}', \vec{x})\frac{\rho(\vec{x}')}{\epsilon_0}$$

$$+ \oint_{S(V)} d\vec{S}' \cdot \left[\phi(\vec{x}')\,\vec{\nabla}' G(\vec{x}', \vec{x})\right]$$

$$- \oint_{S(V)} d\vec{S}' \cdot \left[G(\vec{x}', \vec{x})\,\vec{\nabla}'\phi(\vec{x}')\right]$$





- We can now use our freedom to choose the Green's function:

(i) Given Dirichlet b.c., have $G \to G_D(\vec{x}', \vec{x})$ with $G_D\left(\vec{x}', \vec{x}\right)\Big|_S = 0$

(ii) Given Neumann b.c., have $G \to G_N(\vec{x}', \vec{x})$ with $\left[\frac{\partial}{\partial n'} G_N\left(\vec{x}', \vec{x}\right)\right]_S = 0$

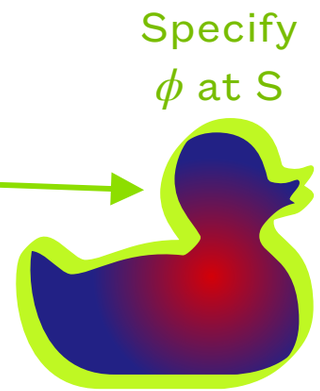
Green's Theorem

- The first case (Dirichlet) is quite simpler:

$$\Rightarrow \quad \phi(\vec{x}) = -\int_V d^3x'\, G_D(\vec{x}', \vec{x})\,\frac{\rho(\vec{x}')}{\epsilon_0}$$

$$+ \oint_{S(V)} d\vec{S}' \cdot \left[\phi(\vec{x}')\,\vec{\nabla}' G_D(\vec{x}', \vec{x})\right]$$

$$- \oint_{S(V)} d\vec{S}' \cdot \left[G_D(\vec{x}', \vec{x})\,\vec{\nabla}'\phi(\vec{x}')\right]$$

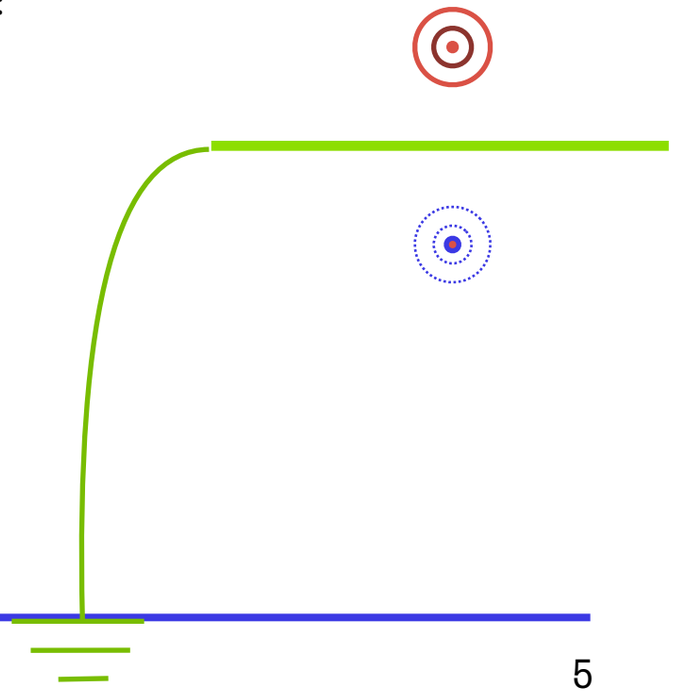


- You can see immediately how this works out for the case of the "method of images". In that case we have $\phi \to 0$ at the surface (the plane $z = 0$), and the choice of Green's function that satisfies the condition that $G_D \to 0$ at that same surface is:

$$G_D = -\frac{1}{4\pi}\frac{1}{|\vec{x}' - \vec{x}|} + \frac{1}{4\pi}\frac{1}{|\vec{x}' - \vec{x}_i|} \quad , \quad \text{where} \quad \vec{x} = \{x, y, z\} \quad \text{and} \quad \vec{x}_i = \{x, y, -z\}$$

$$\Rightarrow \quad \phi(\vec{x}) = \int_V d^3x' \left[\frac{1}{4\pi}\frac{1}{|\vec{x}' - \vec{x}|} - \frac{1}{4\pi}\frac{1}{|\vec{x}' - \vec{x}_i|}\right] \times q\,\delta(\vec{x}' - Z_0\,\hat{z})$$

$$= \frac{q}{4\pi}\left(\frac{1}{\sqrt{x^2 + y^2 + (z - Z_0)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z + Z_0)^2}}\right)$$



Green's Theorem

- Neumann boundary conditions are a bit more subtle. Let's say we pick G_N such that $\hat{n}' \cdot \vec{\nabla}' G_N \to 0$ at the surface. We would have:

$$\Rightarrow \quad \phi(\vec{x}) = -\int_V d^3x'\, G_N(\vec{x}', \vec{x})\frac{\rho(\vec{x}')}{\epsilon_0} + \oint_{S(V)} d\vec{S}' \cdot \left[\phi(\vec{x}')\, \vec{\nabla}' G_N(\vec{x}', \vec{x})\right] - \oint_{S(V)} d\vec{S}' \cdot \left[G_N(\vec{x}', \vec{x})\, \vec{\nabla}' \phi(\vec{x}')\right]$$

?

