PGF5112 - Plasma Physics I

By

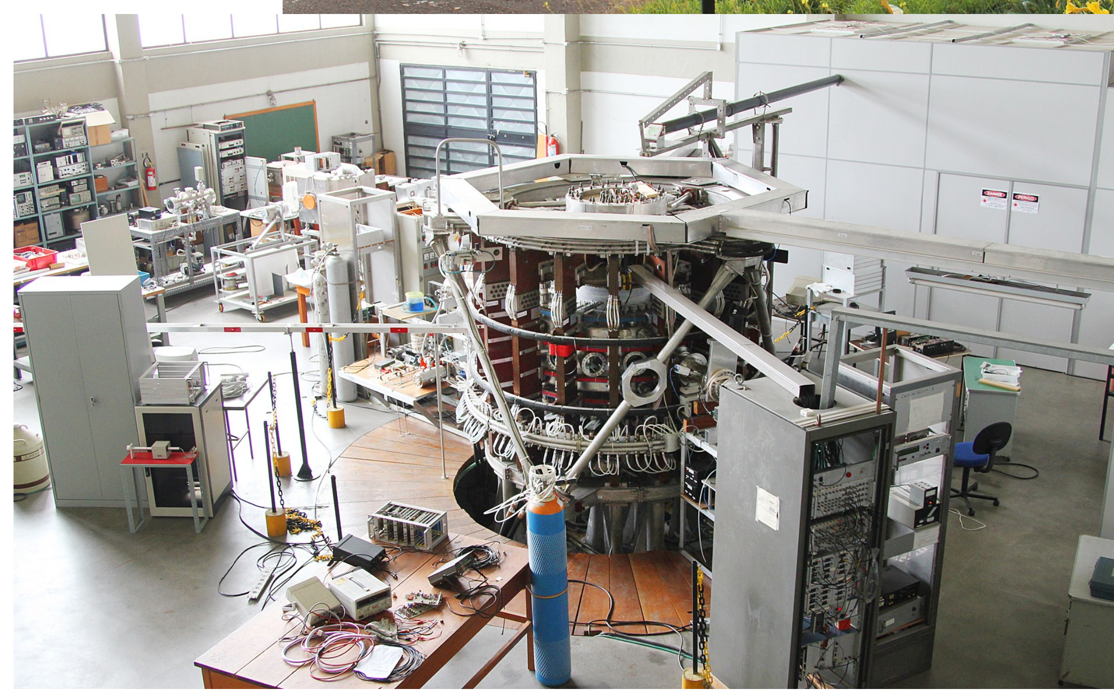
Prof. Gustavo Paganini Canal

Plasma Physics Laboratory

Department of Applied Physics

Institute of Physics

University of São Paulo - Brazil



Postgraduate course ministered remotely from the

Institute of Physics of the University of São Paulo

IFUSP
Instituto de Física da USP

e-mail: canal@if.usp.br

São Paulo - SP, 15 April 2021

- **Tokamak engineering**
  - *Magnetic forces*
  - *Toroidal field coils*
  - *The central solenoid*
  - *Poloidal field coils*
  - *The vertical plasma instability and the RZIP model*

PGF5112 - Plasma Physics I

- **Tokamak engineering**
  - *Magnetic forces*
  - *Toroidal field coils*
  - *The central solenoid*
  - *Poloidal field coils*
  - *The vertical plasma instability and the RZIP model*

Magnetic forces: the virtual work formalism

- Establishing a magnetic field requires energy and this is a direct consequence of Faraday's induction law

$$\epsilon = -\frac{d\Phi}{dt}$$

- If a power supply of voltage V is connected to a circuit, then the current in the circuit can be expressed as $V + \epsilon = RI$, where ϵ accounts for the presence of an induced EMF

- The work done by the power supply to displace an amount of charge $dq = Idt$ is

$$Vdq = VIdt = RI^2dt - \epsilon Idt = RI^2dt + Id\Phi$$

- While the RI^2dt term represents the irreversible conversion of electrical energy into heat, the $Id\Phi$ term represents the work done by the power supply against the induced EMF. Therefore, the amount of work done ON the system to modify its magnetic field

$$dW_{ext} = \sum_{j=1}^{N} I_j d\Phi_j$$

Magnetic forces: the virtual work formalism

- **From classical electrodynamics, we know that the magnetic energy stored in a linear system is given by**

$$U = \frac{1}{2}\sum_{j=1}^{N} I_j \Phi_j$$

- **Suppose that one allows part of the system to move under the action of a magnetic force, but at constant currents, then the work done BY the force is**

$$dW = \mathbf{F} \cdot d\mathbf{r}$$

- **Under this circumstance, the work done BY the system has two contributions**

$$dW = dW_{ext} - dU$$

  - *Here, dU is the variation of the magnetic energy of the system and dW_{ext} is the work done BY the power supplies to keep the currents constant*

- **If the geometry of the system is modified, but the currents remain the same,**

$$dU = \frac{1}{2}\sum_{j=1}^{N} I_j d\Phi_j$$

Magnetic forces: the virtual work formalism

- **Therefore, one finds that $dW_{ext} = 2dU$, which implies that $dW = dU$. Using $dW = \mathbf{F} \cdot d\mathbf{r} = dU$ one have that**

$$dU = \mathbf{F} \cdot d\mathbf{r} \qquad \rightarrow \qquad \mathbf{F} = \nabla U$$

- **If the system is forced to rotate around some axis, them we also have than**

$$dU = \boldsymbol{\tau} \cdot d\boldsymbol{\theta} \qquad \rightarrow \qquad \boldsymbol{\tau} = \nabla_{\boldsymbol{\theta}} U$$

- **The magnetic energy of a conductor with current I and self-inductance L is**

$$U = \frac{1}{2} L I^2$$

while the magnetic energy of, for example, a pair of conductors is

$$U = \frac{1}{2} \left(L_1 I_1^2 + L_2 I_2^2 \pm 2 M_{12} I_1 I_2 \right)$$

Here, M_{12} is the mutual inductance

- **Therefore, the force and torque that cause the magnetic energy of the system to decrease (keeping the current constant) are**

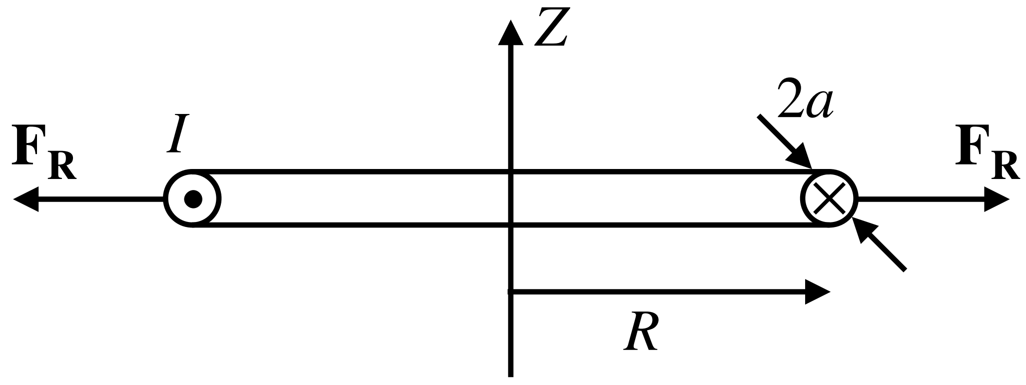
$$F_i = \frac{\partial U}{\partial x_i} \qquad \text{and} \qquad \tau_i = \frac{\partial U}{\partial \theta_i}$$

Magnetic forces: the hoop force

- **The self-inductance of a circular current loop with $R \gg a$ is**

$$L = \mu_0 R \left[\ln\left(\frac{8R}{a}\right) - \frac{7}{4} \right]$$

  - *Therefore, the magnetic force F_i acting on the system due to a change on the variable x_i is given by*



$$F_i = \frac{\partial}{\partial x_i}\left(\frac{1}{2} L I^2\right) = \frac{I^2}{2}\frac{\partial L}{\partial x_i}$$

  - *Supposing that the diameter of the wire ($2a$) does not change, the only force acting on a circular current loop is*

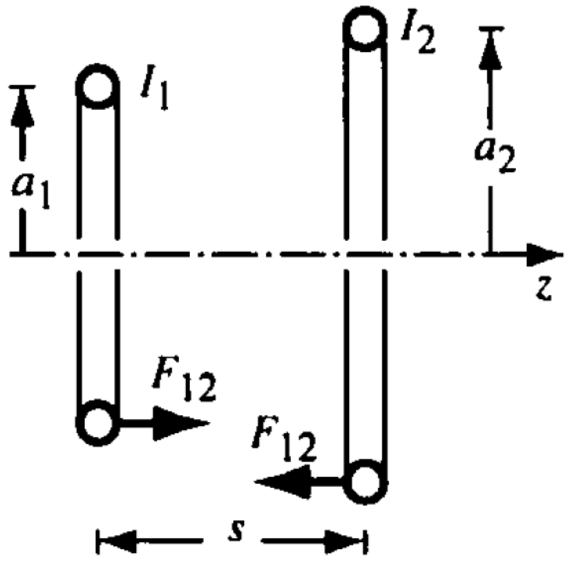
$$F_R = \frac{I^2}{2}\frac{\partial L}{\partial R} = \frac{\mu_0 I^2}{2}\left[\ln\left(\frac{8R}{a}\right) - \frac{3}{4} \right]$$

This is the so-called hoop force

  - *This force points radially outwards and it tends to increase the loop radius*

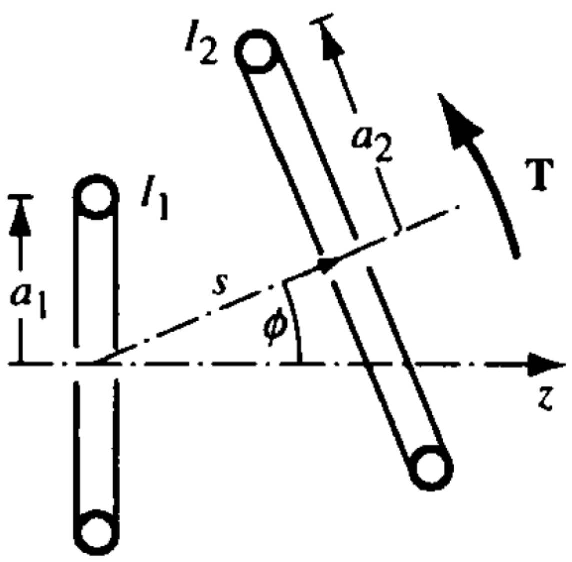
Magnetic forces: the virtual work formalism

- **The force between two coaxial circular current loops**
  - *Supposing that the diameter of the wires and the radii of the current loops (a_1 and a_2) do not change, the only force acting on the circular current loops is due to a change in their separation distance (s):*



$$F_s = \frac{1}{2}\frac{\partial}{\partial s}\left(L_1 I_1^2 + L_2 I_2^2 \pm 2M_{12} I_1 I_2\right) = \pm I_1 I_2 \frac{\partial M_{12}}{\partial s}$$

- **The torque between two inclined circular current loops**
  - *Supposing that the diameter of the wires and the radii of the current loops (a_1 and a_2) do not change, the only torque acting on the circular current loops is due to a change in their orientation angle (ϕ):*



$$\tau_\phi = \frac{1}{2}\frac{\partial}{\partial \phi}\left(L_1 I_1^2 + L_2 I_2^2 \pm 2M_{12} I_1 I_2\right) = \pm I_1 I_2 \frac{\partial M_{12}}{\partial \phi}$$

Exercise

- **The mutual inductance between two coaxial circular current loops of radius a_1 and a_2 , and separation distance s, is given by**

$$M_{12} = \mu_0 \sqrt{a_1 a_2} \left[\left(\frac{2}{k} - k \right) K(k^2) - \frac{2}{k} E(k^2) \right] \qquad \text{with} \qquad k^2 = \frac{4 a_1 a_2}{(a_1 + a_2)^2 + s^2}$$

where $K(k^2)$ and $E(k^2)$ are the complete elliptic integrals of first and second kind, respectively:

$$K(k^2) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \qquad E(k^2) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \phi}\, d\phi$$

Expanding these expressions up to 2nd order in k^2 ($s \gg a_1$ and $s \gg a_2$) yields

$$M_{12} = \frac{\mu_0 \pi a_1^2 a_2^2}{2 s^3} \left(1 + 3 \frac{a_1 a_2}{s^2} + \frac{75}{8} \frac{a_1^2 a_2^2}{s^4} + \dots \right)$$

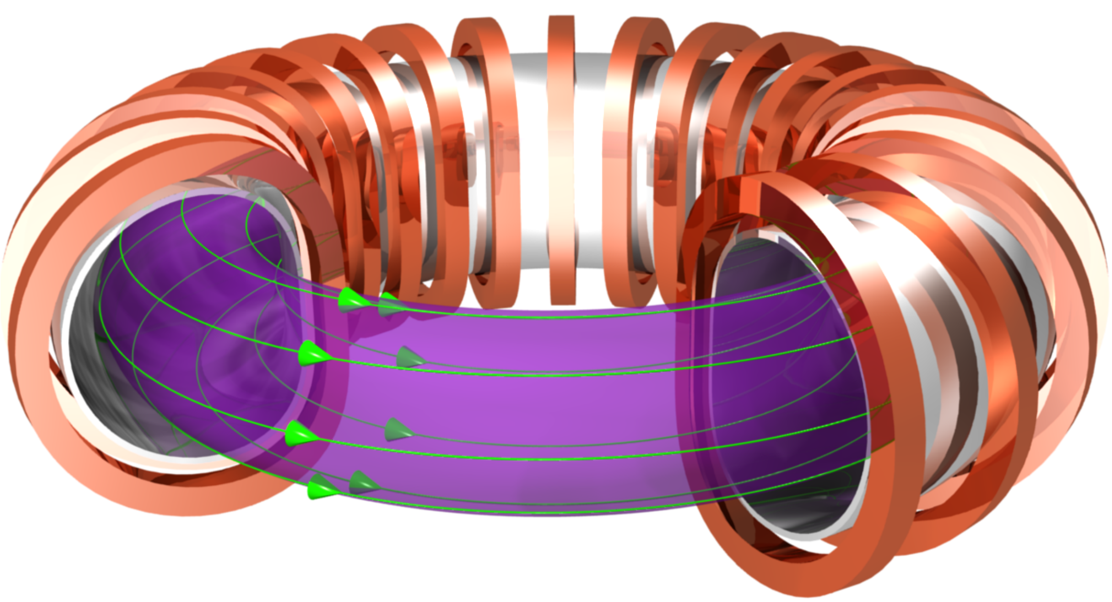
Calculate the force between two coaxial circular current loops up to 2nd order and show that the force is attractive for currents running in the same direction

