PGF5112 - Plasma Physics I

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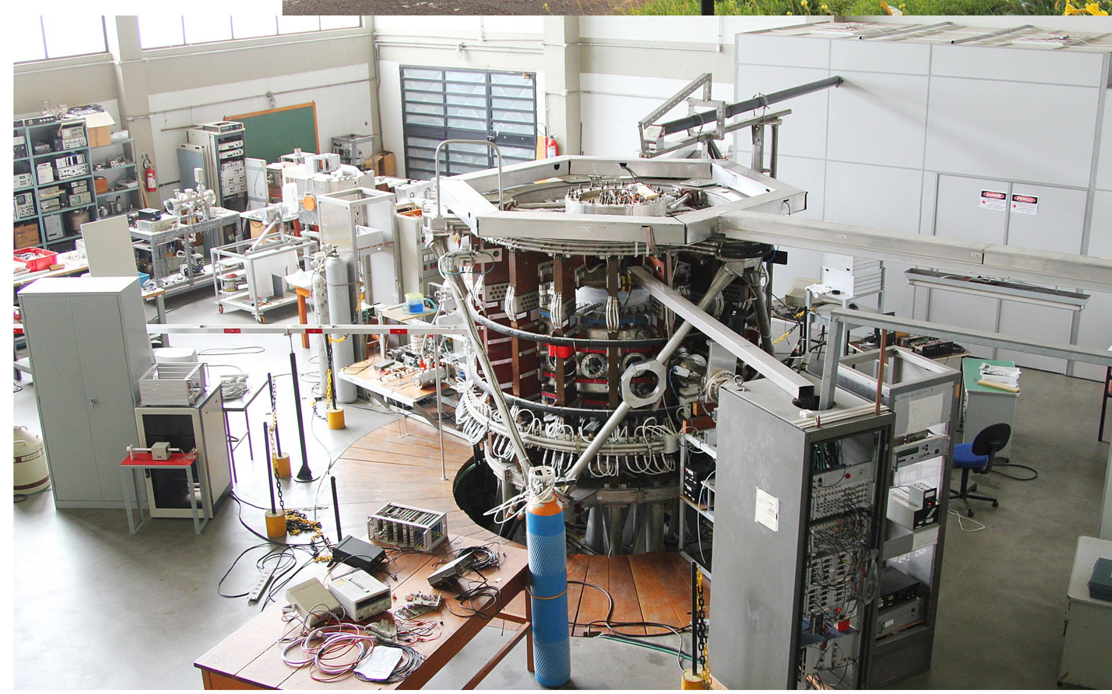

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- **Fundamentals of plasma kinetic theory**
  - *Introduction*
  - *Phase space*
  - *Distribution function*
  - *The Klimontovich distribution function*
  - *The Boltzmann equation*
  - *The Vlasov equation*

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Fundamental of plasma kinetic theory: introduction

- **Plasmas are systems with a very large number of interacting charged particles**
  - *It is appropriate and convenient to use a statistical approach*

- **In this lecture, basic elements of kinetic theory, such as phase space and distribution function, are presented**

- **All the physically interesting information about the system is contained in the distribution function**
  - *From knowledge of the distribution function, macroscopic variables of physical interest, such as mass density, gas flow velocity and pressure, can be systematically deduced*

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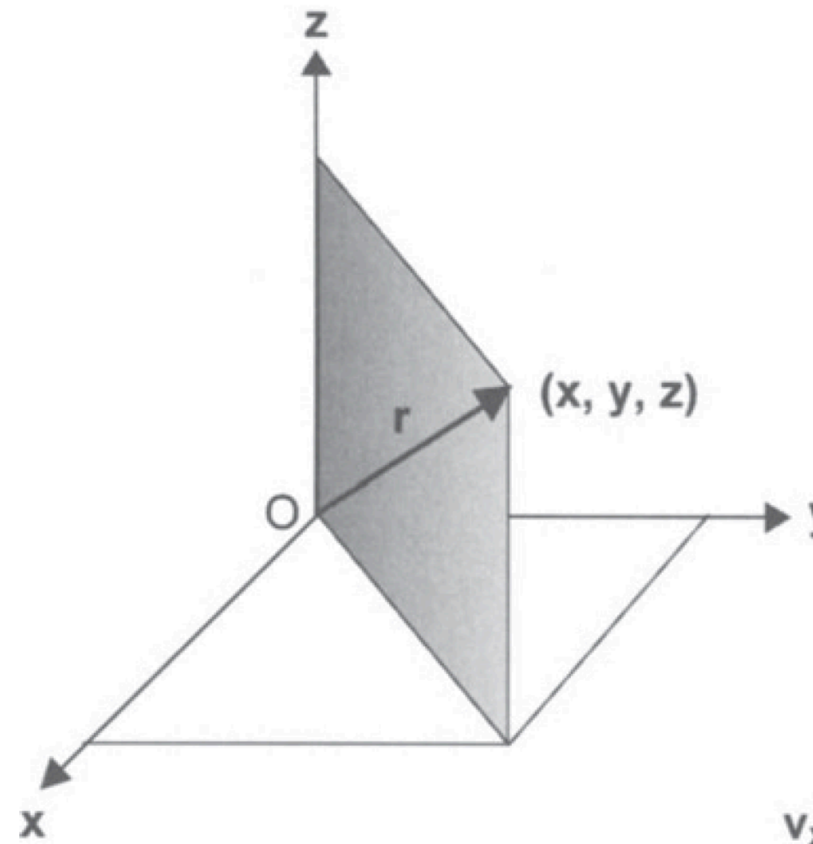
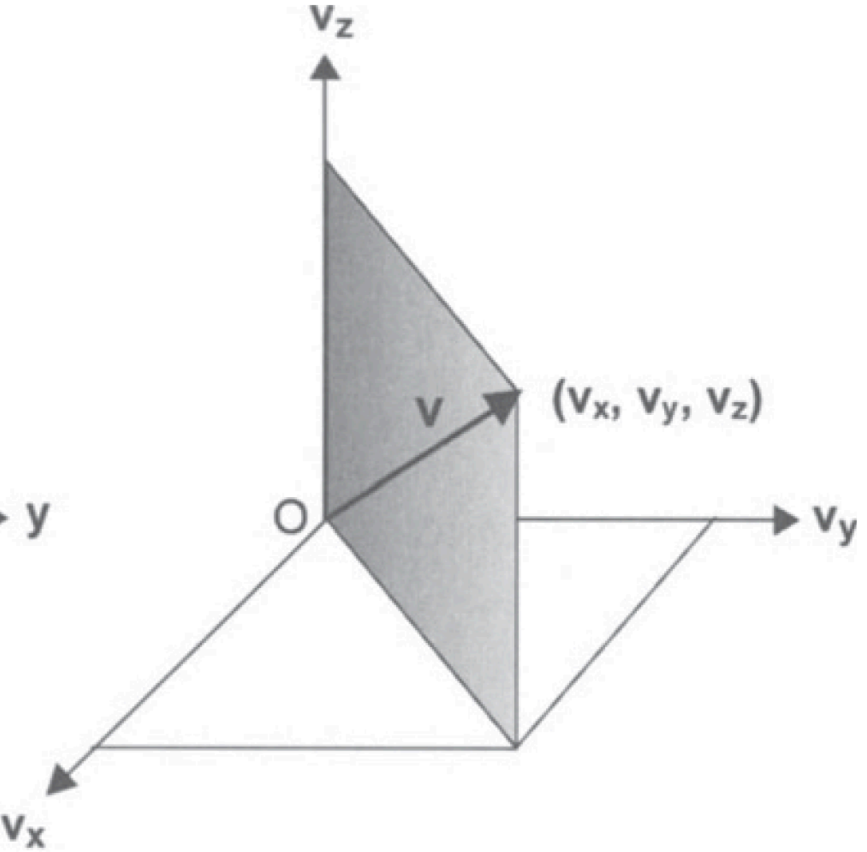
Fundamental of plasma kinetic theory: phase space