- Sounds OK, right? Well, actually… not quite! To appreciate the issue, consider that:

$$\int_V d^3x' \nabla'^2 G(\vec{x}', \vec{x}) = \int_V d^3x'\, \delta(\vec{x}' - \vec{x}) = 1$$

- But a trivial application of Gauss's theorem tells us that:

$$\int_V d^3x' \nabla'^2 G(\vec{x}', \vec{x}) = \oint_{S(V)} d\vec{S}' \cdot \vec{\nabla}' G(\vec{x}', \vec{x}) = 1$$

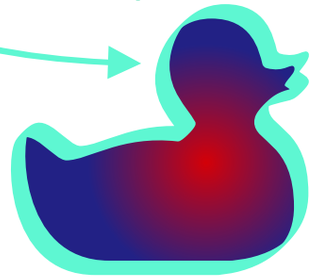
- Hence, we simply **cannot** pick any Green's function that has $\hat{n}' \cdot \vec{\nabla}' G_N \to 0$. The simplest choice would be something like:

$$\hat{n}' \cdot \vec{\nabla}' G_N(\vec{x}', \vec{x}) = \frac{1}{S} \quad , \quad \text{so that we would get} \quad \oint_{S(V)} d\vec{S}' \cdot \left[\phi(\vec{x}')\, \vec{\nabla}' G_N(\vec{x}', \vec{x})\right] = \frac{1}{S}\oint_{S(V)} dS' \cdot \phi(\vec{x}') = \langle\phi\rangle_S$$

- The final result is that for Neumann b.c. we get something like:

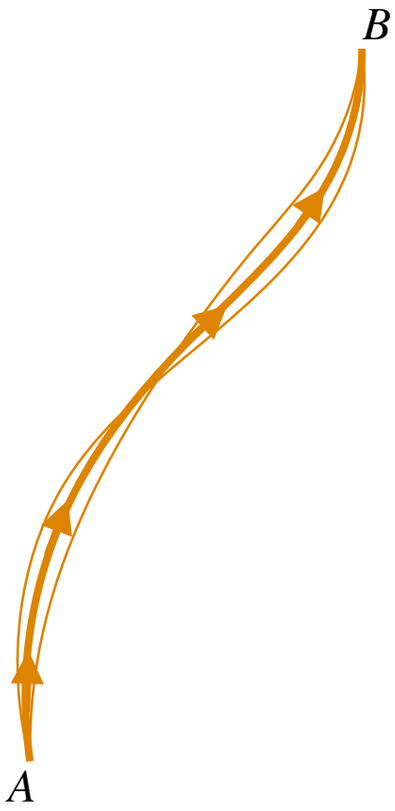
$$\Rightarrow \quad \phi(\vec{x}) = -\int_V d^3x'\, G_N(\vec{x}', \vec{x})\frac{\rho(\vec{x}')}{\epsilon_0} + \langle\phi\rangle_S - \oint_{S(V)} d\vec{S}' \cdot \left[G_N(\vec{x}', \vec{x})\, \vec{\nabla}' \phi(\vec{x}')\right]$$

Specify $\partial\phi/\partial n$ at S



In this sense, Neumann b.c. need "completion": like a further constant of integration. See the discussion in Jackson.

The variational approach



- From Classical Mechanics we know that basically all of Physics can be written in terms of a **Lagrangean**:

$$L(q,\dot{q}) = K - U$$

where, for a point particle typically we have something like $K \to \frac{1}{2}\dot{q}^2$ and the potential energy $U \to U(q)$.

- The action is then defined as:

$$S = \int_A^B dt\, L$$

- By minimizing the action between two fixed points we arrive at the **Euler-Lagrange equation**, which tells us that:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \quad \Rightarrow \quad \frac{d}{dt}(\dot{q}) + \frac{dU}{dq} = 0 \quad \Rightarrow \quad \ddot{q} = -\frac{dU}{dq} = F_q$$

- The idea is that, somehow, the electric potential is itself something that finds the least energy configuration. We can make a first try by associating:

$$K \to \frac{1}{2}\left(\vec{\nabla}\phi\right)^2$$

$$U \to \frac{\rho(\vec{x})}{\epsilon_0}\phi(\vec{x})$$ (the ϵ_0 here is for K and U to have the same units! Alternatively, multiply both by ϵ_0.)

- So, our "guess" is that we have a functional for the potential that we write as:

$$\mathscr{L}[\phi, \vec{\nabla}\phi] = \frac{1}{2}\left(\vec{\nabla}\phi\right)^2 - \frac{\rho}{\epsilon_0}\phi$$

The variational approach

- We now minimize the “action" that we write in terms of this “Lagrangean":

$$\mathcal{S} = \int d^3x \mathscr{L} = \int d^3x \left[\frac{1}{2} \left(\vec{\nabla} \phi \right)^2 - \frac{\rho}{\epsilon_0} \phi \right]$$

- The total variation of the action is written as:

$$\delta\mathcal{S} = \int d^3x \left[\frac{1}{2} \times 2 \, \vec{\nabla} \phi \cdot \delta(\vec{\nabla} \phi) - \frac{\rho}{\epsilon_0} \delta \phi \right] = \int d^3x \left[\vec{\nabla} \phi \cdot \vec{\nabla} (\delta\phi) - \frac{\rho}{\epsilon_0} \delta \phi \right]$$

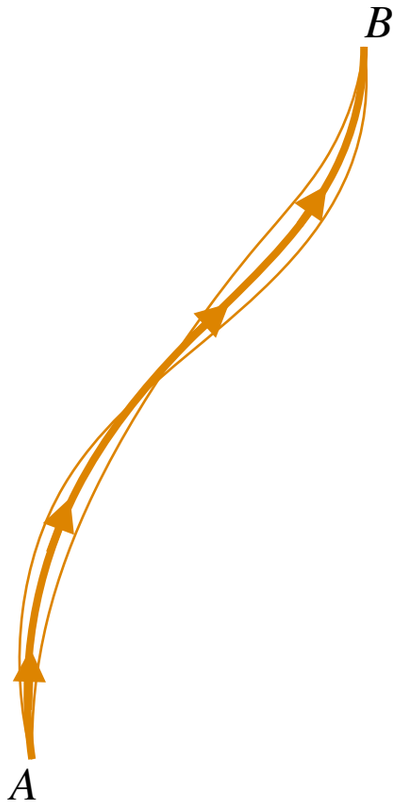
- We now integrate by parts the first term, to obtain:

$$\delta\mathcal{S} = \int d^3x \left[\vec{\nabla} [(\vec{\nabla} \phi) \, \delta\phi] - (\vec{\nabla}^2 \phi) \, \delta\phi - \frac{\rho}{\epsilon_0} \delta \phi \right]$$

- But from the Poisson equation we know that $\nabla^2 \phi = - \rho / \epsilon_0$, so the last two terms cancel, we are left with the expression:

$$\delta\mathcal{S} = \oint d\vec{S} \cdot [\delta\phi \, \vec{\nabla} \phi]$$

- Just like in usual Lagrangeans, we fix the variations to zero at the ends of the integral, $\delta\phi \rightarrow 0$ on the boundary. Hence, we find that the extrema of the “action" above is a field configuration that minimizes (or, better, extremizes) that "action".