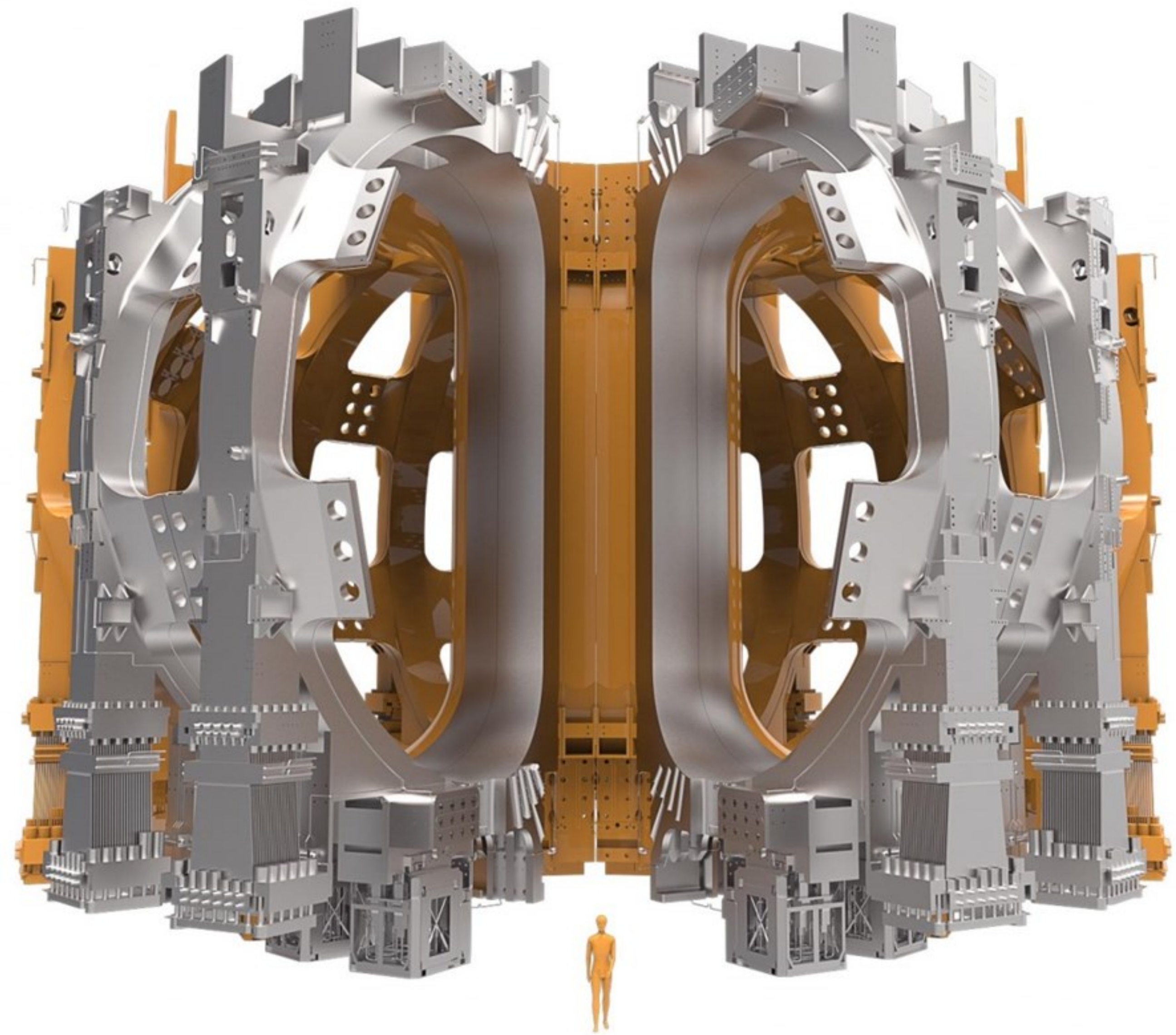
PGF5112 - Plasma Physics I

- **Tokamak engineering**
  - *Magnetic forces*
  - *Toroidal field coils*
  - *The central solenoid*
  - *Poloidal field coils*
  - *The vertical plasma instability and the RZIP model*

Toroidal field coils

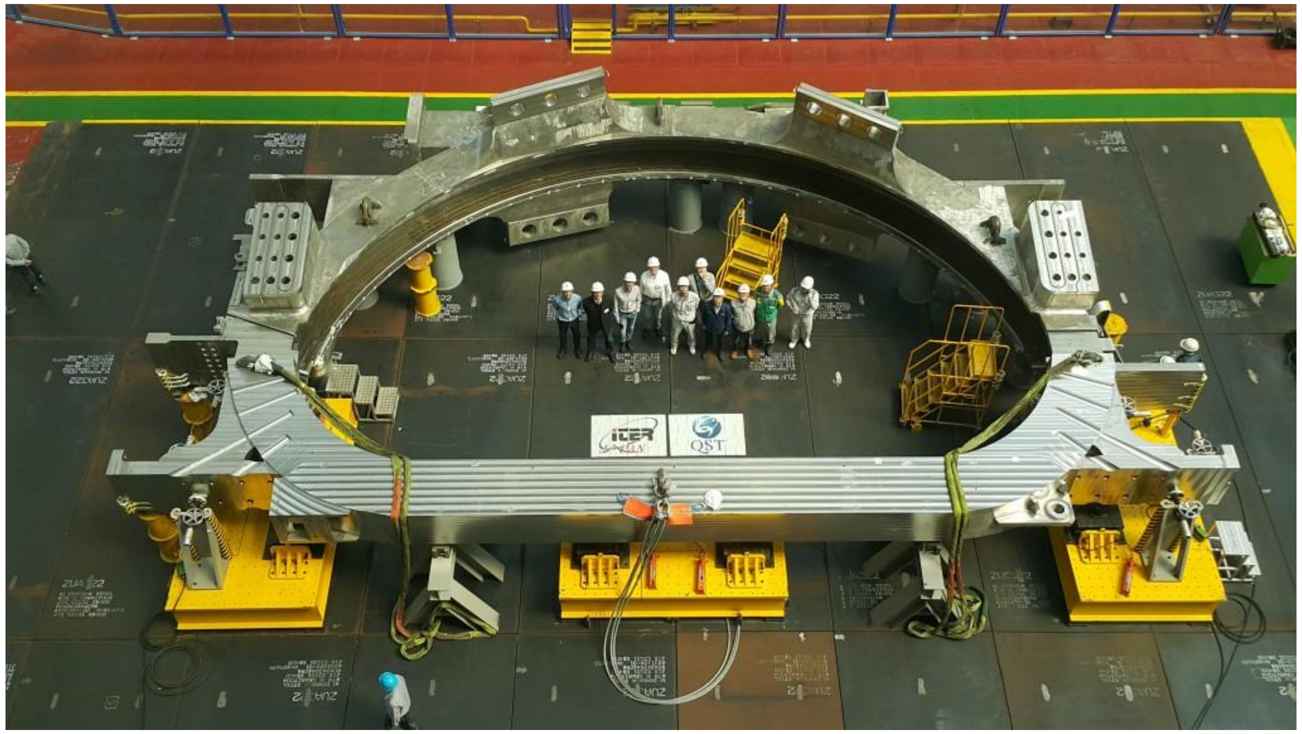
- **The toroidal field coils produce the plasma confining magnetic field, which is the quantity the has the strongest impact on the fusion power to be produced**
  - *The fusion power in a tokamak is* $P_{\text{fus}} \propto B_0^4 V$





The toroidal field coils of ITER

- **The list of applicable superlatives about the ITER toroidal field coils is long**
  - *The toroidal field coils are the largest and most powerful superconducting magnets ever designed, with a stored magnetic energy of 41 GJ*
  - *Together, they weigh in at over 6,000 tonnes*
  - *They required the production of 500 tonnes of Nb_3Sn superconducting strand (100,000 km) required for the toroidal field superconducting cables*



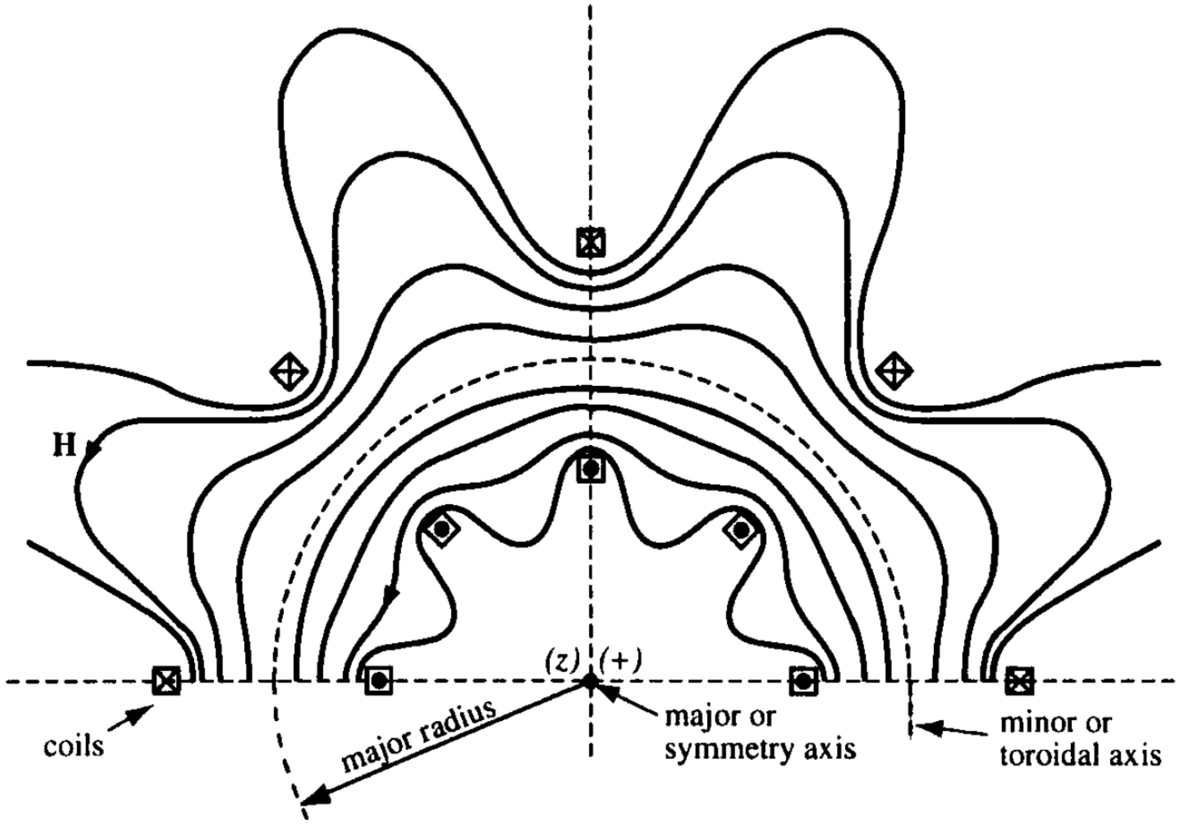
Toroidal field ripple due to the discrete number of coils

- **Due to the discrete number of toroidal field coils, the magnetic field close to the coils is not purely toroidal. This fluctuation is called ripple**
  - *Toroidal field ripple increases plasma losses and, therefore, decreases confinement and plasma performance*

- **Magnetic field lines at the equatorial plane (Z = 0) view from the top**
  - *Field line trajectories can be found by integrating the field line equations*

$$\mathbf{B} \times d\mathbf{l} = 0$$

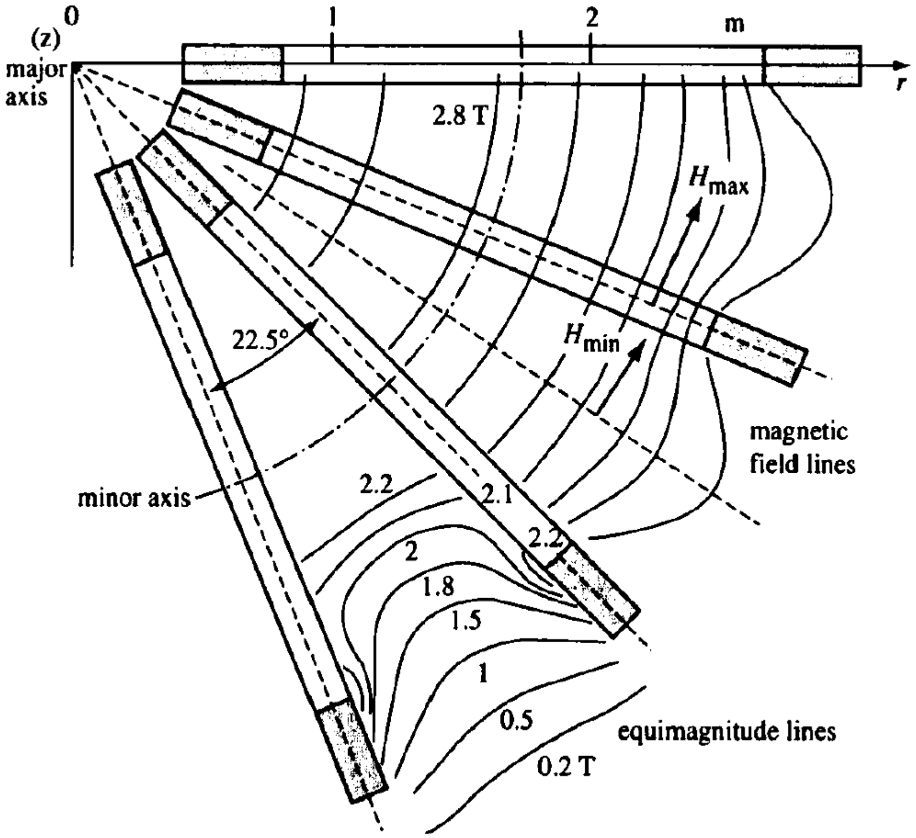
$$\frac{dR}{B_R} = \frac{Rd\phi}{B_\phi} \qquad \frac{dZ}{B_Z} = \frac{Rd\phi}{B_\phi}$$



Toroidal field ripple due to the discrete number of coils

- **Due to the discrete number of toroidal field coils, the magnetic field close to the coils is not purely toroidal. This fluctuation is called ripple**
  - *Toroidal field ripple increases plasma losses and, therefore, decreases confinement and plasma performance*

- **Magnetic field lines at the equatorial plane (Z = 0) view from the top**
  - *Field line trajectories can be found by integrating the field line equations*



- **The magnetic field ripple can be characterized be the parameter**

$$\epsilon = \frac{H_{max} - H_{min}}{H_0} = \frac{B_{max} - B_{min}}{B_0}$$

where

$$H_0 = \frac{H_{max} + H_{min}}{2} \qquad B_0 = \frac{B_{max} + B_{min}}{2}$$

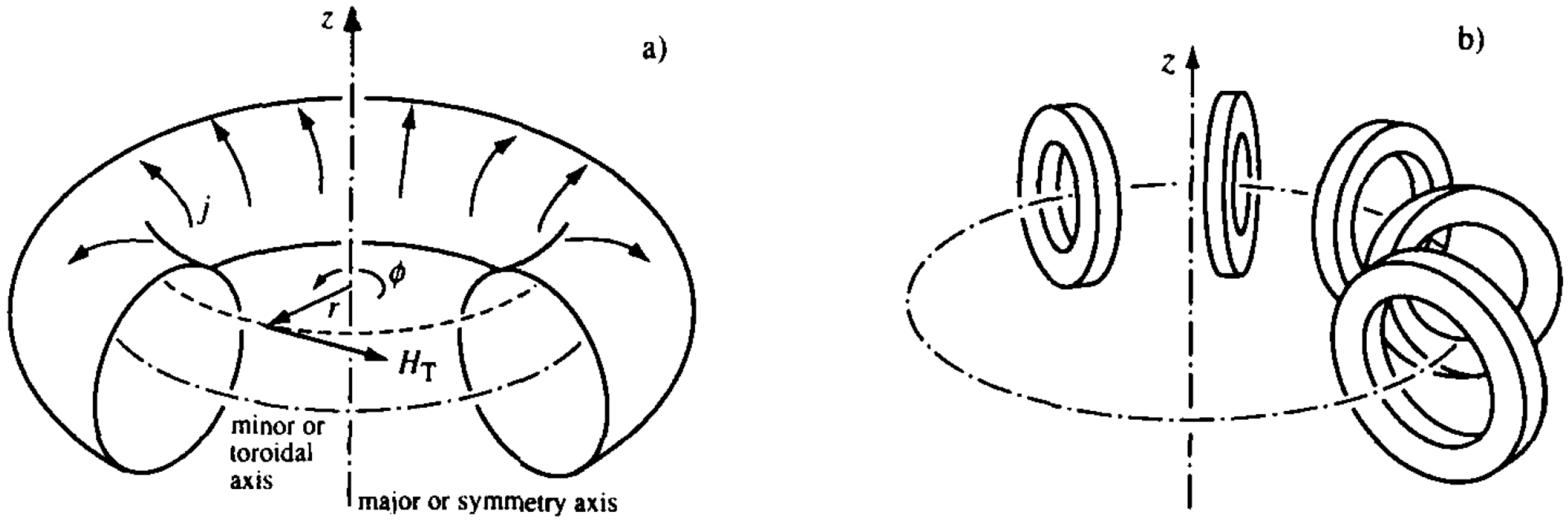
  - *Acceptable values of field ripple are* $\epsilon < 0.1\,\%$