- **At any instant of time, each particle of the plasma can be localized by its position vector, for example, $\mathbf{r} = x\hat{\mathbf{e}}_\mathbf{x} + y\hat{\mathbf{e}}_\mathbf{y} + z\hat{\mathbf{e}}_\mathbf{z}$ in a Cartesian coordinate system, and each particle have a velocity vector, $\mathbf{v} = v_x\hat{\mathbf{e}}_\mathbf{x} + v_y\hat{\mathbf{e}}_\mathbf{y} + v_z\hat{\mathbf{e}}_\mathbf{z}$**

- **In analogy with the configurational space (real space), defined by the position coordinates (x,y,z), it is convenient to introduce the velocity space**
  - *The velocity vector in the velocity space is seen as the position vector in the configurational space*





Single-particle phase space

- **From the point of view of classical mechanics, the instantaneous dynamic state of each particle is specified by its position and velocity vectors**
  - *It is convenient, therefore, to consider the phase space defined by the 6 coordinates* (x, y, z, v_x, v_y, v_z)
- **In this 6-dimensional (6D) space, the dynamic state of each particle is appropriately represented by a single point**
  - *The coordinates* $(\mathbf{r}, \mathbf{v})$ *of the point give the instantaneous particle position and velocity*
- **When the particle moves, its representative point describes a trajectory in phase space**
- **At each instant of time, the dynamic state of a system of N particles is represented by N points in phase space**
  - *This single-particle phase space also sometimes called* μ*-space*

Many-particle phase space

- **One can also define a phase space for the whole system of particles**
  - *In this case, a system consisting of N particles, with no internal degrees of freedom, is represented by a single point in a 6N-dimensional space*

- **This many-particle phase space, also called Γ-space, is defined by the 3N positions $(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots, \mathbf{r}_N)$ and by the 3N velocities $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_N)$**
  - *Thus, a point in this 6N-D space represents the single microscopic state of the whole system*
  - *This many-particle is used in statistical mechanics and advanced kinetic theory*

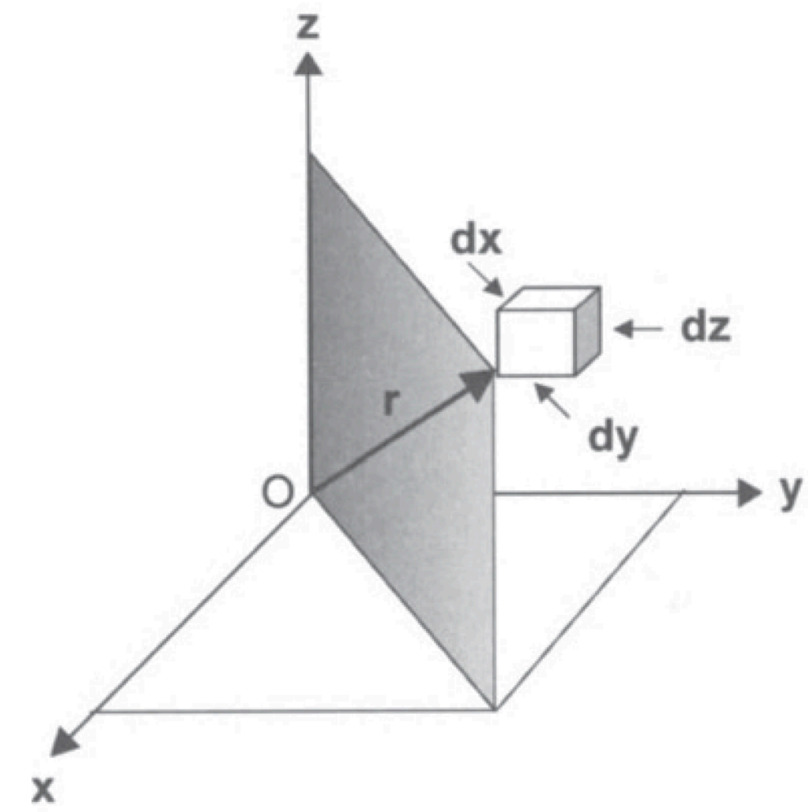
- **The single-particle phase space is the one normally used in elementary kinetic theory and basic plasma physics, and is the one that will be used here**

Volume elements

- **A small element of volume in configuration space is** $dV = d^3r = d\mathbf{r} = dxdydz$
  - *Note that this dV should not be taken literally as a mathematically infinitesimally quantify, but as a finite volume that is sufficiently large, to contain a very large number of particles, yet sufficiently small, in comparison with the plasma characteristic spatial variation lengths*

- **For example, in a gas containing 10^{18} molecules/m³, if we take $dV = 10^{-12}\ m^{-3}$, which in a macroscopic scale can be considered as a point, there are still 10^6 particles inside dV**

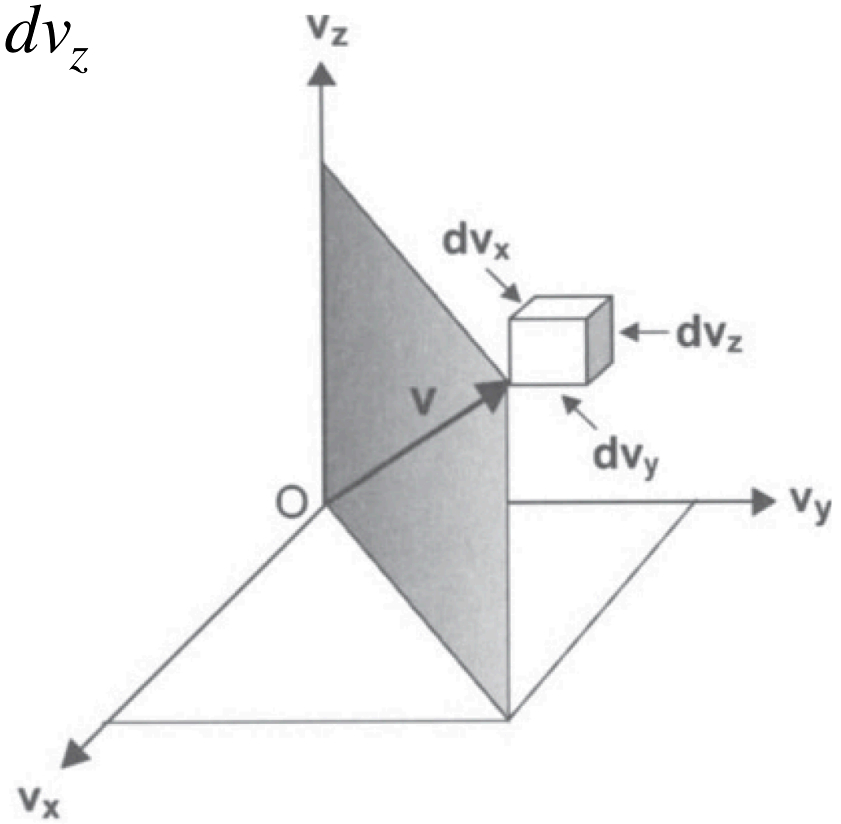
Plasmas that do not allow the choice of such a differential volumes as indicated cannot be analyzed statistically