The variational approach

- OK, but this is not simply a formal re-derivation of Poisson's equations. This "action" and "Lagrangean" allow us to find analytical approximations for real problems.

$$\mathcal{S} = \int d^3x \mathcal{L} = \int d^3x \left[\frac{1}{2} \left(\vec{\nabla} \phi \right)^2 - \frac{\rho}{\epsilon_0} \phi \right]$$

- The idea is that, given some function $\rho = \rho(\vec{x})$, we "guess" a functional form for $\phi(\vec{x}) = \sum_i c_i \, f_i(\vec{x} \,|\, \alpha_i, \beta_i, \ldots)$, with parameters $\theta_i = \{c_i, \alpha_i, \beta_i, \ldots\}$, and basis functions f_i which can be "easily" integrated in the domain:

$$\int d^3x \, \nabla f_i \cdot \nabla f_j = F_{ij}(\alpha_i, \beta_i, \ldots; \alpha_j, \beta_j, \ldots) \quad , \quad \text{and}$$

$$\int d^3x \, \rho \, f_i = G_i(\alpha_i, \beta_i, \ldots)$$

- Then, minimization of the action reduces to an algebraic set of equations:

$$\frac{\partial}{\partial \theta_k} \left(\frac{1}{2} \sum_i \sum_j c_i c_j F_i F_j - \sum_i c_i G_i \right) = 0$$

- Jackson has a nice example using cylindrical coordinates; if *you* can work out another good example, let me know!

The variational approach

- As you can imagine, the “action" above is incredibly close to what we would call an actual action.
- In fact, the Lagrangean for the electric field is basically the one above, all we need to do is to fix the dimensions:

$$\mathscr{L} \to \epsilon_0 \mathscr{L} \quad \Longrightarrow \quad \mathscr{L} = \frac{\epsilon_0}{2}\left(\vec{\nabla}\phi\right)^2 - \rho\,\phi = \frac{\epsilon_0}{2}\vec{E}^2 - \rho\,\phi$$

- In fact, we will see later in this course that the Lagrangean for Electrodynamics is:

$$\mathscr{L} = \frac{\epsilon_0}{2}(\vec{E}^2 - c^2\vec{B}^2) - (\rho\,\phi - \vec{J}\cdot\vec{A})$$

- We will see later in this course an even more elegant form for this Lagrangean, and how it in fact leads to the Maxwell equations!

The Laplace Equation

- The homogeneous version of the Poisson equation is the Laplace equation:

$$\nabla^2 \phi = 0$$

- Besides being itself of interest, any solution of the Laplace equation can be added to a solution of the Poisson equation — as long, of course, as the sum of the two satisfy the boundary conditions:

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0} \quad \rightarrow \quad \phi_1$$

$$\nabla^2 \phi = 0 \quad \rightarrow \quad \phi_2$$

$$\Longrightarrow \quad \left(\phi_1 + \phi_2\right)_S = \Phi_S \quad , \quad \text{where } \Phi_S \text{ expresses the boundary conditions at some surface } S \, .$$

Here enters the fact that Electromagnetism is linear: any linear combination of two solutions to the EM equations is also a solution!

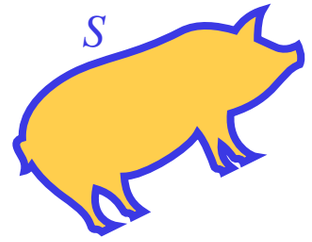
- The Green function methods is useful for finding solutions to the inhomogeneous (Poisson) equation. As for the Laplace equation, it is useful to find direct solutions to it, even if it is to help us find appropriate Green's functions given our boundary value problem.

- So, let's review now some of the methods whereby we can find solutions to the Laplace equations.

The Laplace Equation

- The most general Laplace equation problem has the same two types of boundary conditions:

$$\nabla^2 \phi = 0 \quad , \quad \text{with } \phi_S \text{ for Dirichlet b.c., and } \left(\hat{n} \cdot \vec{\nabla} \phi \right)_S \text{ for Neumann b.c.}$$



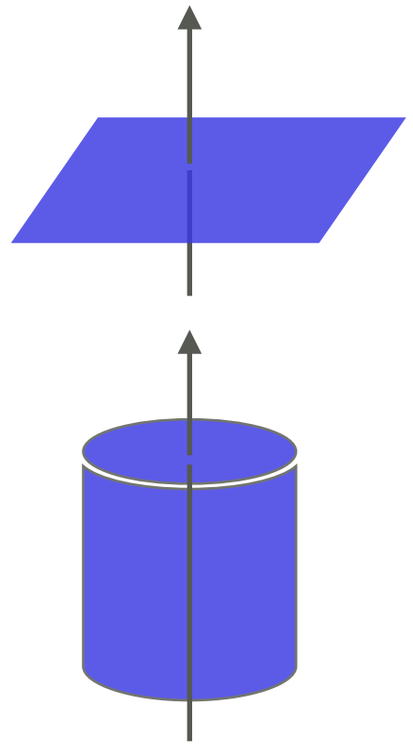
- It is important to recall the ***meaning*** of the Laplace equation: even in the absence of any sources, the Laplace equation ***propagates the boundary conditions*** to the whole space, generating the potential at all points — even where there is ***zero charge***.