Toroidal field coil forces: the centering force

- **The inductance of a evenly wound toroidal sheet magnet of circular radius $\bar{a}$ is**

$$L = \mu_0 N^2 \left(R_0 - \sqrt{R_0^2 - \bar{a}^2} \right)$$

  – *When the toroidal magnet is made of single coils, this expression must be corrected. However, this effect can be neglected when* $N > 10$



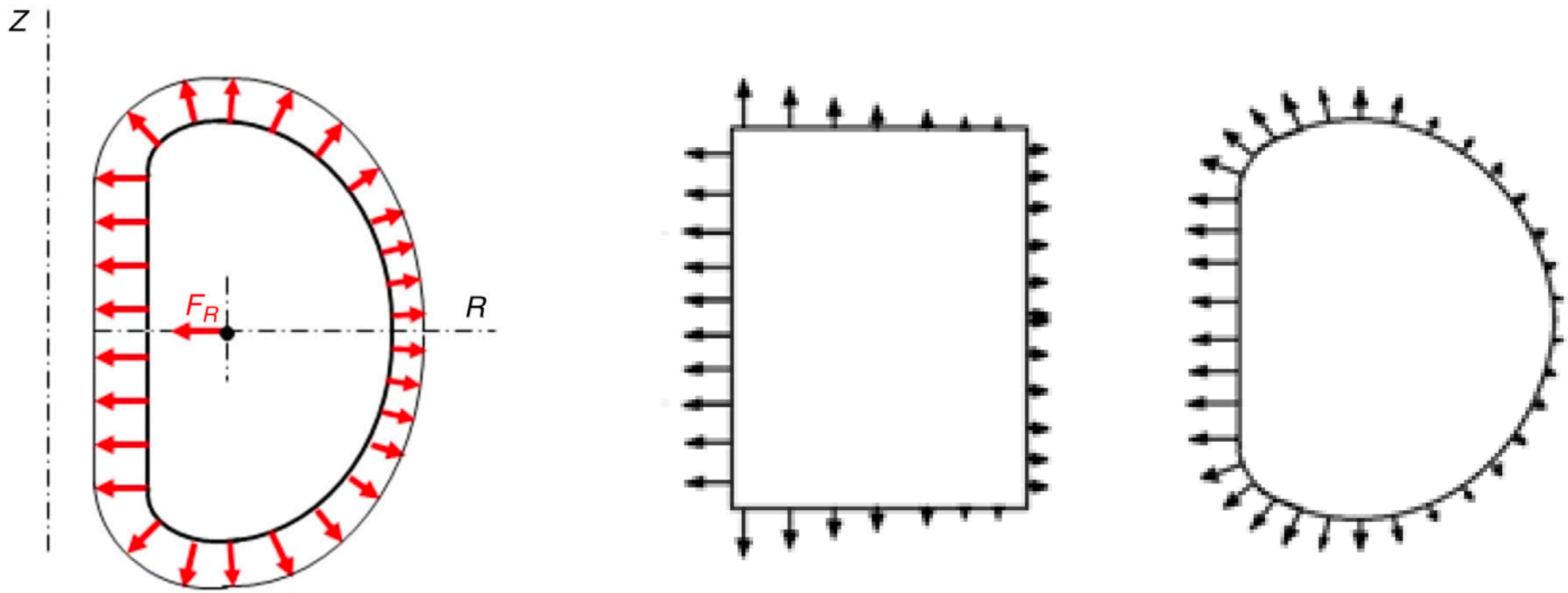
- **The force on a toroidal field coils can be calculated as**

$$F_R\Big|_{\text{per coil}} = \frac{I^2}{2N}\frac{\partial}{\partial R_0}\left[\mu_0 N^2 \left(R_0 - \sqrt{R_0^2 - \bar{a}^2} \right)\right] = -\frac{\mu_0 N I^2}{2}\left(\frac{R_0}{\sqrt{R_0^2 - \bar{a}^2}} - 1 \right)$$

Centering Force

Toroidal field coil forces: the centering force

- **In tokamaks, there is a centering force acting on each toroidal field coils**
  - *Depending on the shape of the TF coils, regions with large mechanical stress can arise*



- **Is there a shape that makes the TF coils to have a constant stress?**
  - *YES, that is why TF coils in large tokamaks have a D-shape*

Toroidal field coil forces: the D-shape

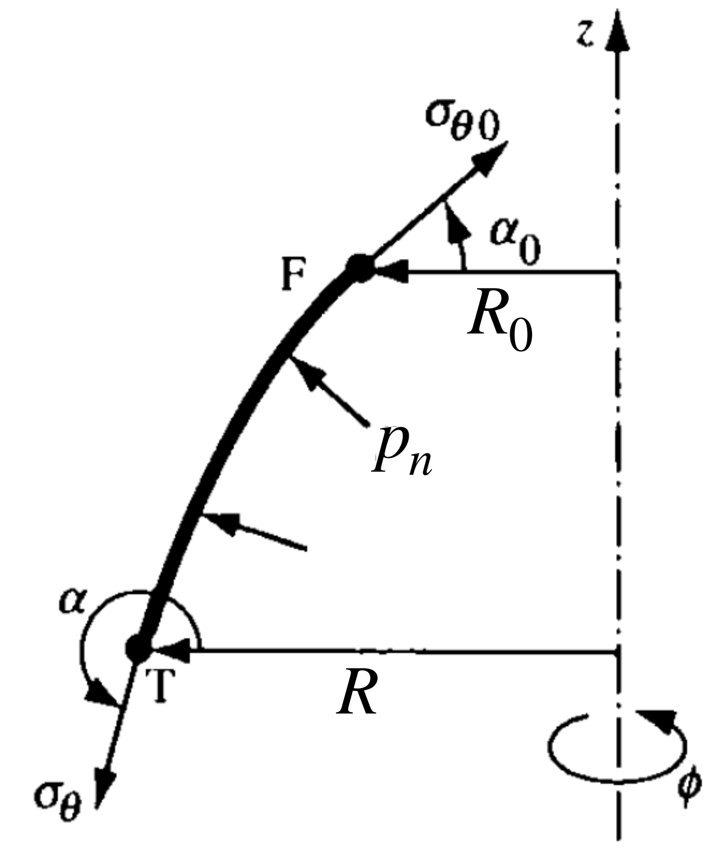
- **To find the TF coil shape that leads to constant stress, let's consider an axisymmetric toroidal shell of thickness d**

- **One can show that, in equilibrium, the vertical component of the total tangential stress in the toroidal shell, acting on F and T,**

$$S_Z = 2\pi d(R_0\sigma_{\theta 0}\sin\alpha_0 - R\sigma_\theta \sin\alpha)$$

must balance the vertical component of the normal pressure load p_n

$$F_Z = 2\pi \int_R^{R_0} p_z(s) R ds = 2\pi \int_R^{R_0} p_n(R) R dR$$



  - *The last integral is independent of the integration path, i.e. the coil shape*
  - *Here, s is a coordinate along the shell and R_0 is the radial position of the maximum heigh of the shell, i.e. the position where $dZ/dR = \tan\alpha = 0$*

Toroidal field coil forces: the D-shape

- **Taking the normal pressure as being caused by the toroidal field**

$$p = p_1 \frac{R_1^2}{R^2} = \frac{B_1^2 R_1^2}{2\mu_0 R^2}$$

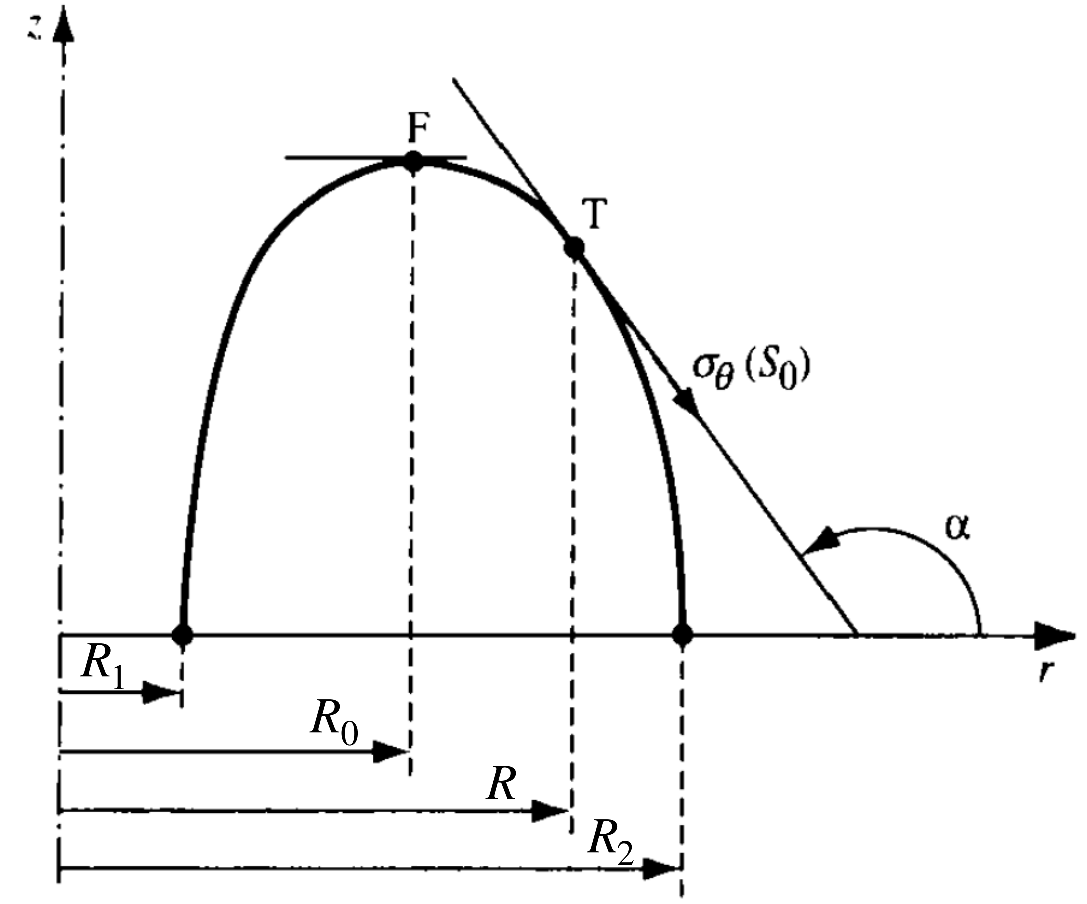
(Results from the $\mathbf{J} \times \mathbf{B}$ force in a toroidal current sheet)

with $r_1 = R_0 - a$, the vertical component of the normal load is

$$F_Z = \frac{\pi B_1^2 R_1^2}{\mu_0} \ln \frac{R_0}{R}$$

- **To find the curve $Z = Z(R)$ which gives constant $\sigma_\theta(R, Z)$, let's impose that, in equilibrium, one has $F_Z + S_Z = 0$, and that $\sigma_\theta = \sigma_0$ is constant, therefore,**

$$\sin\alpha = \frac{p_1 R_1^2}{\sigma_0 d} \frac{1}{R} \ln \frac{R}{R_0}$$



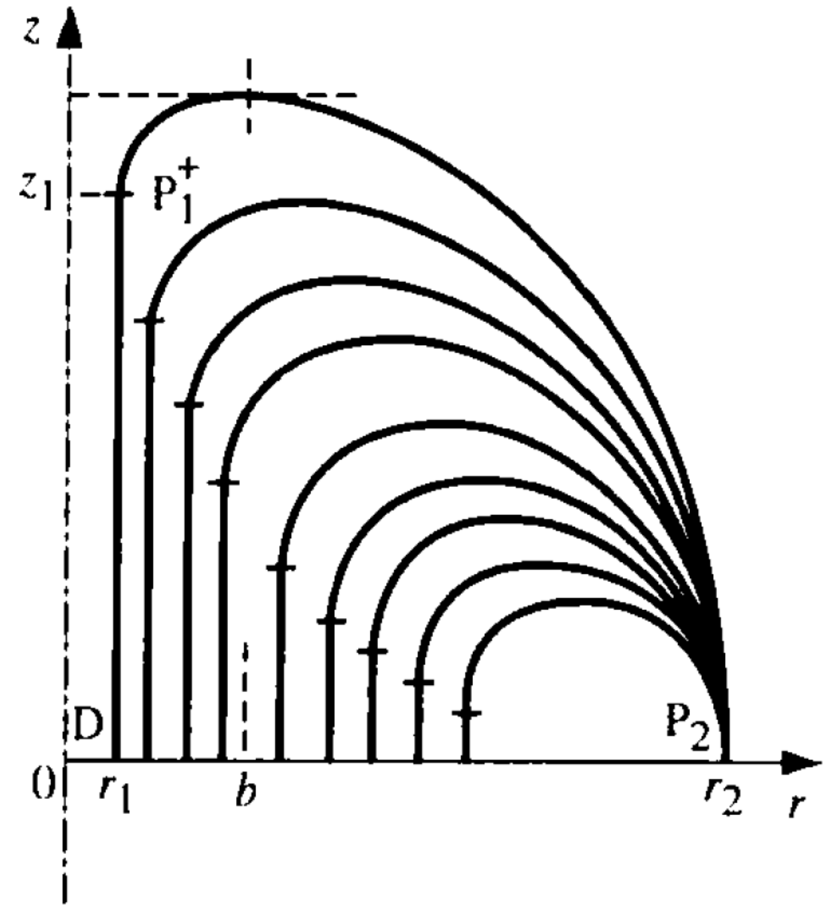
Toroidal field coil forces: the D-shape

- **Using that $dZ/dR = \tan\alpha$ and also the trigonometric identity**

$$\sin\alpha = \frac{\pm\tan\alpha}{\sqrt{1+\tan^2\alpha}}$$

one can show that

$$\frac{dZ}{dR} = \frac{\pm\ln(R/R_0)}{\sqrt{k^2R^2/R_0^2 - \ln^2(R/R_0)}} \qquad k^2 = \frac{\sigma_0^2 d^2 R_0^2}{p_1^2 R_1^4}$$



- **The condition $\left.\frac{dZ}{dR}\right|_{R\to R_1} = \left.\frac{dZ}{dR}\right|_{R\to R_2} = \infty$ determines**

$$\ln R_0 = \frac{R_2\ln R_1 - R_1\ln R_2}{R_2 - R_1} \quad \textbf{and} \quad \sigma_0 = \frac{p_1R_1}{d}\frac{\ln(R_2/R_1)}{R_2/R_1 - 1}$$