Volume elements

- **When we refer to a particle as being situated inside dV and at $\mathbf{r}$, means that its x-coordinate is between x and $x + dx$, its y-coordinate is between y and $y + dy$ and its z-coordinate is between z and $z + dz$**
  - *Note that particles at $\mathbf{r}$ and inside dV may have completely arbitrary velocities, which would be represented by scattered points in velocity space*

- **When we refer to a particle as being situated inside $d3v$ and at $\mathbf{v}$, means that its v_x-component is between v_x and $v_x + dv_x$, its v_y-component is between v_y and $v_y + dv_y$ and its v_z-component is between v_z and $v_z + dv_z$**

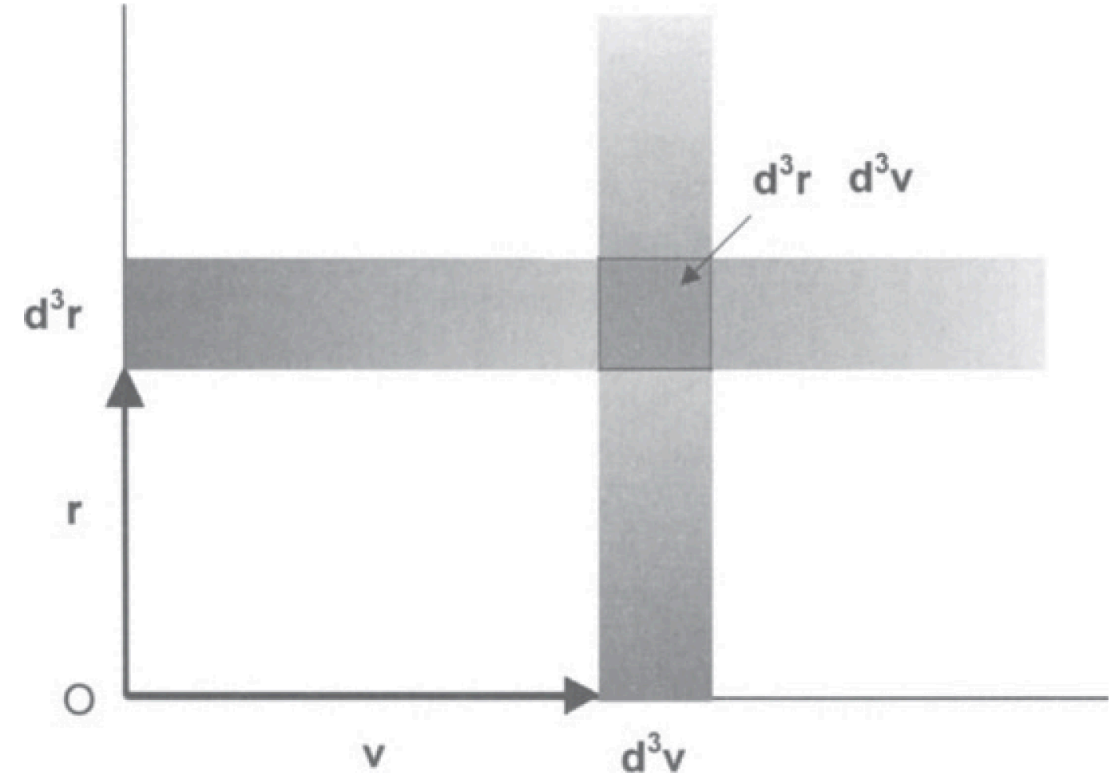


Volume elements

- **In phase space (μ-space), a differential element of volume may be imagined as a 6D cube, represented by**

Phase space volume element

$$d^3r d^3v = dx dy dz dv_x dv_y dv_z$$



  - *Note that inside $d^3r d^3v$ there are only particle inside d^3r around $\mathbf{r}$ whose velocities lie inside d^3v about $\mathbf{v}$*
  - *Here, the coordinates $\mathbf{r}$ and $\mathbf{v}$ are considered to be independent variables since, together, they represent the position of individual volume elements in phase space*

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Distribution function

- **Let $d^6N_\alpha(\mathbf{r},\mathbf{v},t)$ denote the number of particles of type α inside the volume element d^3rd^3v, around the phase space coordinates $(\mathbf{r},\mathbf{v})$, at the instant t**

- **The function distribution in phase space, $f_\alpha(\mathbf{r},\mathbf{v},t)$, is defined as the density of representative points of type α particles in phase space, that is,**

$$f_\alpha(\mathbf{r},\mathbf{v},t) = \frac{d^6N_\alpha(\mathbf{r},\mathbf{v},t)}{d^3rd^3v}$$

- **In volume elements d^3rd^3v with very large velocity coordinates (v_x, v_y, v_z), the number of representative points is relatively small since, in any macroscopic system, there exists few particles with very large velocities**
  - *The function distribution must tend to zero as the velocity gets infinitely large*

- **When $f_\alpha(\mathbf{r},\mathbf{v},t)$ depends on $\mathbf{r}$, the function distribution is inhomogeneous**
  - *In the absence of external forces, an inhomogeneous distribution function evolves, due to particle collisions, to an homogeneous distribution function $f_\alpha(\mathbf{v})$ that does not depend on $\mathbf{r}$ anymore*

Distribution function

- **When $f_\alpha(\mathbf{r}, \mathbf{v}, t)$ depends on the orientation of the vector $\mathbf{v}$, the distribution function can be anisotropic**
  - *When $f_\alpha(\mathbf{r}, \mathbf{v}, t)$ depends only on the magnitude of the vector, i.e. on $v = |\mathbf{v}|$, the distribution function $f_\alpha(\mathbf{r}, v, t)$ is isotropic*

- **The statistical description of different types of plasma requires the use of inhomogeneous and/or anisotropic distribution functions**

- **In a statistical sense, $f_\alpha(\mathbf{r}, \mathbf{v}, t)$ contains all the information about the system**
  - *Knowing $f_\alpha(\mathbf{r}, \mathbf{v}, t)$ allows us to deduce all the macroscopic variable of physical interest for particles of type α*

- **One of the primary problems of kinetic theory consists on determining $f_\alpha(\mathbf{r}, \mathbf{v}, t)$**
  - *Therefore, one needs to find an equation for the evolution of $f_\alpha(\mathbf{r}, \mathbf{v}, t)$*

Macroscopic variables: number density and average velocity

- **The number density, $n_\alpha(\mathbf{r}, t)$, is a macroscopic variable defined, in the configurational space, as the number of particles of type α per unit volume, irrespective of velocity**

$$n_\alpha(\mathbf{r}, t) = \int_v \frac{d^6 N_\alpha(\mathbf{r}, \mathbf{v}, t)}{d^3 r} = \int_v f_\alpha(\mathbf{r}, \mathbf{v}, t)\, d^3 v$$

  - *Here, the single integral sign represents in fact a triple integral extending over the entire velocity space*