- In one dimension, with planar symmetry (i.e., with symmetry in the $x - y$ plane), the general solution of the Laplace equation given some b.c. (e.g., at $z = 0$) is given by:

$$\phi \to c + \alpha z \quad \text{(check!)}$$

- In one dimension, with axial symmetry, the Laplace equation in cylindrical coordinates give us the general solution:

$$\phi \to c + \alpha \log z \quad \text{(check!)}$$



The Laplace Equation

- Let's try another, more interesting boundary value problem, but now effectively in 2D.
- Suppose that the $z = 0$ plane has a surface charge distribution such that the normal electric field is given by:

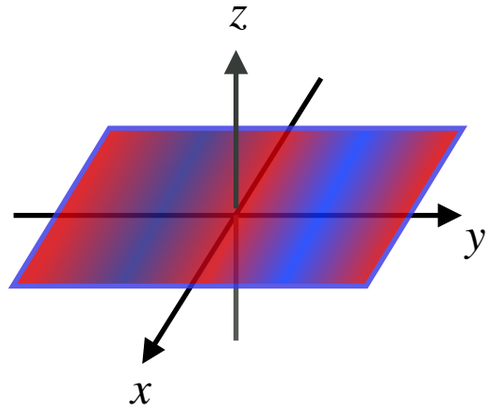
$$\vec{E}_{\perp} = E_0 \, \hat{z} \, \cos k y$$ (check that the surface charge density is $\sigma = \epsilon_0 E_0 \cos k y$!)

- Therefore, at the boundary ($z = 0$), the condition is that:

$$\frac{\partial \phi}{\partial n} = \hat{n} \cdot \vec{\nabla} \phi = - E_0 \, \cos k y$$

- Clearly, in this problem we have full symmetry along the x axis, so in Cartesian coordinates we have:

$$\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$



- This is the typical problem where we could make an attempt at the method of separation of variables: $\phi(y, z) = \sum_i Y_i(y) \, Z_i(z)$,

  where the sum takes advantage of the fact that the equation is *linear*. We then have:

$$\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \sum_i \left[Z_i(z) \frac{\partial^2 Y_i(y)}{\partial y^2} + Y_i(y) \frac{\partial^2 Z_i(z)}{\partial z^2} \right] = 0$$

- SInce each term must be zero, we divide by $Y_i \, Z_i$ and get:

$$\frac{1}{Y_i(y)} \frac{\partial^2 Y_i(y)}{\partial y^2} + \frac{1}{Z_i(z)} \frac{\partial^2 Z_i(z)}{\partial z^2} = 0 \quad , \quad \text{so} \quad \frac{1}{Y_i(y)} \frac{\partial^2 Y_i(y)}{\partial y^2} = - \frac{1}{Z_i(z)} \frac{\partial^2 Z_i(z)}{\partial z^2} = - k_i^2$$ (where k_i may be complex)

The Laplace Equation

- The basic equation we must solve is now much simpler:

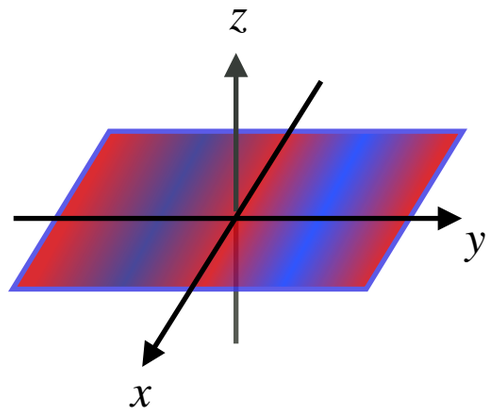
$$\frac{\partial^2 Y_i}{\partial y^2} + k_i^2 Y_i = 0 \quad ,$$

with the obvious solutions $Y_i \rightarrow \cos(k_i y + \varphi_i)$, where φ_i are phases.

- Similarly, for the other basis functions we have:

$$\frac{\partial^2 Z_i}{\partial z^2} - k_i^2 Z_i = 0 \quad ,$$

with the obvious solutions $Z_i \rightarrow \exp(\pm k_i z + \alpha_i)$, where α_i are also some "phases".



- It is not hard to see what is going to happen. At large distances from the plane, an observer will see a zero net charge: the positive and negative average out, and there is no Electric field. Therefore, the solution that satisfies the boundary conditions both at $z = 0$ and at $z \rightarrow \pm\infty$ are:

$$\phi(y,z) = \frac{E_0}{k}\, e^{-k|z|}\, \cos(ky)$$

- The kind of solution found above would also work for Dirichlet boundary conditions. Suppose we specified $\phi(z=0) = \phi_0 \cos(ky)$. Then would would immediately find the solution:

$$\phi(y,z) = \phi_0\, e^{-k|z|}\, \cos(ky)$$

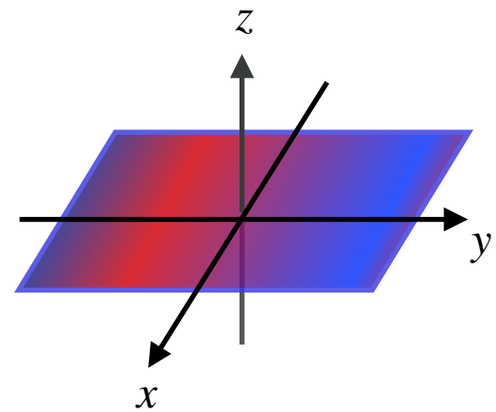
The Laplace Equation

- The solution above also suggests how to tackle a more complicated problem.
  Let's say that we have the boundary conditions:

$$\phi(z=0) = f(y) \quad , \quad \text{with} \quad \langle f \rangle_{z=0} \to 0$$