- **Finally, the shape of the shell (or TF coil) is**

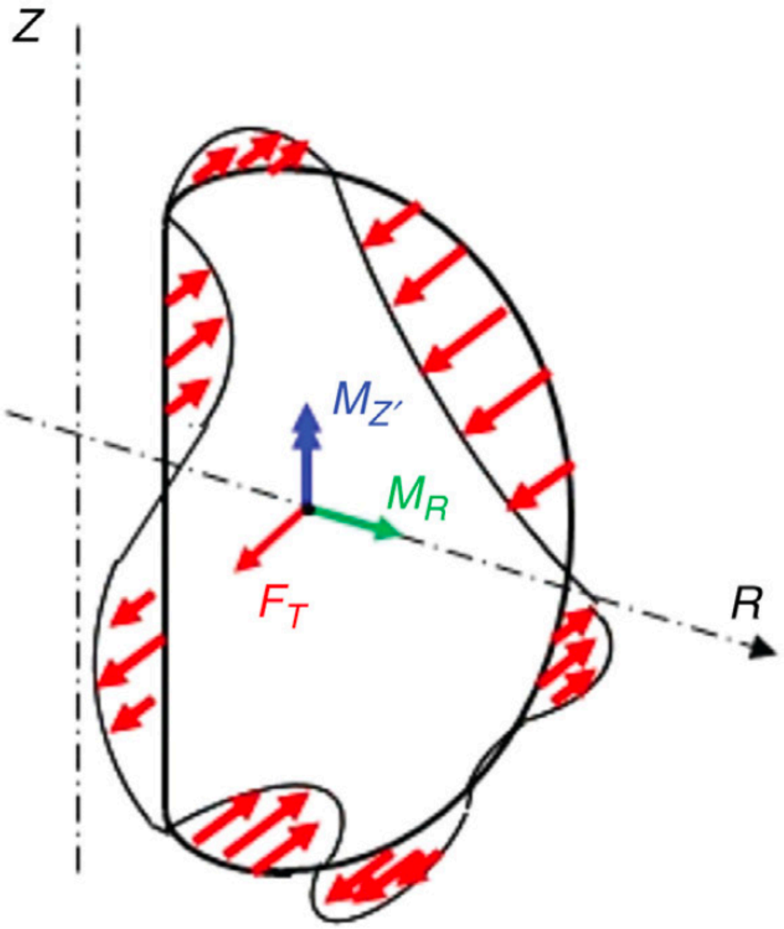
$$Z(R) = \pm\int_{R_1\ or\ R_2}^{R} \frac{\ln(R'/R_0)}{\sqrt{k^2R'^2/R_0^2 - \ln^2(R'/R_0)}}dR'$$

Toroidal field coil forces: out-of-plane forces

- **Toroidal field coils are also under the action of forces perpendicular to the plane of the coil**

- **These out-of-plane forces are due to the interaction between the poloidal magnetic field from the plasma current and the toroidal field coil current**
  - *The specific details of the forces depends on the plasma scenario*

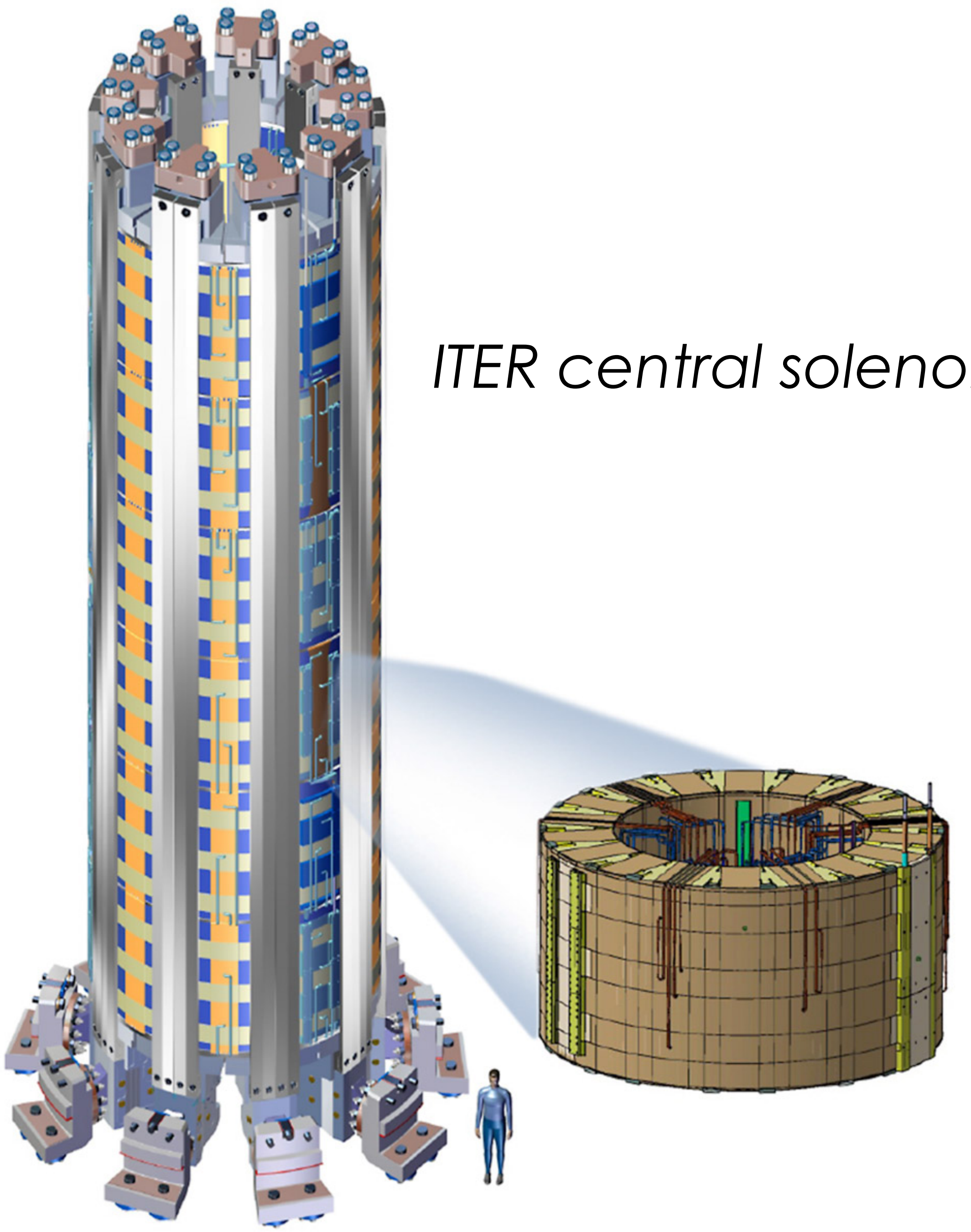
- **Imbalance between these toroidal forces can cause a toroidal force ($\mathbf{F_T}$), which leads to a vertical torque ($\mathbf{M_Z}$)**
  - *This imbalance can also cause a torque in the radial direction ($\mathbf{M_R}$)*



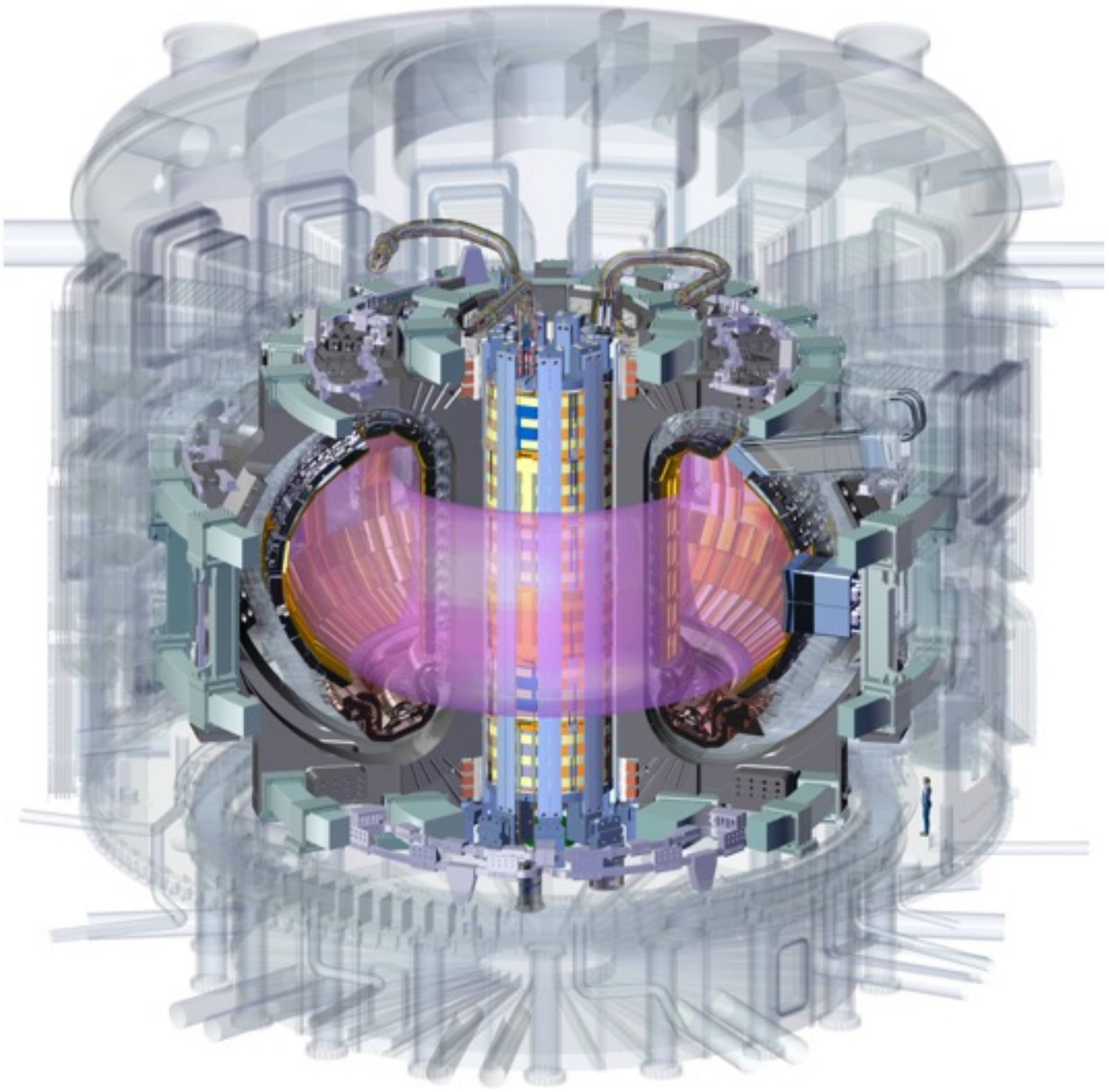
- **Tokamak engineering**
  - *Magnetic forces*
  - *Toroidal field coils*
  - *The central solenoid*
  - *Poloidal field coils*
  - *The vertical plasma instability and the RZIP model*

The central solenoid

- **The central solenoid is responsible for plasma breakdown and for driving the plasma current**

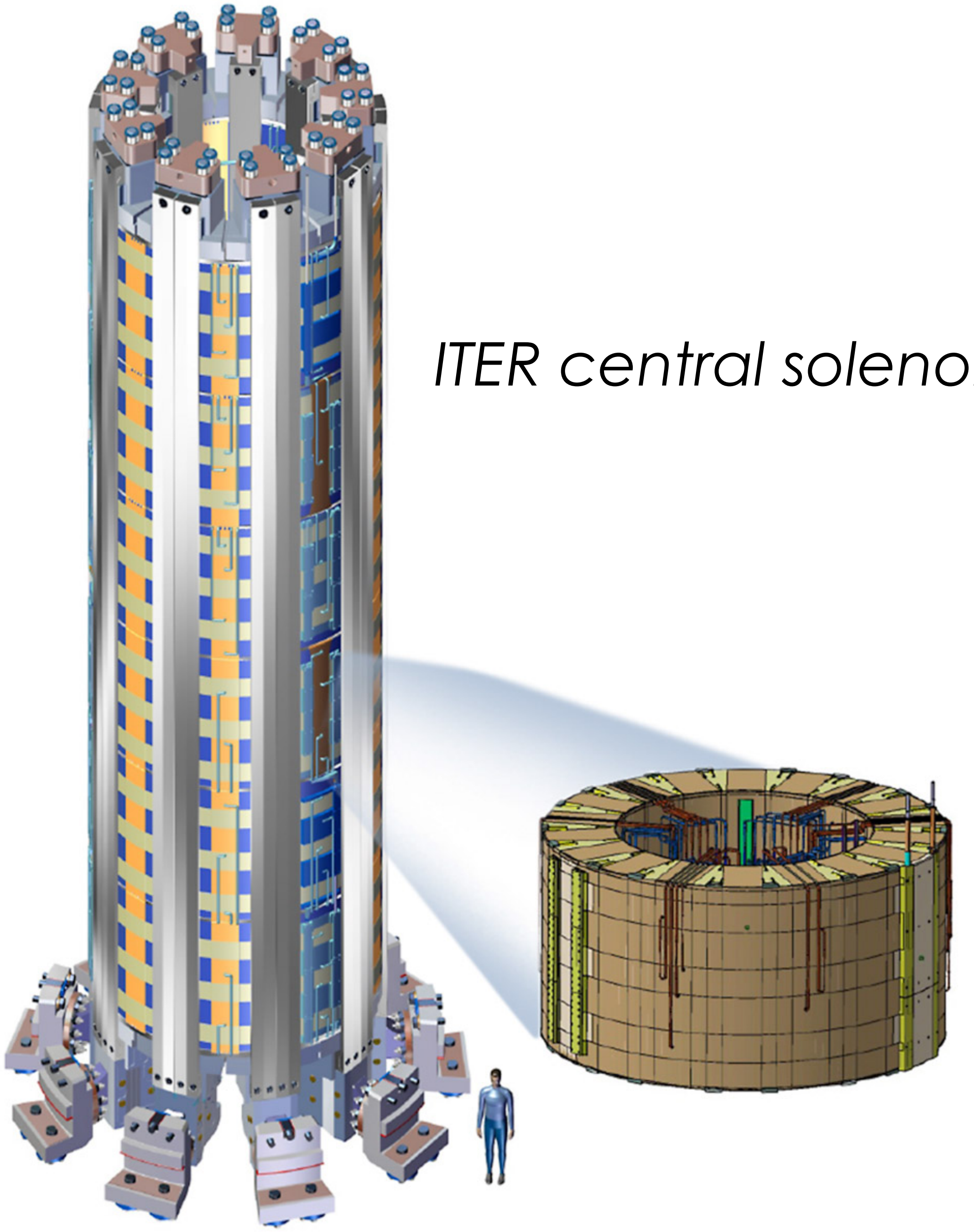


ITER central solenoid



The central solenoid

- **The central solenoid is responsible for plasma breakdown and for driving the plasma current**



ITER central solenoid

A module of the ITER CS being manufactured at General Atomics, USA

The central solenoid is responsible for driving the plasma current

- **The plasma current depends on the OH coil current ramp rate**

- **From Faraday's induction law**

$$\oint_l \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot \hat{\mathbf{n}} dS$$