- **The average velocity, $\mathbf{u}_\alpha(\mathbf{r}, t)$, is a macroscopic variable defined, in the configurational space, as the macroscopic flow of particles of type α in the neighborhood of position $\mathbf{r}$ at the instant t**

$$\mathbf{u}_\alpha(\mathbf{r}, t) = \frac{1}{n_\alpha(\mathbf{r}, t)} \int_v \mathbf{v}\, \frac{d^6 N_\alpha(\mathbf{r}, \mathbf{v}, t)}{d^3 r} = \frac{1}{n_\alpha(\mathbf{r}, t)} \int_v \mathbf{v}\, f_\alpha(\mathbf{r}, \mathbf{v}, t)\, d^3 v$$

  - *Note that $n_\alpha(\mathbf{r}, t)$ and $\mathbf{u}_\alpha(\mathbf{r}, t)$ depend only on $\mathbf{r}$ and t*

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The Klimontovich distribution function

- **As a first attempt to describe a system using kinetic theory, let's write the equations of motion for the j[th] particle of type α**

$$\frac{d\mathbf{r}_{\alpha,\mathbf{j}}}{dt} = \mathbf{v}_{\alpha,\mathbf{j}}$$

$$m_\alpha \frac{d\mathbf{v}_{\alpha,\mathbf{j}}}{dt} = q_\alpha \left[\mathbf{E_m}(\mathbf{r}_{\alpha,\mathbf{j}}, t) + \mathbf{v}_{\alpha,\mathbf{j}} \times \mathbf{B_m}(\mathbf{r}_{\alpha,\mathbf{j}}, t)\right]$$

$\mathbf{E_m}$ and $\mathbf{B_m}$ are the microscopic electric and magnetic fields

- **Naturally, the orbits of particles can be described by the exact distribution function**

$$f_\alpha^{exact}(\mathbf{r}, \mathbf{v}, t) = \sum_{j=1}^{N_\alpha} \delta[\mathbf{r} - \mathbf{r}_{\alpha,\mathbf{j}}(t)]\delta[\mathbf{v} - \mathbf{v}_{\alpha,\mathbf{j}}(t)]$$

  - *Here, N_α is the total number of particles of type α*

- **The normalization of the exact distribution function**

$$\int_r \int_v f_\alpha^{exact}(\mathbf{r}, \mathbf{v}, t) d^3r d^3v = \sum_{j=1}^{N_\alpha} \int_r \delta[\mathbf{r} - \mathbf{r}_{\alpha,\mathbf{j}}(t)] d^3r \int_v \delta[\mathbf{v} - \mathbf{v}_{\alpha,\mathbf{j}}(t)] d^3v = \sum_{j=1}^{N_\alpha} 1 = N_\alpha$$

The Klimontovich distribution function

- **The number density for this distribution function is**

$$n_\alpha(\mathbf{r},t) = \int_v f_\alpha^{exact}(\mathbf{r},\mathbf{v},t)d^3v = \sum_{j=1}^{N_\alpha} \delta[\mathbf{r}-\mathbf{r}_{\alpha,\mathbf{j}}(t)]\int_v \delta[\mathbf{v}-\mathbf{v}_{\alpha,\mathbf{j}}(t)]d^3v = \sum_{j=1}^{N_\alpha} \delta[\mathbf{r}-\mathbf{r}_{\alpha,\mathbf{j}}(t)]$$

- **In the same way, the average velocity**

$$\mathbf{u}_\alpha(\mathbf{r},t) = \int_v \mathbf{v} f_\alpha^{exact}(\mathbf{r},\mathbf{v},t)d^3v = \sum_{j=1}^{N_\alpha} \delta[\mathbf{r}-\mathbf{r}_{\alpha,\mathbf{j}}(t)]\int_v \mathbf{v}\,\delta[\mathbf{v}-\mathbf{v}_{\alpha,\mathbf{j}}(t)]d^3v = \sum_{j=1}^{N_\alpha} \mathbf{v}_{\alpha,\mathbf{j}}(t)\,\delta[\mathbf{r}-\mathbf{r}_{\alpha,\mathbf{j}}(t)]$$

- **In addition, the microscopic electromagnetic fields must satisfy Maxwell's equation**

$$\nabla\cdot\mathbf{E_m} = \frac{\rho_m}{\epsilon_0} \qquad \nabla\cdot\mathbf{B_m} = 0 \qquad \nabla\times\mathbf{E_m} = -\frac{\partial\mathbf{B_m}}{\partial t} \qquad \nabla\times\mathbf{B_m} = \mu_0\mathbf{J_m} + \mu_0\epsilon_0\frac{\partial\mathbf{E_m}}{\partial t}$$

where the microscopic sources must be determined self-consistently:

$$\rho_m(\mathbf{r},t) = \sum_\alpha q_\alpha \int_v f_\alpha^{exact}(\mathbf{r},\mathbf{v},t) = \sum_\alpha q_\alpha \sum_{j=1}^{N_\alpha} \delta[\mathbf{r}-\mathbf{r}_{\alpha,\mathbf{j}}(t)]$$

$$\mathbf{J_m}(\mathbf{r},t) = \sum_\alpha q_\alpha \int_v \mathbf{v} f_\alpha^{exact}(\mathbf{r},\mathbf{v},t) = \sum_\alpha q_\alpha \sum_{j=1}^{N_\alpha} \mathbf{v}_{\alpha,\mathbf{j}}\,\delta[\mathbf{r}-\mathbf{r}_{\alpha,\mathbf{j}}(t)]$$

The Klimontovich distribution function

- **Taking the total time derivative of $f_\alpha^{exact}(\mathbf{r}, \mathbf{v}, t)$, and using the properties of the Dirac delta function, yields**

$$\frac{df_\alpha^{exact}}{dt} = \left(\frac{\partial}{\partial t} + \frac{d\mathbf{r}}{dt}\cdot\frac{\partial}{\partial \mathbf{r}} + \frac{d\mathbf{v}}{dt}\cdot\frac{\partial}{\partial \mathbf{v}}\right)\sum_{j=1}^{N_\alpha}\delta[\mathbf{r}-\mathbf{r}_{\alpha,\mathbf{j}}(t)]\delta[\mathbf{v}-\mathbf{v}_{\alpha,\mathbf{j}}(t)]$$

$$\frac{df_\alpha^{exact}}{dt} = \sum_{j=1}^{N_\alpha}\left(\frac{\partial}{\partial t} + \frac{d\mathbf{r}}{dt}\cdot\frac{\partial}{\partial \mathbf{r}} + \frac{d\mathbf{v}}{dt}\cdot\frac{\partial}{\partial \mathbf{v}}\right)\delta[\mathbf{r}-\mathbf{r}_{\alpha,\mathbf{j}}(t)]\delta[\mathbf{v}-\mathbf{v}_{\alpha,\mathbf{j}}(t)]$$

$$\frac{df_\alpha^{exact}}{dt} = \sum_{j=1}^{N_\alpha}\left(\frac{\partial}{\partial t} - \frac{d\mathbf{r}_{\alpha,\mathbf{j}}}{dt}\cdot\frac{\partial}{\partial \mathbf{r}_{\alpha,\mathbf{j}}} - \frac{d\mathbf{v}_{\alpha,\mathbf{j}}}{dt}\cdot\frac{\partial}{\partial \mathbf{v}_{\alpha,\mathbf{j}}}\right)\delta[\mathbf{r}-\mathbf{r}_{\alpha,\mathbf{j}}(t)]\delta[\mathbf{v}-\mathbf{v}_{\alpha,\mathbf{j}}(t)]$$