- We can then expand this function in a Fourier series:

$$f(y) = \sum_n f_n \cos(k_n y + \varphi_n)$$



- Linearity again comes to the rescue, and for *each* mode n we have a solution:

$$\phi_n(y,z) = f_n\, e^{-k_n|z|}\, \cos(k_n y + \varphi_n)$$

$$\Rightarrow \quad \phi(y,z) = \sum_n \phi_n(y,z)$$

from where it follows that $\phi(y, z=0) = f(y)$

The Laplace Equation

- Now, consider what would happen if the surface was shifted to a different position, say, $z = L$. The only change in the solution would be:

$$\phi_n(y,z) \to f'_n\, e^{-k_n|z-L|}\, \cos(k_n y + \varphi_n)$$

- OK, you can see where we are going with this. Now, suppose we give boundary conditions both at $z = 0$ and at $z = L$. Then, the solution would be:

$$\phi(y,z) = \sum_n \left[g_n\, e^{-k_n|z|}\, \cos(k_n y + \varphi_n) + g'_n\, e^{-k_n|z-L|}\, \cos(k_n y + \varphi_n) \right]$$

- We now need to solve for the constants g_n, g'_n, so that the field satisfies the boundary conditions:

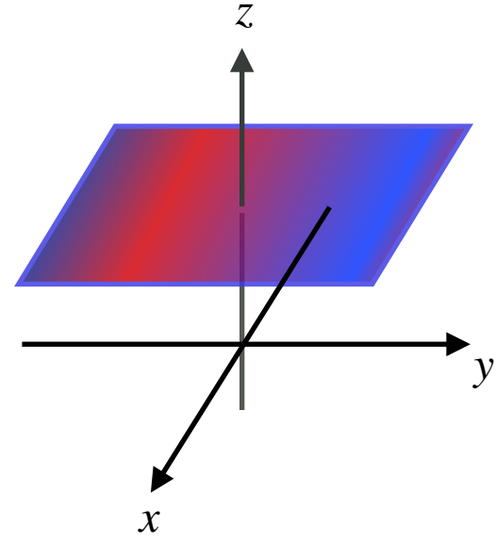
$$\phi(y, z=0) \to g_n\, \cos(k_n y + \varphi_n) + g'_n\, e^{-k_n L}\, \cos(k_n y + \varphi_n) = f_n \cos(k_n y + \varphi_n)$$

$$\phi(y, z=L) \to g_n\, e^{-k_n L}\, \cos(k_n y + \varphi_n) + g'_n\, \cos(k_n y + \varphi_n) = f'_n \cos(k_n y + \varphi_n)$$

- The equations then reduce to the simple linear problem:

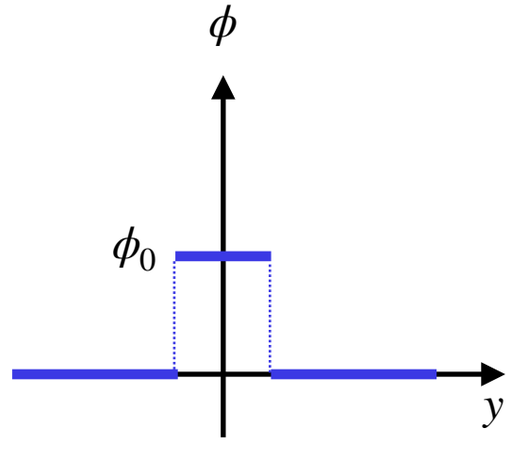
$$g_n + g'_n\, e^{-k_n L} = f_n$$

$$g_n\, e^{-k_n L} + g'_n = f'_n$$



- I will leave it up to you to finish this, if you feel like so!
- Try to think about how to generalize this to a boundary condition at $z = 0$ such that $\phi(x, y, z = 0) = f(x, y)$!

The Laplace Equation



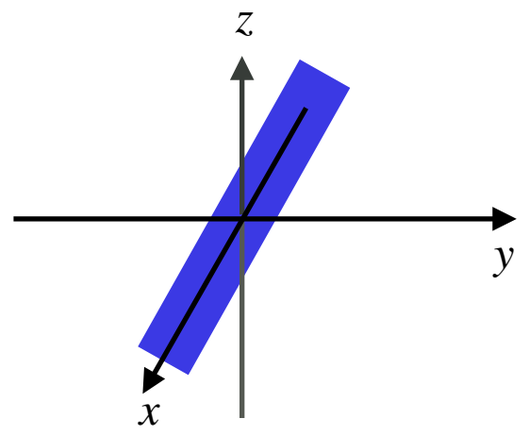
- Let's consider an "advanced" problem that has an exact solution in terms of these basis functions. Take a "strip" of length L in the $z = 0$ plane where we set the potential to:

$$\phi(|y| < L/2\,, z = 0) = \phi_0$$

$$\phi(|y| > L/2\,, z = 0) = 0$$

- You may recall how to write a "step" in terms of a Fourier expansion. This is how:

$$\phi(y, z = 0) = \phi_0 \frac{2}{\pi} \int_0^\infty \frac{dk}{k} \sin\left(\frac{kL}{2}\right) \cos(ky)$$



- Substituting into the Laplace equation and using the solutions that were found there we obtain:

$$\phi(y, z) = \phi_0 \frac{2}{\pi} \int_0^\infty \frac{dk}{k}\, e^{-k|z|}\, \sin\left(\frac{kL}{2}\right) \cos(ky)$$

- But we can write this as:

$$\phi(y, z) = \frac{\phi_0}{\pi} \int_0^\infty \frac{dk}{k}\, e^{-k|z|}\, \left[\sin k(y + L/2) - \sin k(y - L/2)\right]$$

- Now, notice that this integral is exact, in fact:

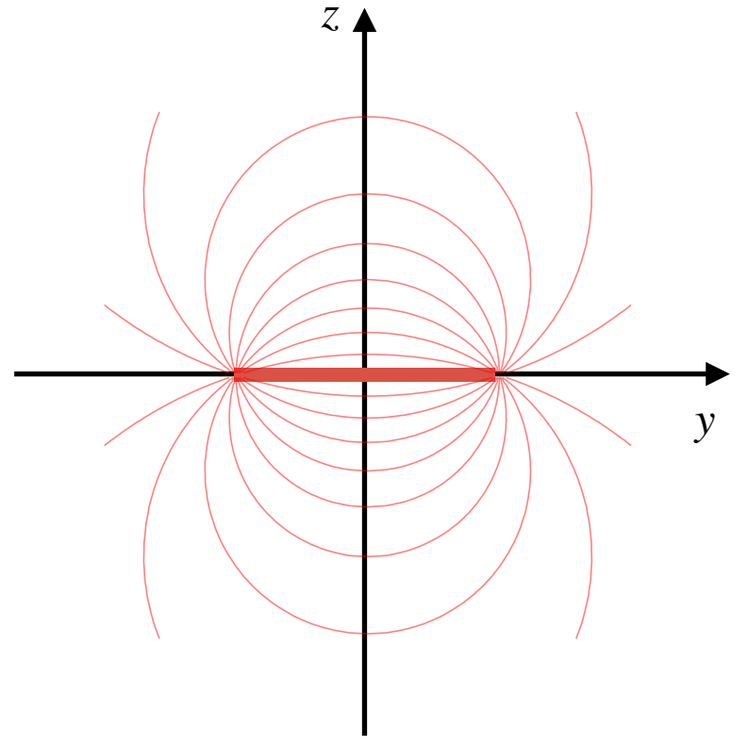
$$\int_0^\infty \frac{dk}{k}\, e^{-k|z|}\, \sin ky = \mathrm{ArcTan}\left(\frac{y}{|z|}\right)$$

The Laplace Equation

- Using the integral above we rewrite the potential for this problem as:

$$\phi(y,z) = \frac{\phi_0}{\pi}\left[\text{ArcTan}\left(\frac{y+L/2}{|z|}\right) - \text{ArcTan}\left(\frac{y-L/2}{|z|}\right)\right]$$

- The shape of the potential is shown in the figure on the right.

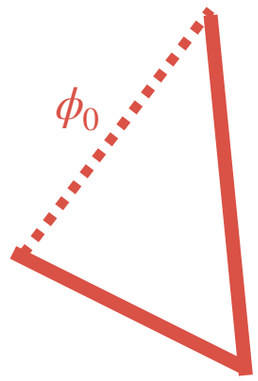


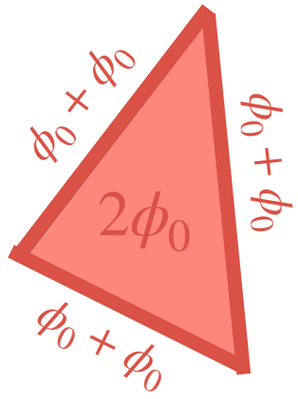
- We could (***I could***!) now start thinking about **combining** strips such as this one, in the $y-z$ plane. But we have to be ***very careful***, because when we **add one strip**, this **changes the field on the other strip**. Hence, the solution that sums the solution of two strips does not obey the original boundary conditions that the potential is ϕ_0 on each strip!

- It is quite fascinating that these arctangent solutions have the following "crazy" property: suppose that we ***join*** two solutions for each strip, each with the same amplitude ϕ_0 — and that the field is the sum of the two fields.

- Then you can show (!) that, even though the **potential at each one of the original strips is not simply** ϕ_0 anymore, the **remaining side** of the **triangle** formed by those two strips has a potential ϕ_0 !!

- This means that if we now **actually** put a stripe in that side of the triangle with potential ϕ_0, and then each two sides would induce a potential ϕ_0 on the remaining side, and we would end up with the following situation:





- Well, now what do you think will happen inside that triangle?…

- Yes, the potential **inside** would be constant — it would in fact be $2\phi_0$, like on the boundary!

- But now we have an ***exact*** solution (inside & out) for any ***triangular shaped*** region of constant potential!

The Laplace Equation

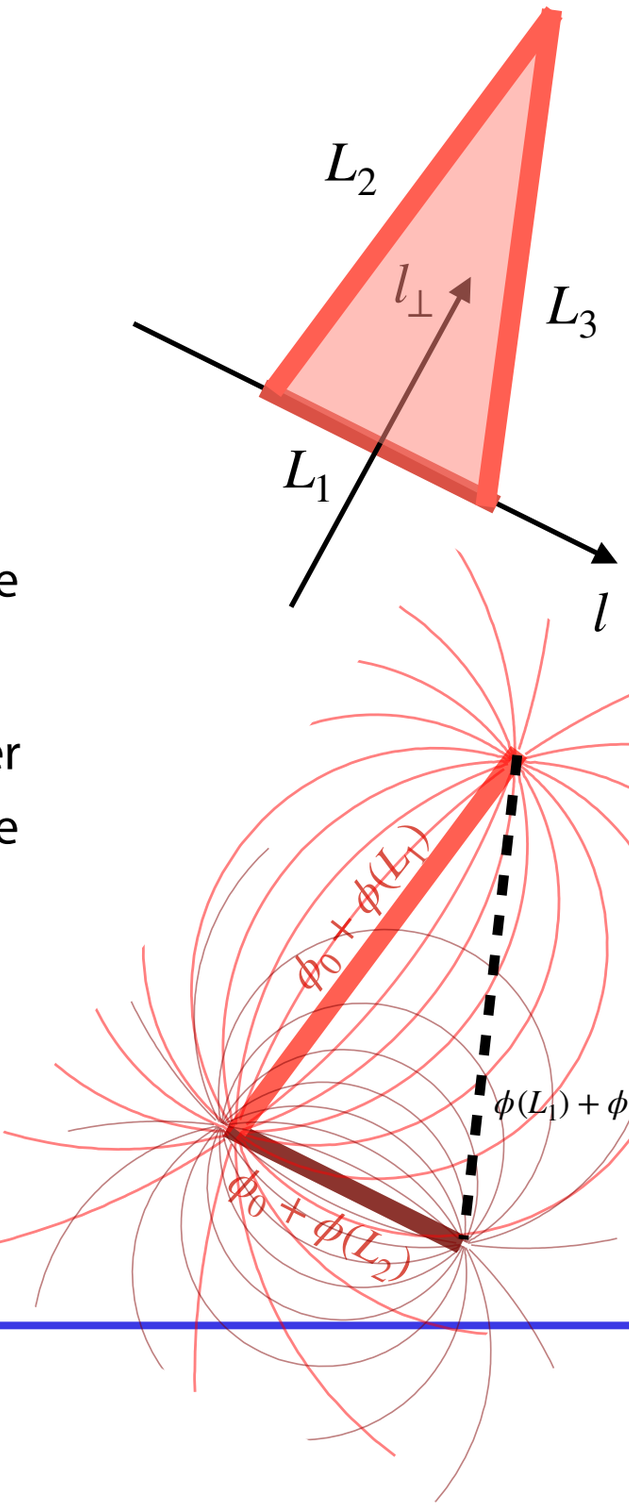
- What is going on here? A formal, exact solution for a strip along the direction l is:

$$\phi^{(L_1)}(l, l_\perp) = \frac{\phi_0}{\pi}\left[\text{ArcTan}\left(\frac{l + L/2}{|l_\perp|}\right) - \text{ArcTan}\left(\frac{l - L/2}{|l_\perp|}\right)\right]$$

- On the other hand, it must be true that, for any **closed surface** with a **fixed value of ϕ as a boundary condition,** the potential inside that surface must be **homogeneous**, and **equal to the value at the boundary.**

- Suppose we start with one line (L_1), and that we close a triangle with two other lines (L_2 and L_3). Then, linearity of the Laplace equation implies that it **must** be true that the sum of all three solutions yield a **constant** inside the triangle.

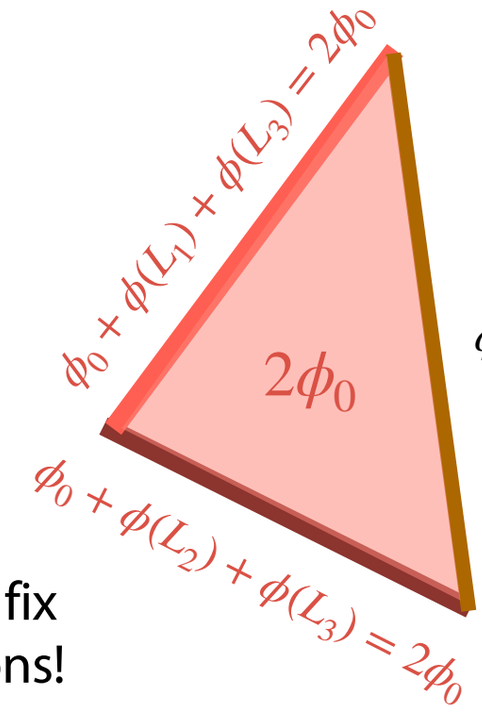
- But if the solution is a **constant inside the triangle**, and the boundary condition is itself a constant line for each solution for one side of the triangle, then it **must be** that the **other two lines contribute a constant** at the **other side** of the triangle!



The Laplace Equation

- The combination of all this is then:

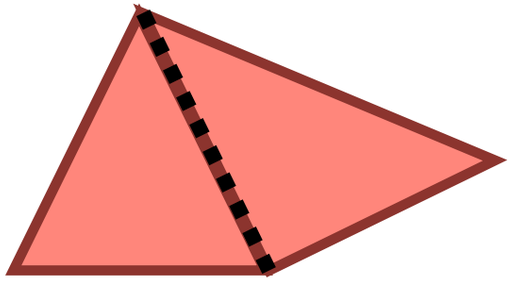
$$\phi(y,z) = \phi^{(L_1)}(y,z) + \phi^{(L_2)}(y,z) + \phi^{(L_3)}(y,z)$$



$$\phi_0 + \phi(L_1) + \phi(L_2) = 2\phi_0$$

- This is the *magic* of the harmonic functions: they "know" how to fix the solution everywhere, while satisfying the boundary conditions!

- Now, a provocation for you: consider the following situation. Suppose we now combine two triangles:



What now? …

Can we write an **exact solution** for any **polygon in 2D** using the solution for the line segment?

The Relaxation Method

- The Laplacian operator tells us that at any point in space the sum total of the second derivatives cancel out:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0$$

- Let's think about what this means in 2D, in terms of a "grid" of points, all spaced by some "cell size" ℓ , where we measure the field:

$$\phi(x_i,y_j)\equiv\phi_{i,j}$$

- The derivative at the point x_i along the x direction can be approximated by:

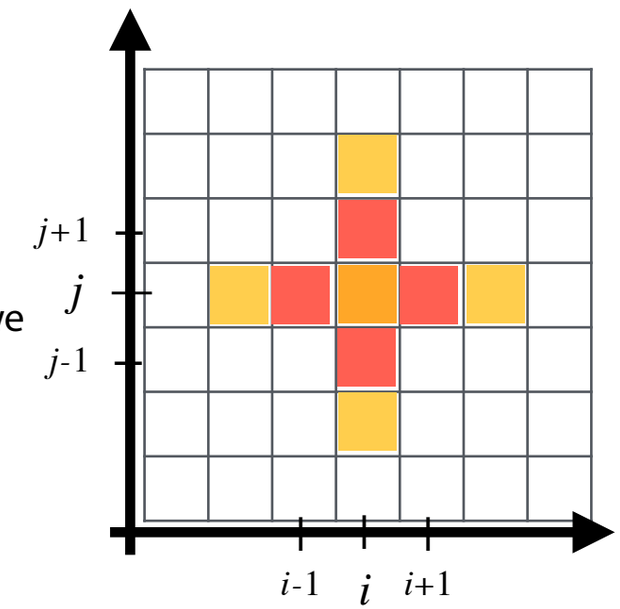
$$\phi^x_{i,j}=\frac{1}{2\ell}\left(\phi_{i+1,j}-\phi_{i-1,j}\right)$$