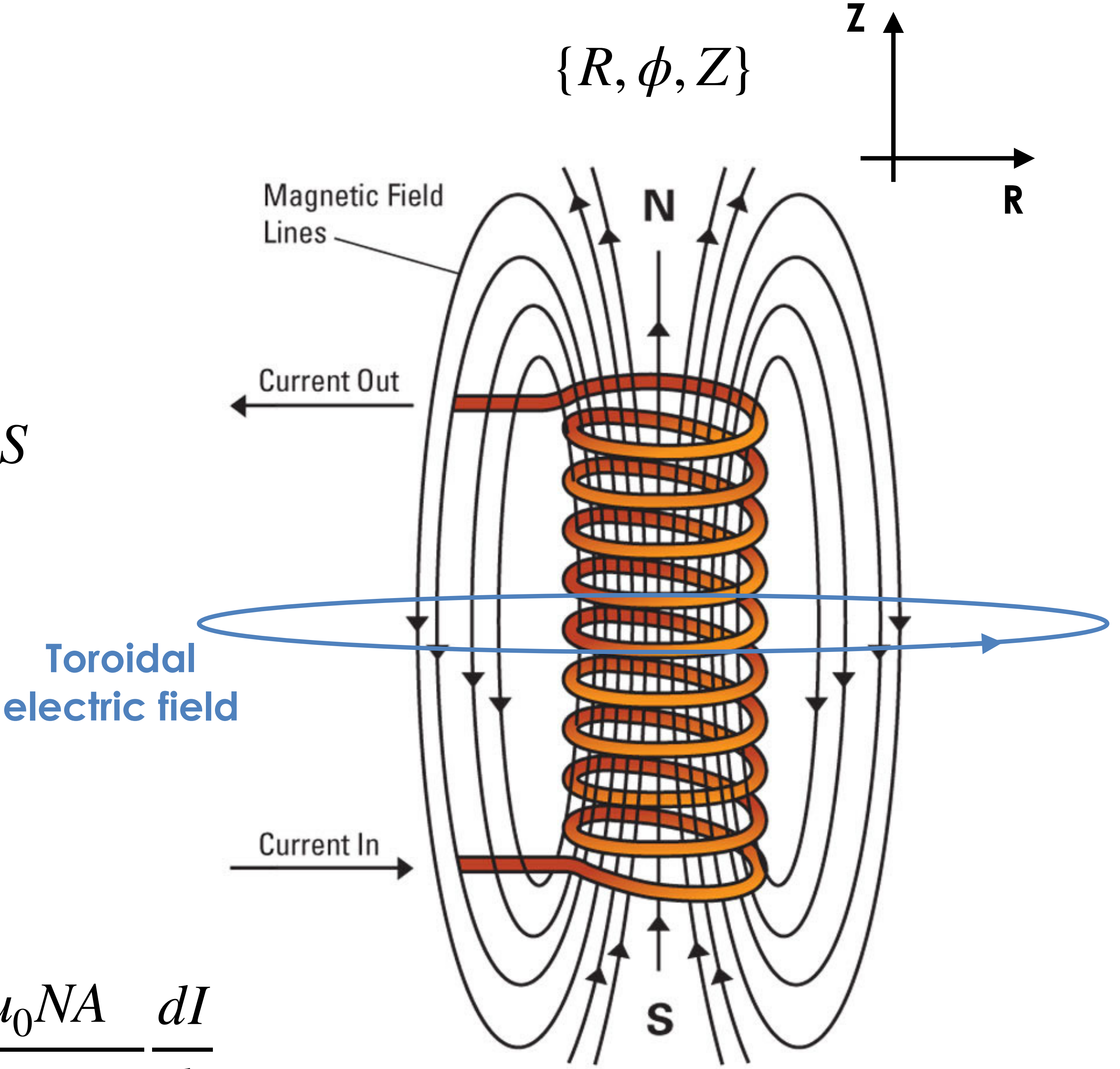
$$\int_0^{2\pi} E_\phi(R)\hat{\mathbf{e}}_\phi \cdot \hat{\mathbf{e}}_\phi R d\phi = -\frac{d}{dt}\int B_Z \hat{\mathbf{e}}_\mathbf{Z} \cdot \hat{\mathbf{e}}_\mathbf{Z} dS$$

$$2\pi R E_\phi(R) = -\frac{d}{dt}\int \frac{\mu_0 N I}{L} dS$$

$$2\pi R E_\phi(R) = V = -\frac{\mu_0 N A}{L}\frac{dI}{dt}$$

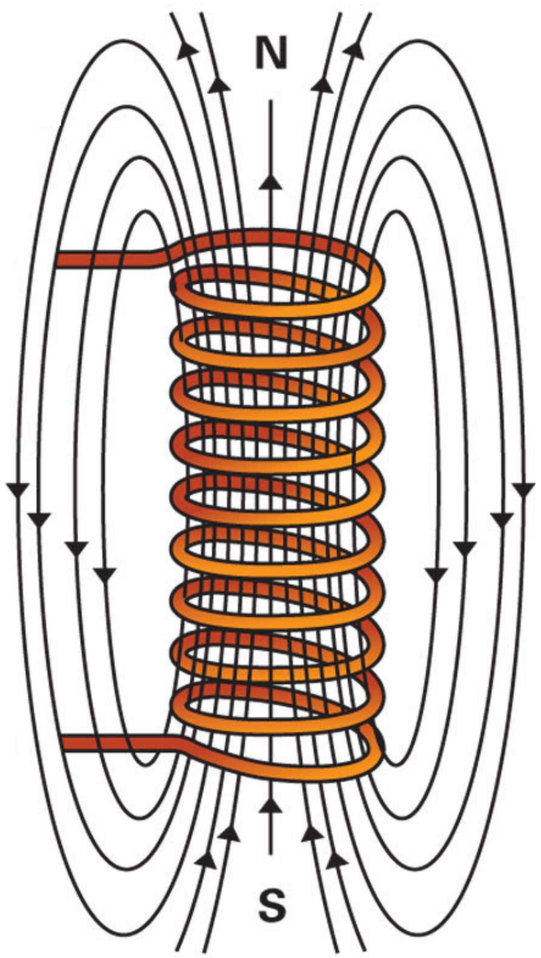
$$V = R_{plasma} I_{plasma} \quad \rightarrow \quad I_{plasma} = -\frac{\mu_0 N A}{R_{plasma} L}\frac{dI}{dt}$$

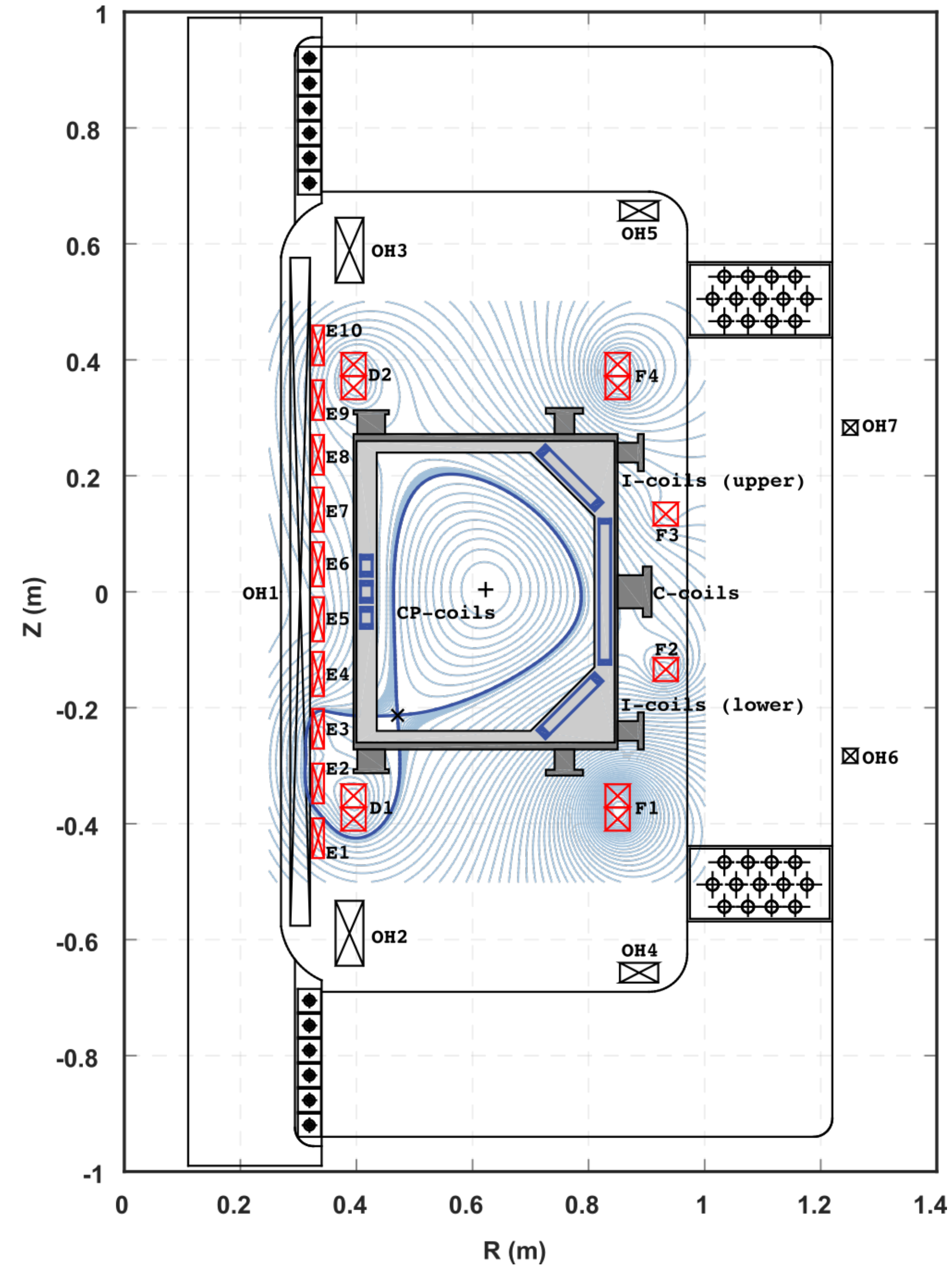
$\{R, \phi, Z\}$



The need for stray field compensation coils

- **The magnetic field lines leaving the ends of the CS tend to pass through the plasma region, causing the induced electric field to decrease**
  - *These (stay) fields are compensated by some additional OH coils, called compensation coils, which guide the field lines to close outside the plasma*





The central solenoid and the flux swing

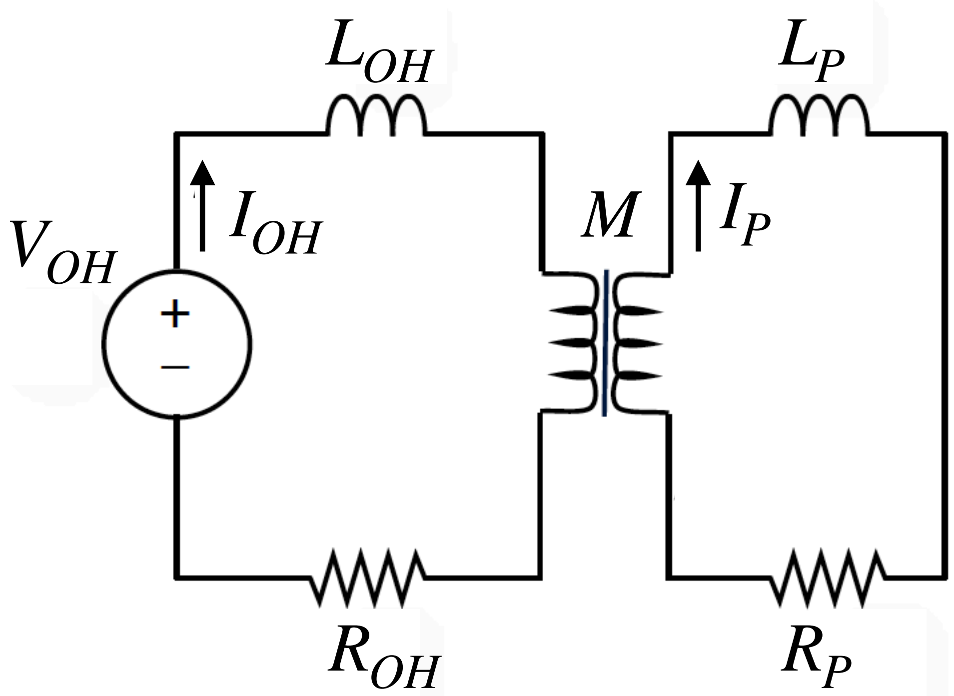
- **An important quantity related to the design of a central solenoid in pulsed tokamaks is the amount of magnetic flux (the flux swing) it can produce**

$$V_{\text{loop}} = -\frac{d\Phi}{dt} \quad \rightarrow \quad d\Phi = -V_{\text{loop}}dt \quad \rightarrow \quad \Delta\Phi = -\int_0^{\Delta t} V_{\text{loop}}(t)dt \quad \textbf{or} \quad \Delta\Phi = -V_{\text{loop}}\Delta t$$

- **The coupling between the central solenoid and the plasma can be modeled, in a very simplified way, by the electric circuit below**
  - *Here, $M > 0$ means that the I_{OH} and the I_P flow in the same toroidal direction*

$$V_{OH} = R_{OH}I_{OH} + L_{OH}\frac{dI_{OH}}{dt} + M\frac{dI_P}{dt}$$

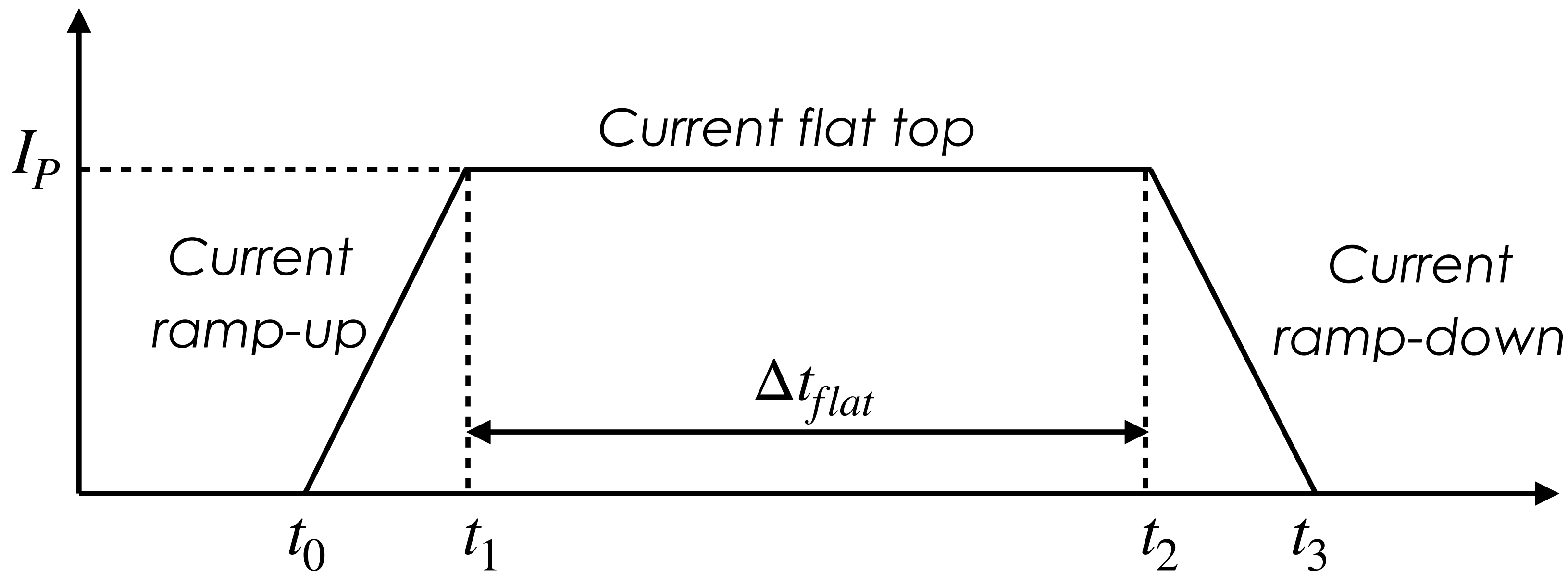
$$0 = R_P I_P + L_P\frac{dI_P}{dt} + \underbrace{M\frac{dI_{OH}}{dt}}_{-V_{\text{loop}}}$$



- **What is the power supply voltage temporal evolution needed to sustain a certain pre-programmed plasma current time trace?**

The central solenoid and the flux swing

- **Let's suppose we want the plasma current to change as indicated below**



  - *The current ramp-up should not be too fast: triggering of resistive instabilities*
  - *The current ramp-up should not be too slow: consumption of the flux swing*