$$\frac{df_\alpha^{exact}}{dt} = \sum_{j=1}^{N_\alpha}\left(\frac{d\mathbf{r}_{\alpha,\mathbf{j}}}{dt}\cdot\frac{\partial}{\partial \mathbf{r}_{\alpha,\mathbf{j}}} + \frac{d\mathbf{v}_{\alpha,\mathbf{j}}}{dt}\cdot\frac{\partial}{\partial \mathbf{v}_{\alpha,\mathbf{j}}} - \frac{d\mathbf{r}_{\alpha,\mathbf{j}}}{dt}\cdot\frac{\partial}{\partial \mathbf{r}_{\alpha,\mathbf{j}}} - \frac{d\mathbf{v}_{\alpha,\mathbf{j}}}{dt}\cdot\frac{\partial}{\partial \mathbf{v}_{\alpha,\mathbf{j}}}\right)\delta[\mathbf{r}-\mathbf{r}_{\alpha,\mathbf{j}}(t)]\delta[\mathbf{v}-\mathbf{v}_{\alpha,\mathbf{j}}(t)] = 0$$

The Klimontovich distribution function

- **Therefore, using the properties of the Dirac delta function, one can write**

$$\frac{\partial f_\alpha^{exact}}{\partial t} + \mathbf{v}\cdot\frac{\partial f_\alpha^{exact}}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha}\left[\mathbf{E_m}(\mathbf{r}_{\alpha,\mathbf{j}}, t) + \mathbf{v}\times\mathbf{B_m}(\mathbf{r}_{\alpha,\mathbf{j}}, t)\right]\cdot\frac{\partial f_\alpha^{exact}}{\partial \mathbf{v}} = 0$$

  - *This equation is known as the Klimontovich equation*
  - *Mathematically, this equation contains all N_α particles equations of motion*

- **We must now be careful about the phase space element volume to be used**
  - *To have small fluctuations inside d^3rd^3v, i.e. $\delta N_{6D}/N_{6D} \approx 1/\sqrt{N_{6D}} \ll 1$, we must have dx large compared to the mean spacing of particles in the plasma, i.e. $dx \gg n_0^{-1/3}$*
  - *However, d^3r should not be so large that macroscopic properties of the plasma (e.g. the average number density) vary significantly within it. Therefore, we must have $dx \ll \lambda_D$, which also allows for collective effects*

- **In summary, to have a representative phase space ($\delta N_{6D}/N_{6D} \ll 1$), we must have $n_0^{-1/3} \ll dx \ll \lambda_D$**

The Klimontovich distribution function

- **With these requirements accounted for, we can write**

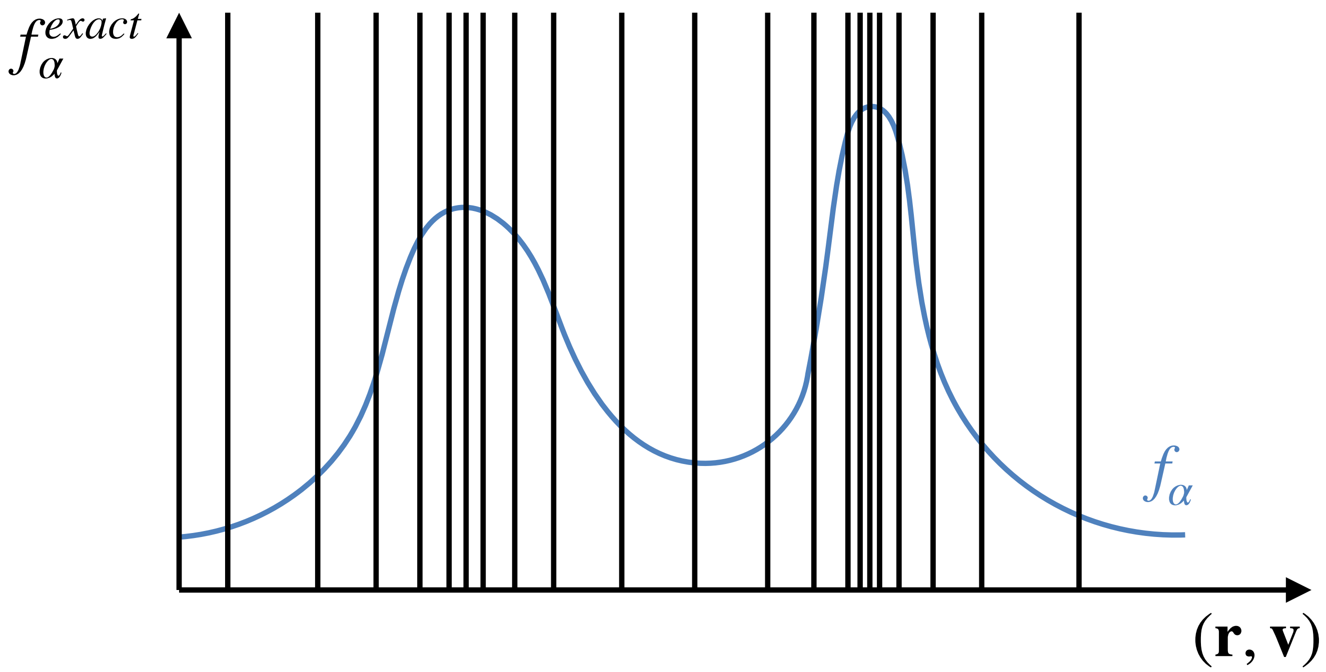
$$\mathbf{E_m} = \mathbf{E} + \delta\mathbf{E_m} \qquad \rho_m = \rho + \delta\rho_m$$

$$f_\alpha^{exact} = f_\alpha + \delta f_\alpha^{exact}$$

$$\mathbf{B_m} = \mathbf{B} + \delta\mathbf{B_m} \qquad \mathbf{J_m} = \mathbf{J} + \delta\mathbf{J_m}$$

  - *Here,* $(\delta\rho_m, \delta\mathbf{J_m}, \delta\mathbf{E_m}, \delta\mathbf{B_m}, \delta f_\alpha)$ *are deviations of the microscopic variables from their respective ensemble average (macroscopic) values* $(\rho_m, \mathbf{J}, \mathbf{E}, \mathbf{B}, f_\alpha)$*, with the fluctuating variables* $\langle\delta\rho_m\rangle = \langle\delta\mathbf{J_m}\rangle = \langle\delta\mathbf{E_m}\rangle = \langle\delta\mathbf{B_m}\rangle = \langle\delta f_\alpha^{exact}\rangle = 0$

- **Ultimately, we are looking for a smoothed distribution function** $f_\alpha(\mathbf{r}, \mathbf{v}, t)$



The Klimontovich distribution function

- **Substitution into the Klimontovich equation, and taking an ensemble average, leads to an evolution equation for $f_\alpha(\mathbf{r}, \mathbf{v}, t)$**