- The derivative at the points x_{i+1} and x_{i-1} along the x direction are then:

$$\phi^x_{i+1,j}=\frac{1}{2\ell}\left(\phi_{i+2,j}-\phi_{i,j}\right)\quad,\quad\text{and}\quad\phi^x_{i-1,j}=\frac{1}{2\ell}\left(\phi_{i,j}-\phi_{i-2,j}\right)$$

- Similarly, the derivative at the points y_{j+1} and y_{j-1} along the y direction are:

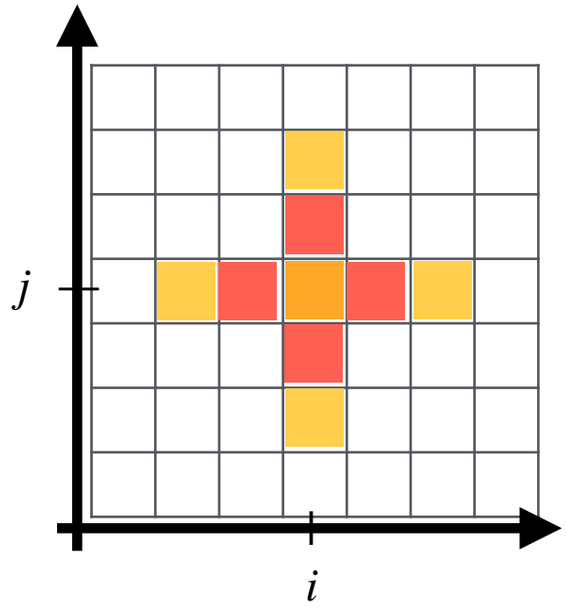
$$\phi^y_{i,j+1}=\frac{1}{2\ell}\left(\phi_{i,j+2}-\phi_{i,j}\right)\quad,\quad\text{and}\quad\phi^x_{i,j-1}=\frac{1}{2\ell}\left(\phi_{i,j}-\phi_{i,j-2}\right)$$



The Relaxation Method

- Now, the *second derivative* at the point x_i and y_j along the x direction is:

$$\phi^{xx}_{i,j} = \frac{1}{2\ell}\left(\phi^{x}_{i+1,j} - \phi^{x}_{i-1,j}\right) = \frac{1}{(2\ell)^2}\left(\phi_{i+2,j} + \phi_{i-2,j} - 2\phi_{i,j}\right)$$



- Similarly also at the same point, the second derivative along the y direction is:

$$\phi^{yy}_{i,j} = \frac{1}{2\ell}\left(\phi^{x}_{i,j+1} - \phi^{x}_{i,j-1}\right) = \frac{1}{(2\ell)^2}\left(\phi_{i,j+2} + \phi_{i,j-2} - 2\phi_{i,j}\right)$$

- The Laplace equation for the cell x_i, y_j is now the statement that:

$$\phi^{xx}_{i,j} + \phi^{yy}_{i,j} = \frac{1}{(2\ell)^2}\left(\phi_{i+2,j} + \phi_{i-2,j} + \phi_{i,j+2} + \phi_{i,j-2} - 4\phi_{i,j}\right) = 0$$

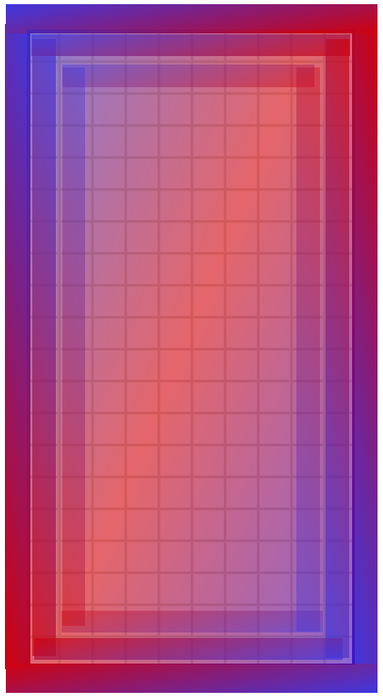
- We can write that equation as:

$$\phi_{i,j} = \frac{1}{4}\left(\phi_{i+2,j} + \phi_{i-2,j} + \phi_{i,j+2} + \phi_{i,j-2}\right) = \langle\phi\rangle$$

The Relaxation Method

- This can be generalized to any form of mean taken in space:

$$\nabla^2\phi = 0 \quad \Longrightarrow \quad \phi(\vec{x}) = \langle\phi\rangle_{\text{near } \vec{x}}$$



- This suggests to us yet another method to solve for the Laplace equation. Let's say we provide a boundary — for simplicity, think of a closed volume enveloped by some boundary where we specify the field (or its derivatives).

- We can start with the field in this volume (outside of the boundary, of course) set to $\phi \to \phi^{(0)}_{i,j} = 0$.

- The first step is to "propagate" the information from the boundary to the cells just next to that boundary:

$$\phi^{(0)} \to \phi^{(1)}_{i,j} = \langle\phi^{(0)}\rangle_{i,j}\,.$$

- We then iterate this process: each time we update the field, it propagates the information further and further inside our volume:

$$\phi^{(n)} \to \phi^{(n+1)}_{i,j} = \langle\phi^{(n)}\rangle_{i,j}$$

- After a few iterations, the field has converged to the "mean" value, and it satisfies the Laplace equation! Let's take a look at a numerical example in "real time". [Go to Mathematica & Python]

Next class:

- Laplace equation in spherical coordinates
- Basis functions in spherical coordinates: Legendre polynomials, Spherical Harmonics and Spherical Bessel functions
- Multipole expansion part 1
- Jackson, Chapters 2 & 3