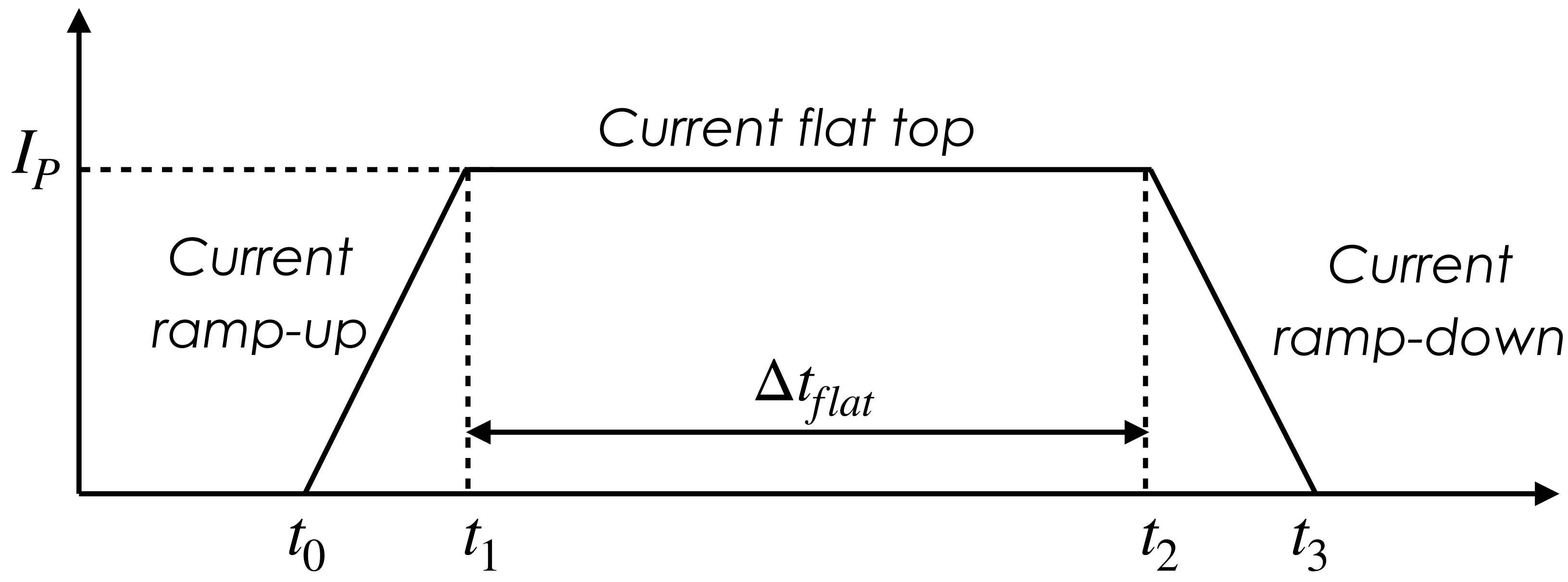
- **The current ramp-up phase**
  - *Integrating the second circuit equation:*

$$\Delta I_{OH,up} = -\frac{1}{M}\left[L_P \Delta I_P + \int_{t_0}^{t_1} R_P(t') I_P(t') dt'\right] \approx -\frac{2L_P \Delta I_P}{M} \quad \rightarrow \quad \Delta \Phi_{up} = M \Delta I_{OH,up} = -2L_P \Delta I_P$$

  - *First term: corresponds to the work needed to change the plasma current*
  - *Second term: resistive flux swing consumption (assumed to be about $L_P \Delta I_P$)*

The central solenoid and the flux swing

- **Let's suppose we want the plasma current to change as indicated below**



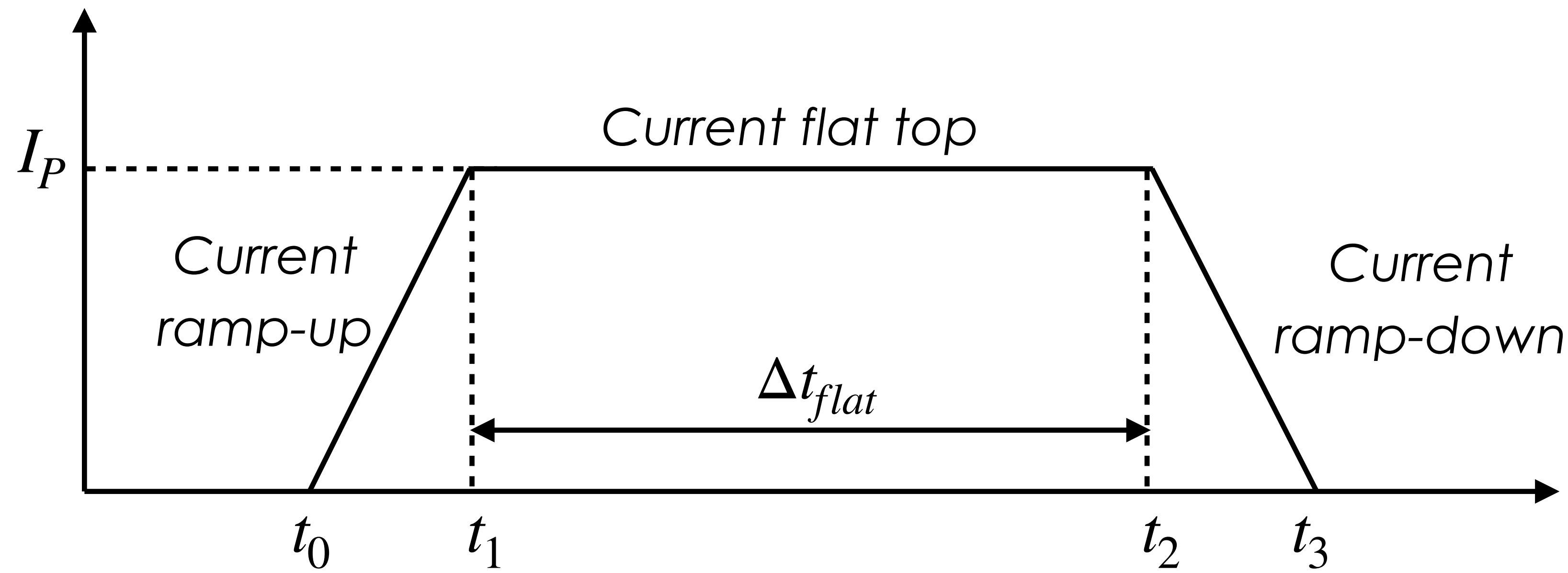
- **The current flat-top phase**
  - *From the loop voltage (with I_P constant and, therefore, V_{loop} constant):*

$$V_{\text{loop}} = -M\frac{dI_{OH}}{dt} \quad \rightarrow \quad \Delta I_{OH,flat} = -\frac{V_{\text{loop}}\Delta t_{flat}}{M} \quad \rightarrow \quad \Delta\Phi_{flat} = M\Delta I_{OH,flat} = -V_{\text{loop}}\Delta t_{flat}$$

  - *From the last expression, flux swing is sometimes also expressed in volt-seconds*

The central solenoid and the flux swing

- **Let's suppose we want the plasma current to change as indicated below**



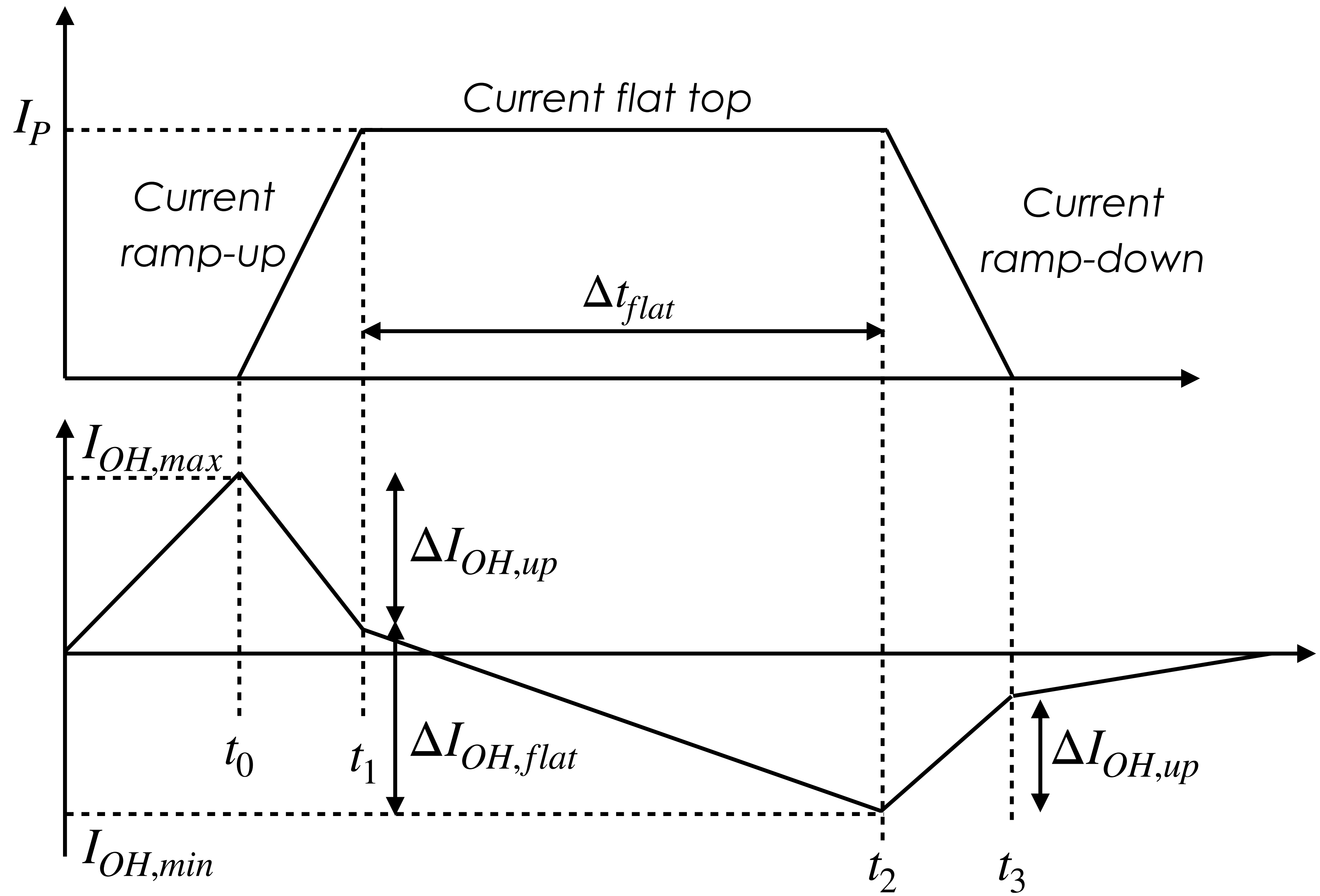
- **The current ramp-down phase**
  - *Again, from integrating the second circuit equation:*

$$\Delta I_{OH,down} = -\frac{1}{M}\left[L_P \Delta I_P + \int_{t_0}^{t_1} R_P(t')I_P(t')dt'\right] \approx -\frac{L_P \Delta I_P}{M} \quad \rightarrow \quad \Delta\Phi_{down} = M\Delta I_{OH,down} = -L_P \Delta I_P$$

  - *First term: corresponds to the work needed to change the plasma current*
  - *The second term is now much smaller compared to the ramp-up phase as the plasma is now hot and, consequently, its resistance is very low*

The central solenoid and the flux swing

- **Time traces of the OH current and the plasma current**



  - *The OH coil must be magnetized before the discharge*
  - *The OH current reverses direction to reduce its peak value*

The central solenoid and the flux swing

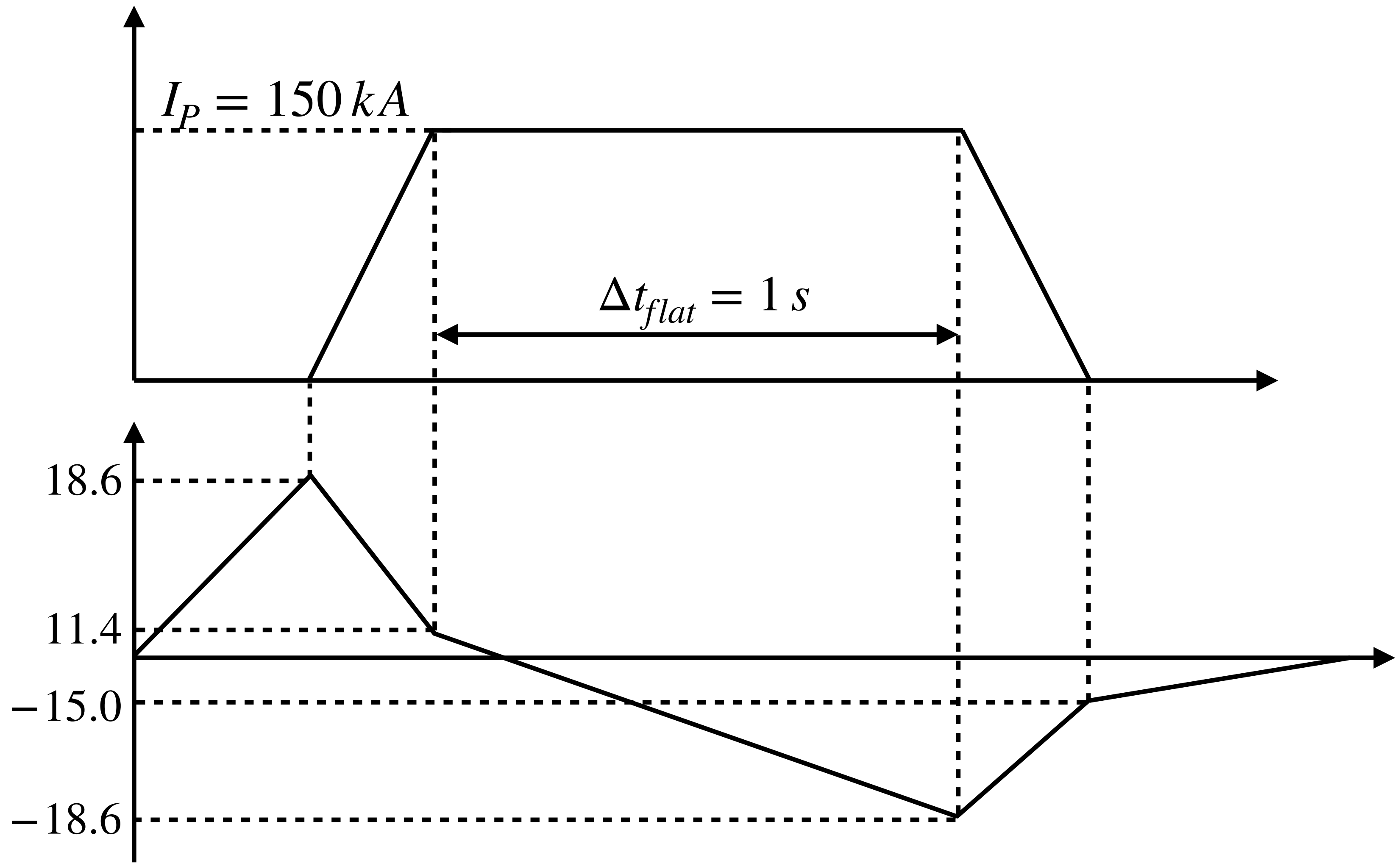
- **Example: TCABR**
  - *Let's take* $R_0 = 0.6\,m$, $a = 0.18\,m$, $M = 50\,\mu H$, $I_P = 150\,kA$, $V_{\text{loop}} = 1.5\,V$, $\Delta t_{flat} = 1\,s$

$$L_P = \mu_0 R_0 \left[\ln\left(\frac{8R_0}{a}\right) - \frac{7}{4}\right] = 1.2\,\mu H$$

$$\Delta I_{OH,up} = -\frac{2L_P \Delta I_P}{M} = -7.2\,kA$$

$$\Delta I_{OH,flat} = -\frac{V_{\text{loop}} \Delta t_{flat}}{M} = -30\,kA$$

$$\Delta I_{OH,down} = -\frac{L_P \Delta I_P}{M} = +3.6\,kA$$



$$I_{OH,max} = I_{OH,min} = \frac{\Delta I_{OH,up} + \Delta I_{OH,flat}}{2} = 18.6\,kA$$