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha}\left[\mathbf{E}(\mathbf{r}, t) + \mathbf{v} \times \mathbf{B}(\mathbf{r}, t)\right] \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} = \left.\frac{\delta f_\alpha}{\delta t}\right|_{coll} \qquad \textit{(Boltzmann equation)}$$

with the RHS given by

$$\left.\frac{\delta f_\alpha}{\delta t}\right|_{coll} = -\frac{q_\alpha}{m_\alpha}\left[\delta \mathbf{E}_{\mathbf{m}}(\mathbf{r}_{\alpha,\mathbf{j}}, t) + \mathbf{v} \times \delta \mathbf{B}_{\mathbf{m}}(\mathbf{r}_{\alpha,\mathbf{j}}, t)\right] \cdot \frac{\partial}{\partial \mathbf{v}}(\delta f_\alpha^{exact})$$

  - *The terms on the LHS describe the evolution of the smoothed, average distribution function in response to the smoothed, average electric and magnetic fields due to the plasma and due to external sources*
  - *As a consequence, particles are, in some sense, interacting/colliding via the smoothed/macroscopic $\mathbf{E}$ and $\mathbf{B}$ fields*
  - *The RHS term represents, and is dominated by, the effect on $f_\alpha(\mathbf{r}, \mathbf{v}, t)$ of small angle Coulomb collision inside their respective Debye spheres*

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The Boltzmann equation

- **The Boltzmann equation**

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha} \left[\mathbf{E}(\mathbf{r}, t) + \mathbf{v} \times \mathbf{B}(\mathbf{r}, t) \right] \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} = \left. \frac{\delta f_\alpha}{\delta t} \right|_{coll}$$

can be derived in several ways

- **In this equation, collisional effects are incorporated in the RHS through a general collision term**

- **There exist several forms for the collision term**
  - *The Krook relaxation model*
  - *The Boltzmann collision integral*
  - *The Fokker-Planck equation*

The Krook relaxation model for the collision term

- **A very simple method for taking into account collision effects is provided by the relaxation model (see Bittencourt ch. 5, section 6)**
  - *Collisions tend to restore local thermodynamic equilibrium, which is characterized by a a local equilibrium distribution function* $f_{\alpha,0}(\mathbf{r},\mathbf{v})$

- **In the absence of external forces, it is assumed that the plasma is characterized by** $f_\alpha(\mathbf{r},\mathbf{v},t)$**, which is not far from equilibrium, and reaches equilibrium as as result of collisions with a relaxation time** τ
  - *This model was originally proposed by Krook and can be expressed as*

$$\left.\frac{\delta f_\alpha}{\delta t}\right|_{coll} = -\frac{(f_\alpha - f_{\alpha,0})}{\tau}$$

- **Relaxation times for the average velocity and momentum are the same (**τ**) but, for average thermal energy, it is approximately** $(m_\beta/m_\alpha)\tau$
  - *This model is strictly applicable to collision between particles of same mass*
  - *Due to its simplicity, this relaxation model is useful to provide insight into weakly ionized plasmas in which only charge-neutral collisions are important*

Exercise

- **Solve the Boltzmann equation in the absence of external forces and spatial gradients using the Krook relaxation model for the collision term and show that**

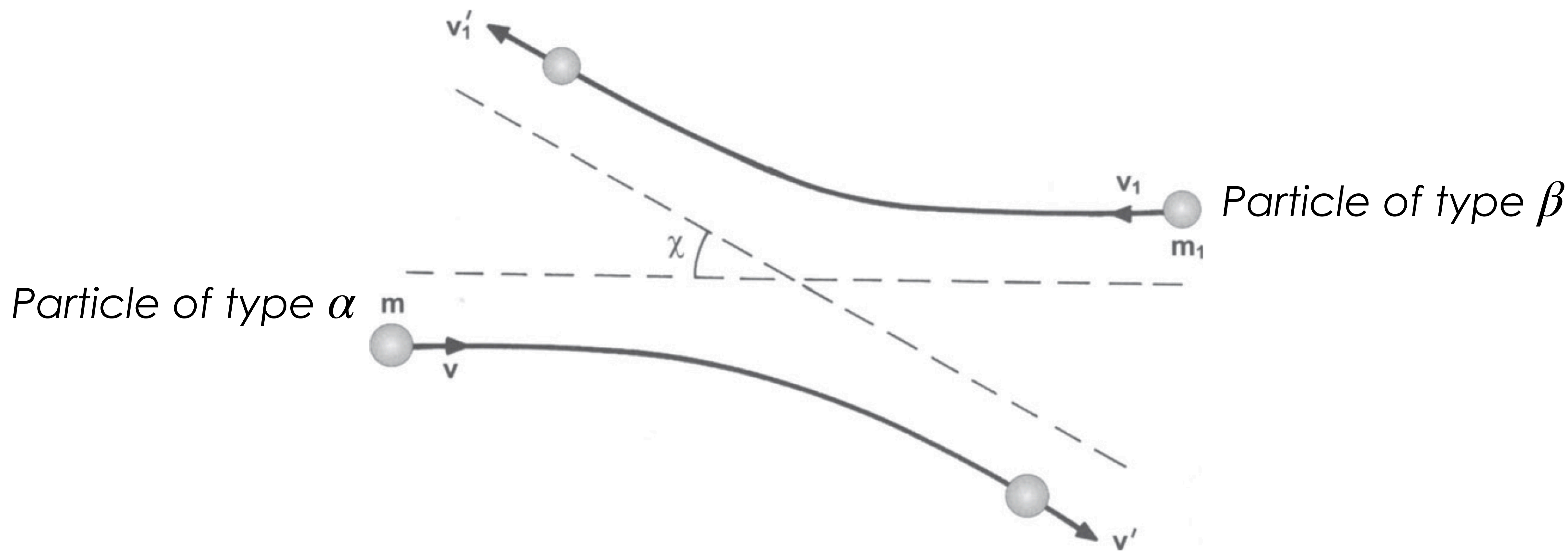
$$f_\alpha(\mathbf{v},t) = f_{\alpha,0}(\mathbf{v}) + \left[f_\alpha(\mathbf{v},0) - f_{\alpha,0}(\mathbf{v})\right] e^{-\frac{t}{\tau}}$$

The Boltzmann collision integral

- **The Boltzmann collision integral is given by (see Bittencourt ch. 21, section 2)**

$$\left.\frac{\delta f_\alpha}{\delta t}\right|_{coll} = \sum_\beta \int_{v_1} \int_\Omega (f'_\alpha f'_\beta - f_\alpha f_\beta) d^3v \,|\mathbf{v_1} - \mathbf{v}|\, \sigma(\Omega) d\Omega$$

  – *Here,* $f'_\alpha = f_\alpha(\mathbf{r}, \mathbf{v}', t)$, $f'_{\beta 1} = f_\beta(\mathbf{r}, \mathbf{v}'_1, t)$, $f_\alpha = f_\alpha(\mathbf{r}, \mathbf{v}, t)$ *and* $f_{\beta 1} = f_\beta(\mathbf{r}, \mathbf{v_1}, t)$



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  - *Here,* $f'_\alpha = f_\alpha(\mathbf{r}, \mathbf{v}', t)$, $f'_{\beta 1} = f_\beta(\mathbf{r}, \mathbf{v}'_\mathbf{1}, t)$, $f_\alpha = f_\alpha(\mathbf{r}, \mathbf{v}, t)$ *and* $f_{\beta 1} = f_\beta(\mathbf{r}, \mathbf{v_1}, t)$

- **The derivation of the Boltzmann collision integral involves four basic assumptions**
  - *The distribution function does not vary appreciably over a distance of the order of the range of the interparticle force law*
  - *Effects of the external force, on the magnitude of the collision cross section, are ignored*
  - *Only binary collisions are taken into account*
  - *The velocities of the interacting particles, before the collision, are assumed to be uncorrelated*

The Boltzmann collision integral

- **With this collision term, the Boltzmann equation becomes an integro-differential equation involving integrals and partial derivatives**

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\cdot\frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha}\left[\mathbf{E}(\mathbf{r},t) + \mathbf{v}\times\mathbf{B}(\mathbf{r},t)\right]\cdot\frac{\partial f_\alpha}{\partial \mathbf{v}} = = \sum_\beta \int_{v_1}\int_\Omega (f'_\alpha f'_\beta - f_\alpha f_\beta)d^3v\,|\mathbf{v_1} - \mathbf{v}|\,\sigma(\Omega)d\Omega$$

- **For example, in a ionized gas composed by electrons, positive ions and neutral particles, we have a system of three Boltzmann equations, coupled through the collision term, plus Maxwell's equations**

$$\frac{\partial f_e}{\partial t} + \mathbf{v}\cdot\frac{\partial f_e}{\partial \mathbf{r}} - \frac{e}{m_e}\left[\mathbf{E}(\mathbf{r},t) + \mathbf{v}\times\mathbf{B}(\mathbf{r},t)\right]\cdot\frac{\partial f_e}{\partial \mathbf{v}} = \sum_{\beta=i,n} \int_{v_1}\int_\Omega (f'_e f'_\beta - f_e f_\beta)d^3v\,|\mathbf{v_1} - \mathbf{v}|\,\sigma(\Omega)d\Omega$$

$$\frac{\partial f_i}{\partial t} + \mathbf{v}\cdot\frac{\partial f_i}{\partial \mathbf{r}} + \frac{e}{m_i}\left[\mathbf{E}(\mathbf{r},t) + \mathbf{v}\times\mathbf{B}(\mathbf{r},t)\right]\cdot\frac{\partial f_i}{\partial \mathbf{v}} = \sum_{\beta=e,n} \int_{v_1}\int_\Omega (f'_i f'_\beta - f_i f_\beta)d^3v\,|\mathbf{v_1} - \mathbf{v}|\,\sigma(\Omega)d\Omega$$

$$\frac{\partial f_n}{\partial t} + \mathbf{v}\cdot\frac{\partial f_n}{\partial \mathbf{r}} = \sum_{\beta=e,i} \int_{v_1}\int_\Omega (f'_n f'_\beta - f_n f_\beta)d^3v\,|\mathbf{v_1} - \mathbf{v}|\,\sigma(\Omega)d\Omega$$

Exercise

- **Derive the Boltzmann collision integral (see Bittencourt ch. 21, section 2)**
- **Derive the Boltzmann collision integral for the collision between electrons and neutrals in a weakly ionized plasma (see Bittencourt ch. 21, section 4)**

The Fokker-Planck collision term

- **The Fokker-Planck collision term is given by (Bittencourt ch. 21, section 5)**

$$\left.\frac{\delta f_\alpha}{\delta t}\right|_{coll} = -\sum_j \frac{\partial}{\partial v_j}\left(f_\alpha \langle \Delta v_j \rangle_{av}\right) + \frac{1}{2}\sum_{j,k} \frac{\partial^2}{\partial v_j \partial v_k}\left(f_\alpha \langle \Delta v_j \Delta v_k \rangle_{av}\right)$$

- **The Fokker-Planck collision term accounts for simultaneous Coulomb interactions between charged particles**
  - *It is assumed that the large angle scattering in a multiple Coulomb interaction can be considered as a series of consecutive weak binary collisions (small angle collisions):* $\mathbf{v}' = \mathbf{v} + \Delta\mathbf{v}$
  - *As a consequence, the Fokker-Planck collision term can be derived from the Boltzmann collision integral, which is valid only for binary collisions*

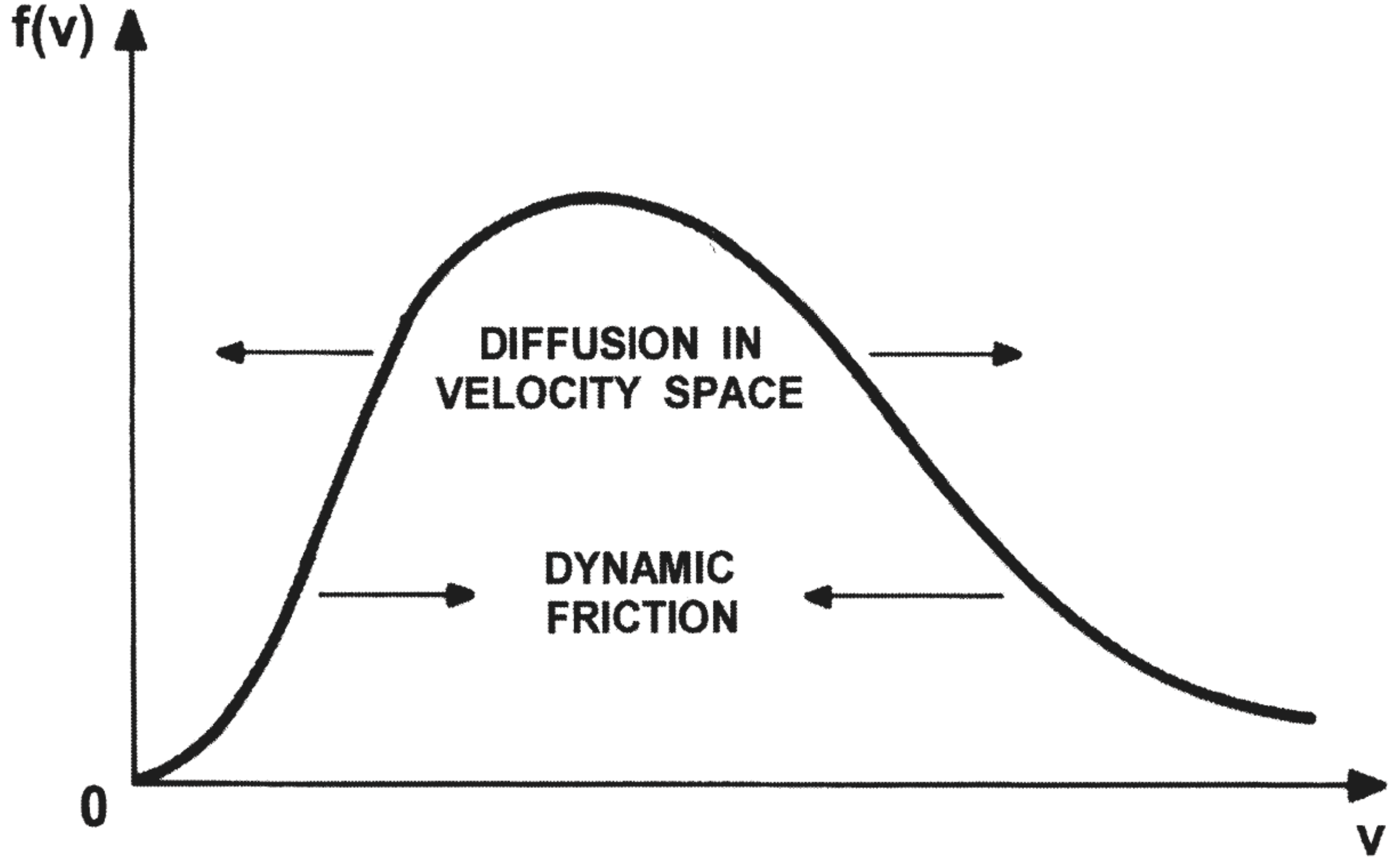
- **The average quantities $\langle \Delta v_j \rangle_{av}$ and $\langle \Delta v_j \Delta v_k \rangle_{av}$ are given by**

$$\langle \Delta v_j \rangle_{av} = \int_\Omega \int_{v_1} \Delta v_j \,|\mathbf{v_1} - \mathbf{v}|\, \sigma(\Omega)\, d\Omega\, f_{\beta,1} d^3 v_1$$

$$\langle \Delta v_j \Delta v_k \rangle_{av} = \int_\Omega \int_{v_1} \Delta v_j \Delta v_k \,|\mathbf{v_1} - \mathbf{v}|\, \sigma(\Omega)\, d\Omega\, f_{\beta,1} d^3 v_1$$