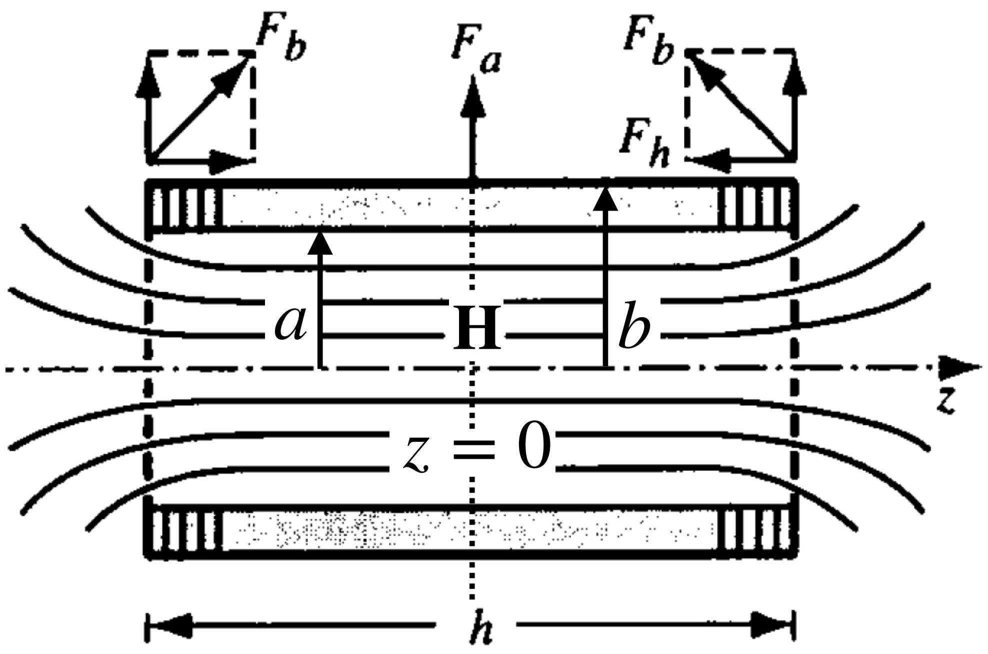
$$\Delta\Phi_{swing} = \Delta\Phi_{up} + \Delta\Phi_{flat} = 1.86\ V \cdot s$$

The forces acting on a solenoid

- **The forces action on a solenoid can also be found using the previous method**
  - *The self-inductance of a solenoid of inner radius a, outer radius b, length h and N turns is given by*

$$L = \frac{\mu_0 N^2 \pi a^2}{h} K_L$$

where the correction factor $K_L = K_L(h/2a)$ is due to its finite length (boundary effect)

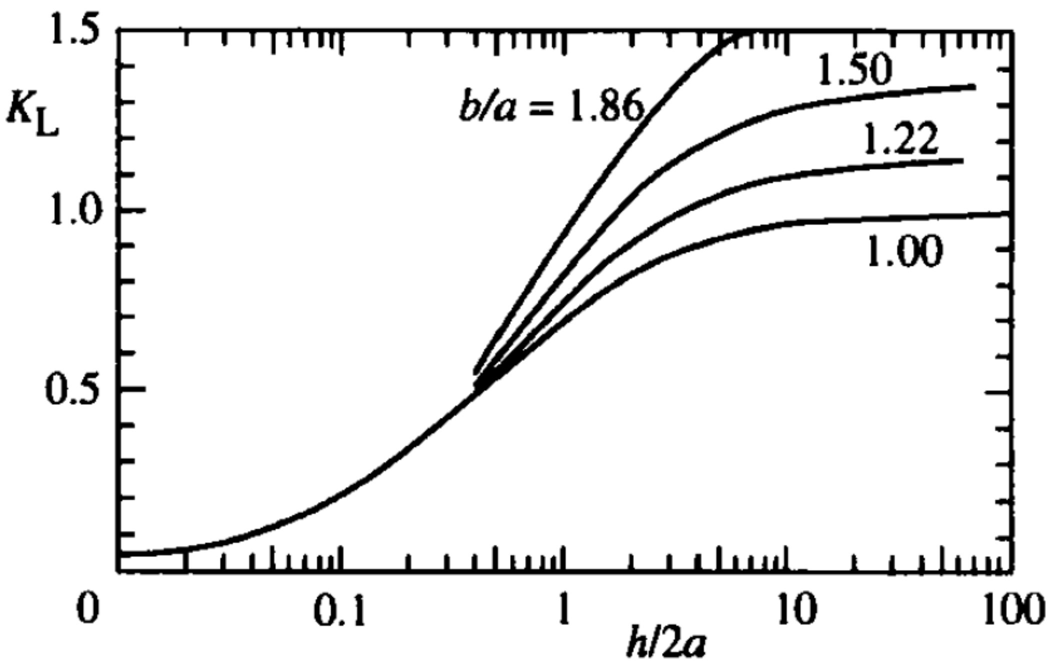


- **Let's define a parameter $x \equiv h/2a$ and calculate**

$$\frac{dK_L}{da} = \frac{dK_L}{dx}\frac{dx}{da} = -\frac{h}{2a^2}\frac{dK_L}{dx}$$

  - *Note that, for $x \gg 1$ we have*

$$\frac{dK_L}{dx} \ll \frac{K_L}{x}$$



The forces acting on a solenoid

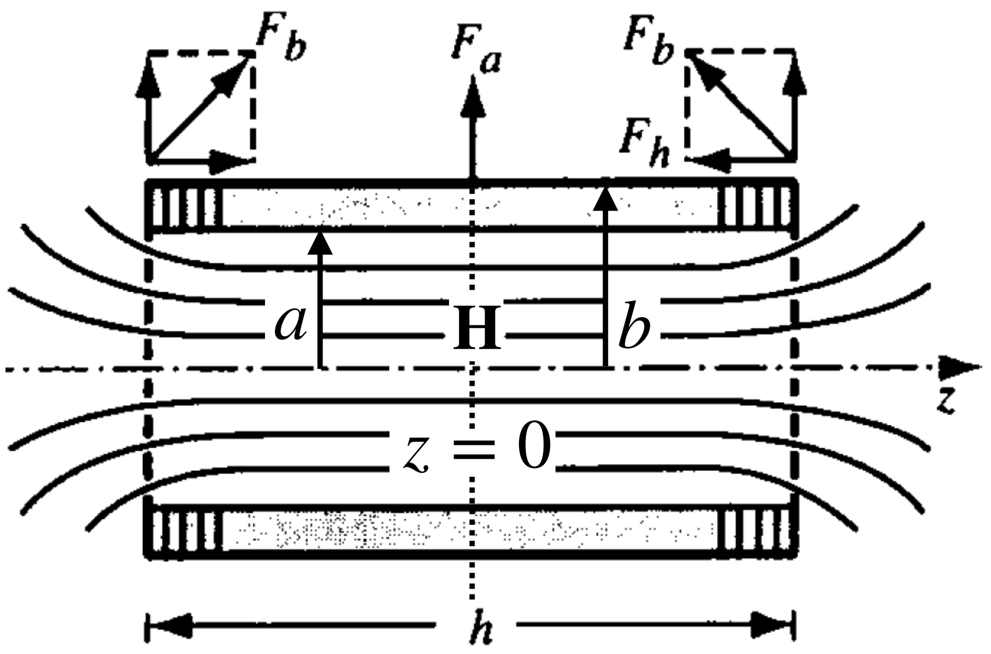
- **The hoop force acting on a solenoid is calculated as**

$$F_a = \frac{I^2}{2}\frac{\partial L}{\partial a} = \frac{I^2}{2}\frac{\partial}{\partial a}\left(\frac{\mu_0 N^2 \pi a^2}{h} K_L\right)$$

$$F_a = \frac{\mu_0 N^2 \pi I^2}{2}\left(\frac{K_L}{x} - \frac{1}{2}\frac{dK_L}{dx}\right)$$

  - *For solenoids with* $x \gg 1$

$$F_a \approx \frac{\mu_0 N^2 \pi a K_L I^2}{h}$$



- **Similarly, the axial (compressional) force acting on a solenoid is calculated as**

$$F_h = \frac{I^2}{2}\frac{\partial L}{\partial h} = \frac{I^2}{2}\frac{\partial}{\partial h}\left(\frac{\mu_0 N^2 \pi a^2}{h} K_L\right) = -\frac{\mu_0 N^2 \pi a I^2}{4h}\left(\frac{K_L}{x} - \frac{\partial K_L}{\partial x}\right)$$

  - *For solenoids with* $x \gg 1$*, and defining the density of turns as* $n = N/h$*:*

$$F_h = -\frac{\mu_0 n^2 \pi a^2 I^2 K_L}{2}$$

The forces acting on a solenoid

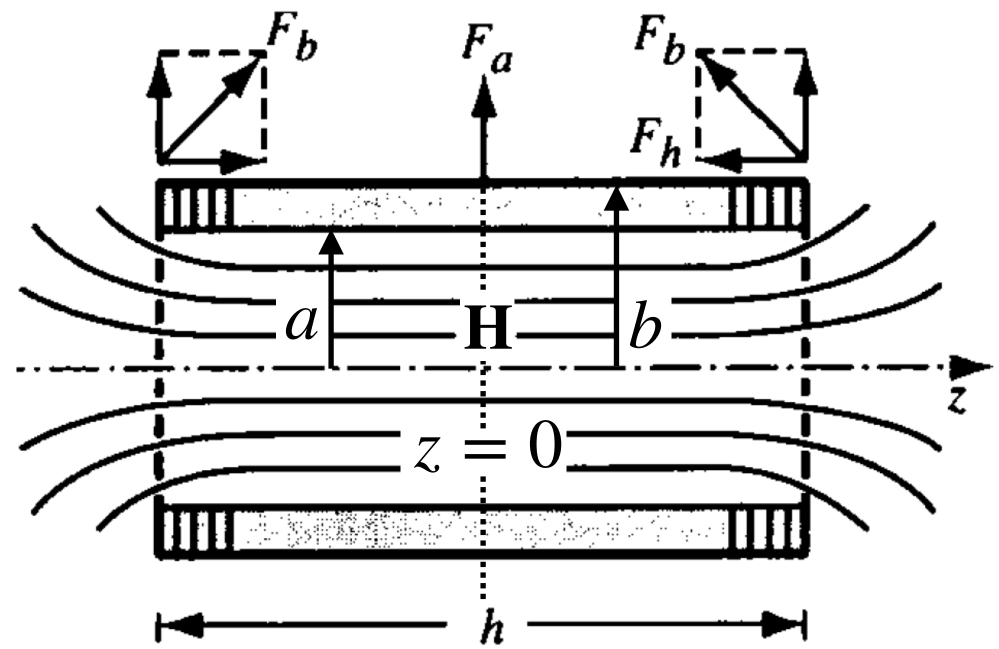
- **The hoop force acting on a solenoid is calculated as**

$$F_a = \frac{I^2}{2}\frac{\partial L}{\partial a} = \frac{I^2}{2}\frac{\partial}{\partial a}\left(\frac{\mu_0 N^2 \pi a^2}{h} K_L\right)$$

$$F_a = \frac{\mu_0 N^2 \pi I^2}{4}\left(\frac{K_L}{x} - \frac{dK_L}{dx}\right)$$

  - *For solenoids with* $x \gg 1$

$$F_a \approx \frac{\mu_0 N^2 \pi a K_L I^2}{2h}$$



- **Similarly, the axial (compressional) force acting on a solenoid is calculated as**

$$F_h = \frac{I^2}{2}\frac{\partial L}{\partial h} = \frac{I^2}{2}\frac{\partial}{\partial h}\left(\frac{\mu_0 N^2 \pi a^2}{h} K_L\right) = -\frac{\mu_0 N^2 \pi a I^2}{4h}\left(\frac{K_L}{x} - \frac{\partial K_L}{\partial x}\right)$$

  - *For solenoids with* $x \gg 1$*, and defining the density of turns as* $n = N/h$*:*

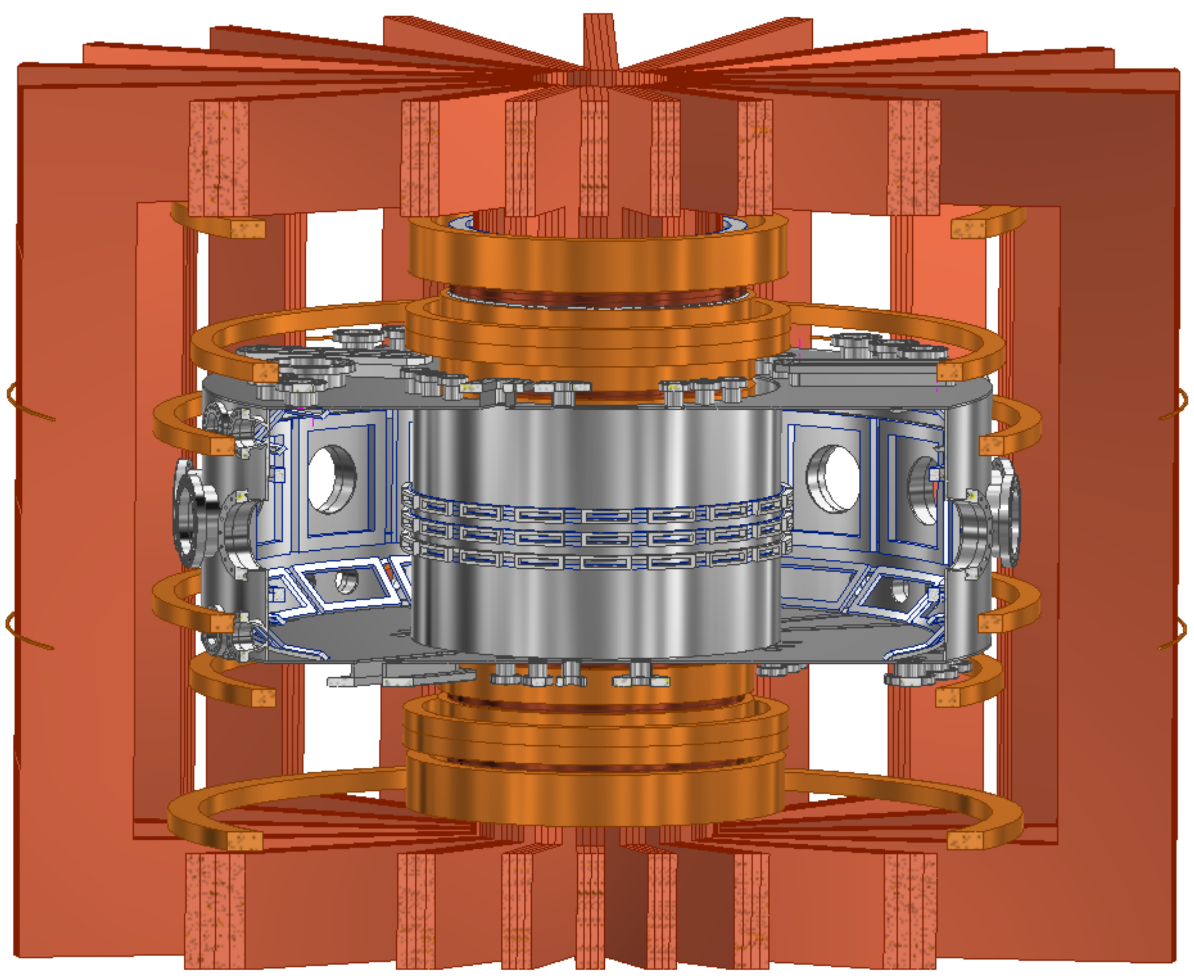
$$F_h(z) \approx \frac{K_L(z/2a)}{K_L(h/2a)} F_h(h/2a) \approx -\frac{\mu_0 n^2 \pi a^2 I^2}{2} K_L(z/2a)$$