The Fokker-Planck collision term

- **The average quantities $\langle \Delta v_j \rangle_{av}$ and $\langle \Delta v_j \Delta v_k \rangle_{av}$ are known as the Fokker-Planck coefficients of dynamic friction and diffusion in velocity space, respectively**



Exercise

- **Derive the Fokker-Planck equations (see Bittencourt ch. 21, section 5)**
- **Derive the Fokker-Planck equations for Coulomb interactions (see Bittencourt ch. 21, section 5.2)**
- **Derive the Fokker-Planck equations for electron-ion Coulomb interactions (see Bittencourt ch. 21, section 5.3)**

PGF5112 - Plasma Physics I

- **Fundamentals of plasma kinetic theory**
  - *Introduction*
  - *Phase space*
  - *Distribution function*
  - *The Klimontovich distribution function*
  - *The Boltzmann equation*
  - *The Vlasov equation*

The Vlasov equation

- **A very useful approximate way to describe the dynamics of a plasma is to consider that the plasma particle motions are governed by externally applied fields plus the macroscopic internal fields due to the plasma particles, but ignoring the effect of particle scattering due to collisions**
  - *The equation used to describe such (hot) plasmas is called Vlasov equation or, sometimes, the collisionless plasma kinetic equation*

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha} \left[\mathbf{E}(\mathbf{r}, t) + \mathbf{v} \times \mathbf{B}(\mathbf{r}, t) \right] \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} = 0$$

- **Although the Vlasov equation does not explicitly include a collision term, i.e. does not accounts for the frequent small angle Coulomb scattering, part of the effect of particle collisions are accounted for via the macroscopic fields**

- **The condition for the neglect of collisional effects is that the frequency of the relevant physical process (ω) be much larger than the collision frequency (ν)**
  - *Here, the frequency ω represents whichever of the various fundamental frequencies involved in the process*

Debye shielding using the Vlasov equation

- **Let's study the Debye shielding problem using the kinetic approach**

- **To determine the steady-state electron and ion distribution functions, $f_e(\mathbf{r},\mathbf{v})$ and $f_i(\mathbf{r},\mathbf{v})$, and the electrostatic potential, $\Phi(\mathbf{r})$, let us consider the steady-state Vlasov equations for electrons and ions (only electric field in the Lorentz force)**

$$\mathbf{v}\cdot\nabla f_e(\mathbf{r},\mathbf{v}) - \frac{e\,\mathbf{E}(\mathbf{r})}{m_e}\cdot\nabla_v f_e(\mathbf{r},\mathbf{v}) = 0 \qquad \mathbf{v}\cdot\nabla f_i(\mathbf{r},\mathbf{v}) + \frac{e\,\mathbf{E}(\mathbf{r})}{m_i}\cdot\nabla_v f_i(\mathbf{r},\mathbf{v}) = 0$$

- **Substituting $\mathbf{E} = -\nabla\Phi$ in these equations leads to**

$$\mathbf{v}\cdot\nabla f_e(\mathbf{r},\mathbf{v}) + \frac{e}{m_e}\nabla\Phi(\mathbf{r})\cdot\nabla_v f_e(\mathbf{r},\mathbf{v}) = 0 \qquad \mathbf{v}\cdot\nabla f_i(\mathbf{r},\mathbf{v}) - \frac{e}{m_i}\nabla\Phi(\mathbf{r})\cdot\nabla_v f_i(\mathbf{r},\mathbf{v}) = 0$$

- **Since $n_\alpha(\mathbf{r},t) = \int_v f_\alpha(\mathbf{r},\mathbf{v},t)\,d^3v$, the total charge density (including the test charge) can be expresses as**

$$\rho(\mathbf{r}) = q_t\delta(\mathbf{r}) - e\int_v (f_e - f_i)\,d^3v$$

Debye shielding using the Vlasov equation

- The Poisson equation for this case becomes

$$\nabla^2\Phi - \frac{e}{\epsilon_0}\int_v (f_e - f_i)\, d^3v = -\frac{q_t}{\epsilon_0}\delta(\mathbf{r})$$

- These three equations constitute a system of equations that need to be solved simultaneously to determine $f_e(\mathbf{r},\mathbf{v})$, $f_i(\mathbf{r},\mathbf{v})$ and $\Phi(\mathbf{r})$

- The solution of the two Vlasov equations (for $f_e(\mathbf{r},\mathbf{v})$ and $f_i(\mathbf{r},\mathbf{v})$) can be given in terms of Maxwellian distribution functions (see Bittencourt ch. 7, section 5) like

$$f_\alpha(\mathbf{r},\mathbf{v}) = f_{\alpha,M}(v)\exp\left[-\frac{q_\alpha\Phi(\mathbf{r})}{k_BT}\right]$$

  the Poisson equation becomes

$$\nabla^2\Phi - \frac{e}{\epsilon_0}\left[\exp\left(\frac{e\Phi}{k_BT}\right)\int_v f_{e,M}(v)\, d^3v - \exp\left(-\frac{e\Phi}{k_BT}\right)\int_v f_{i,M}(v)\, d^3v\right] = -\frac{q_t}{\epsilon_0}\delta(\mathbf{r})$$

Debye shielding using the Vlasov equation

- **Denoting the equilibrium/Maxwellian electron and ion densities as**

$$\int_v f_{e,M}(v)\, d^3v = \int_v f_{i,M}(v)\, d^3v = n_0$$

the Poisson equation becomes

$$\nabla^2\Phi - \frac{en_0}{\epsilon_0}\left[\exp\left(\frac{e\Phi}{k_BT}\right) - \exp\left(-\frac{e\Phi}{k_BT}\right)\right] = -\frac{q_t}{\epsilon_0}\delta(\mathbf{r})$$

  - *This equation is the same solved in the 30th March lecture, whose solution is*

$$\Phi(r) = \frac{1}{4\pi\epsilon_0}\frac{q_t}{r}\exp\left(-\frac{r}{\lambda_D}\right)$$

Exercises

- **Bittencourt: ch. 5**
  - *5.1, 5.3, 5.4, 5.5, 5.6, 5.7 and 5.8*

- **Bittencourt: ch. 21**
  - *21.2 and 21.3*

References

- **Fundamentals of plasma kinetic theory**
  - *Bittencourt, Ch. 5*
  - *Bittencourt, Ch. 21*