The compressional force increases towards the ends

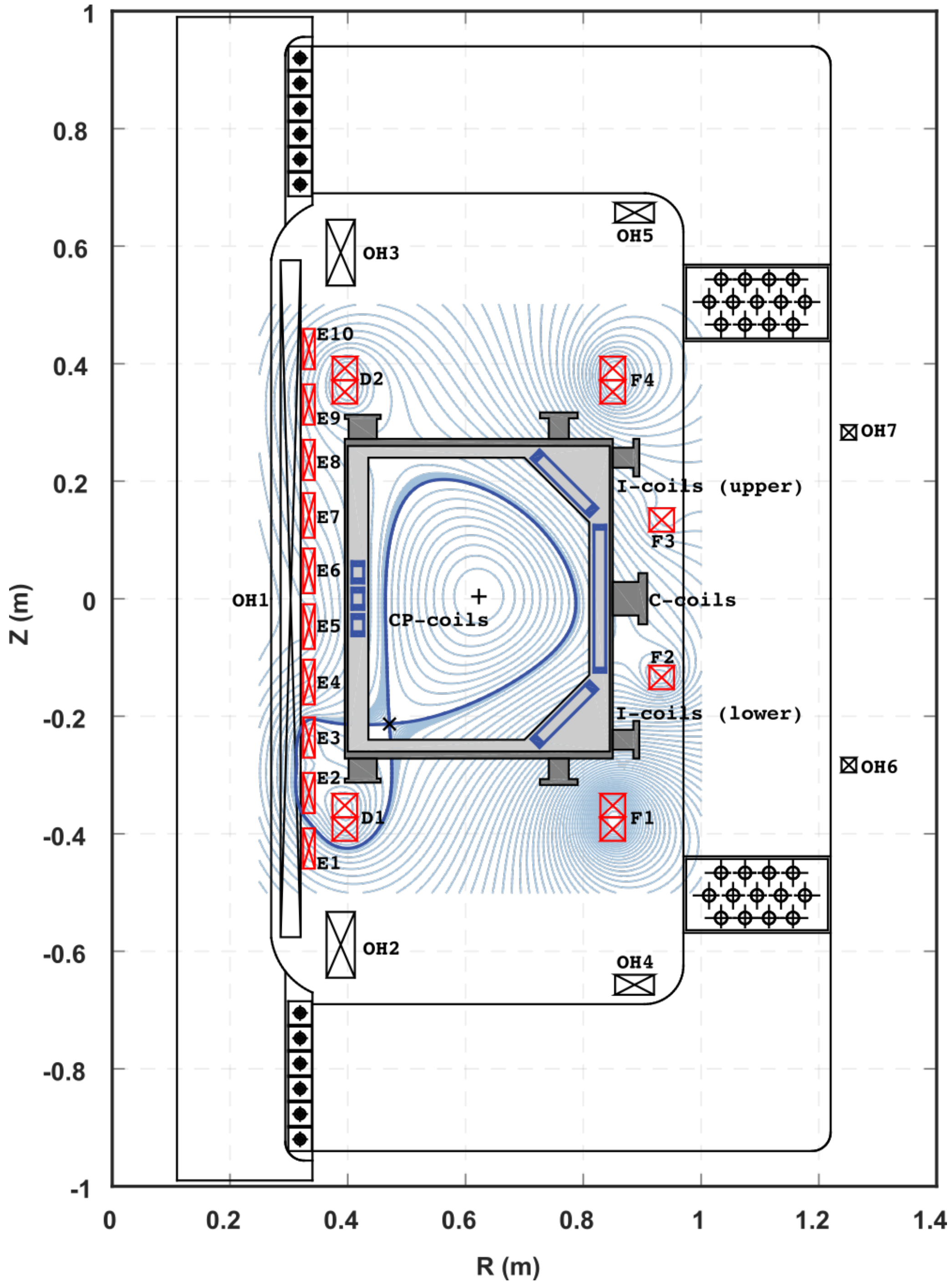
- **Tokamak engineering**
  - *Magnetic forces*
  - *Toroidal field coils*
  - *The central solenoid*
  - *Poloidal field coils*
  - *The vertical plasma instability and the RZIP model*

Poloidal field coils

- **Poloidal field coils (PF) are responsible for shaping the plasma boundary and also for controlling the plasma position**

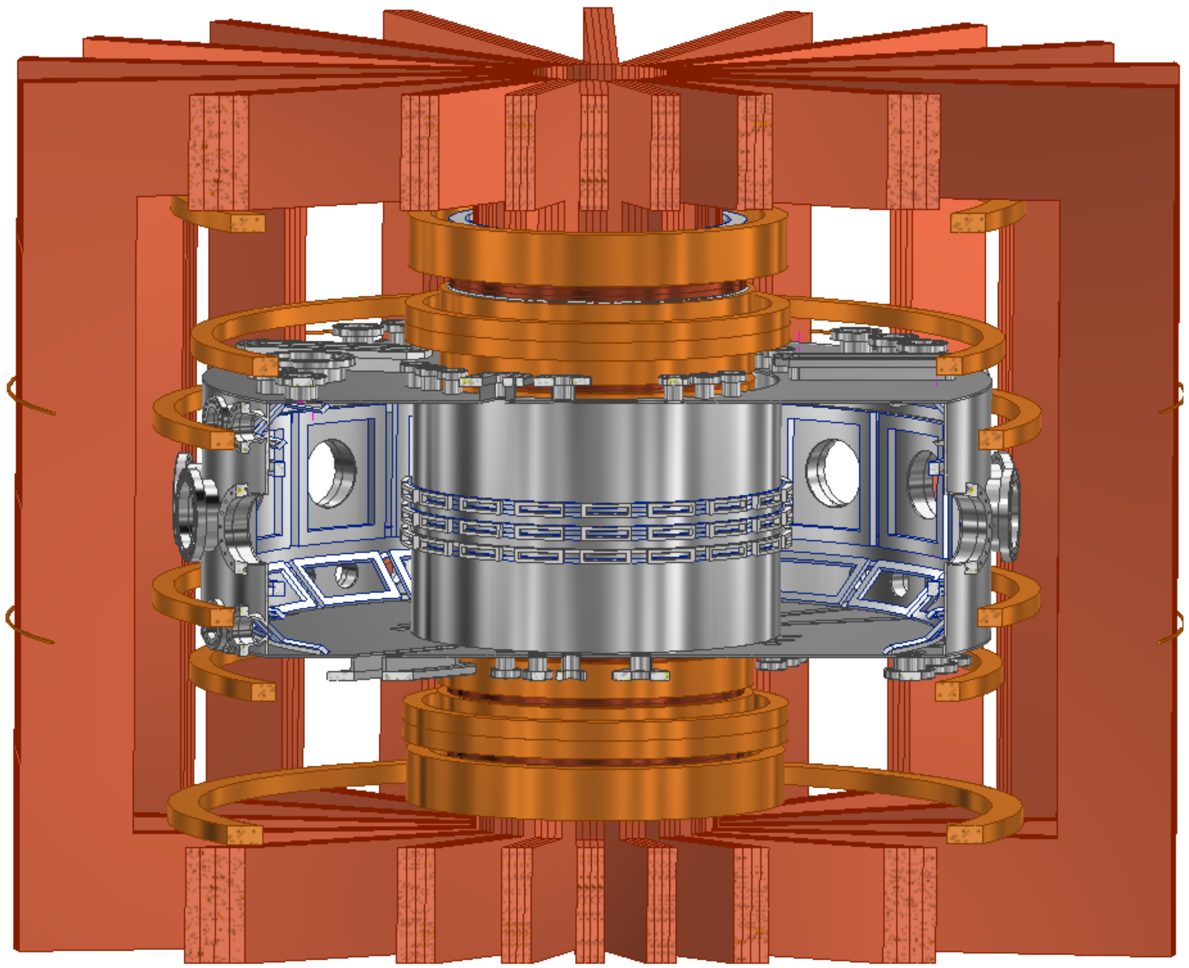


TCABR PF coils

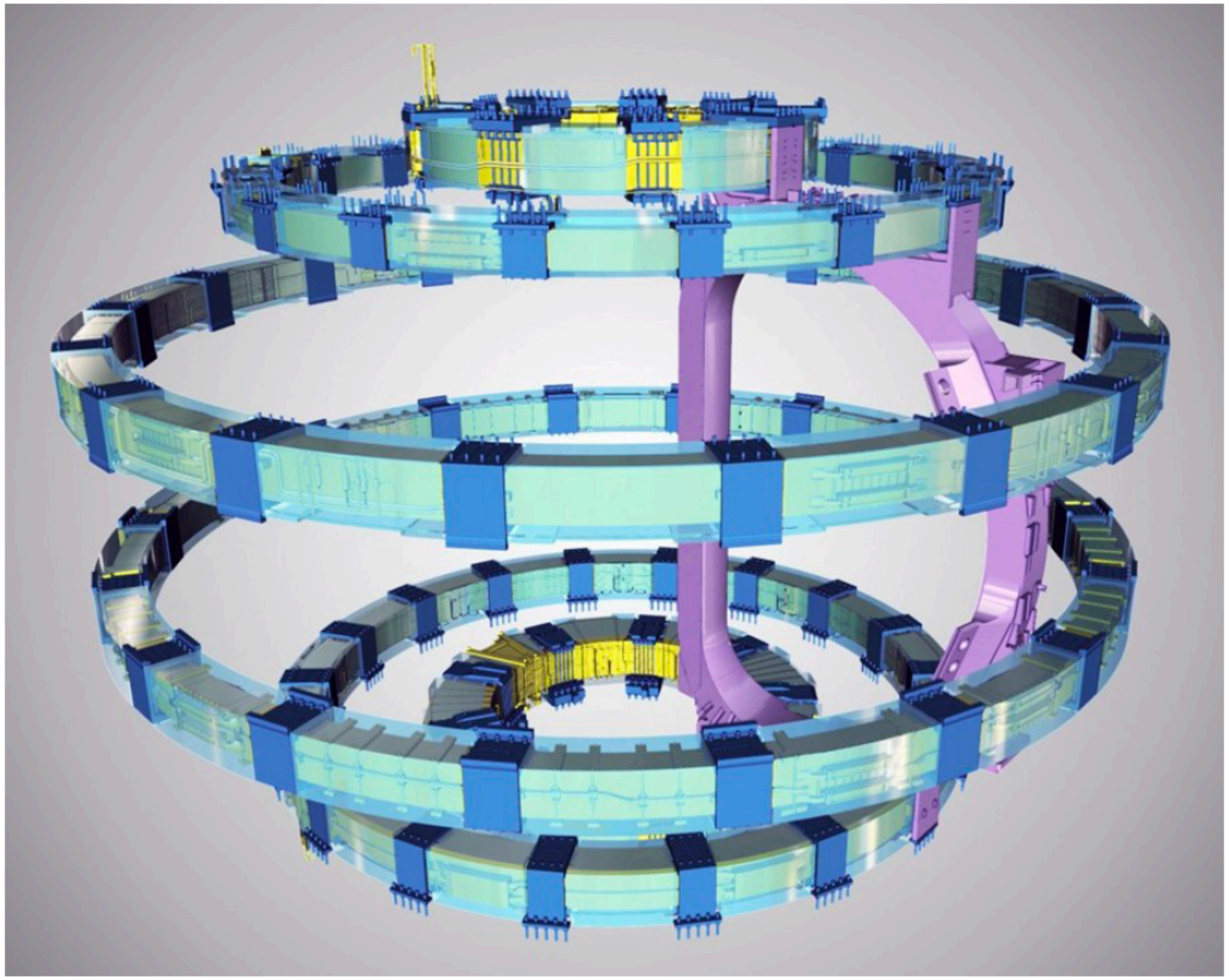


Poloidal field coils

- **Poloidal field coils (PF) are responsible for shaping the plasma boundary and also for controlling the plasma position**



TCABR PF coils



ITER PF coils

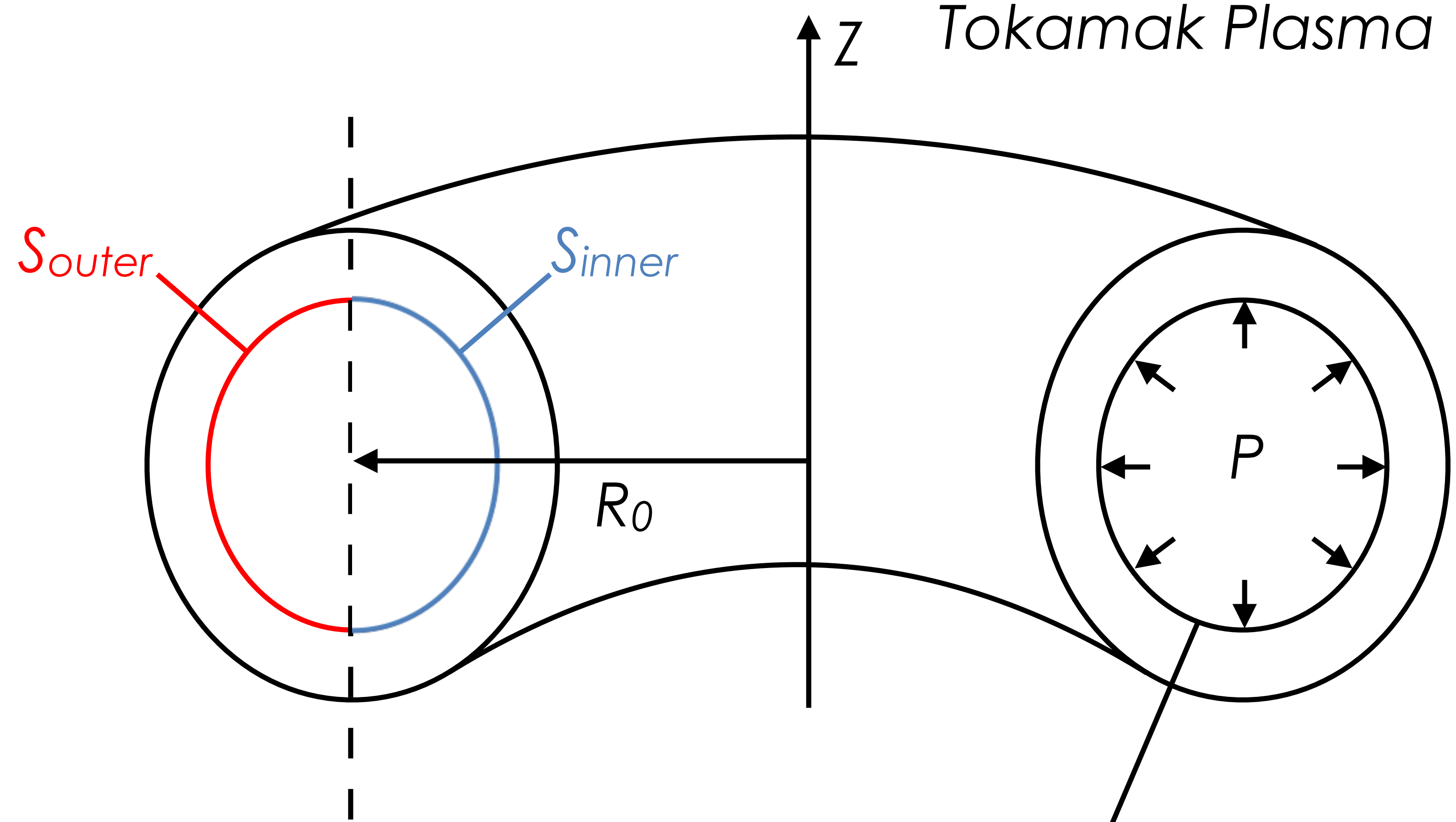
The tire tube force

- **The tire tube force is the results of a constant pressure acting on regions with diference areas**
  - *In a tire tube, as well as in a tokamak plasma, the tire tube force points outwards and, therefore, it tends to increase the system's major radius (R_0)*

$$F_R = F_{outer} - F_{inner} = p_{outer}S_{outer} - p_{inner}S_{inner}$$

$$F_R = p\left(S_{outer} - S_{inner}\right) > 0$$

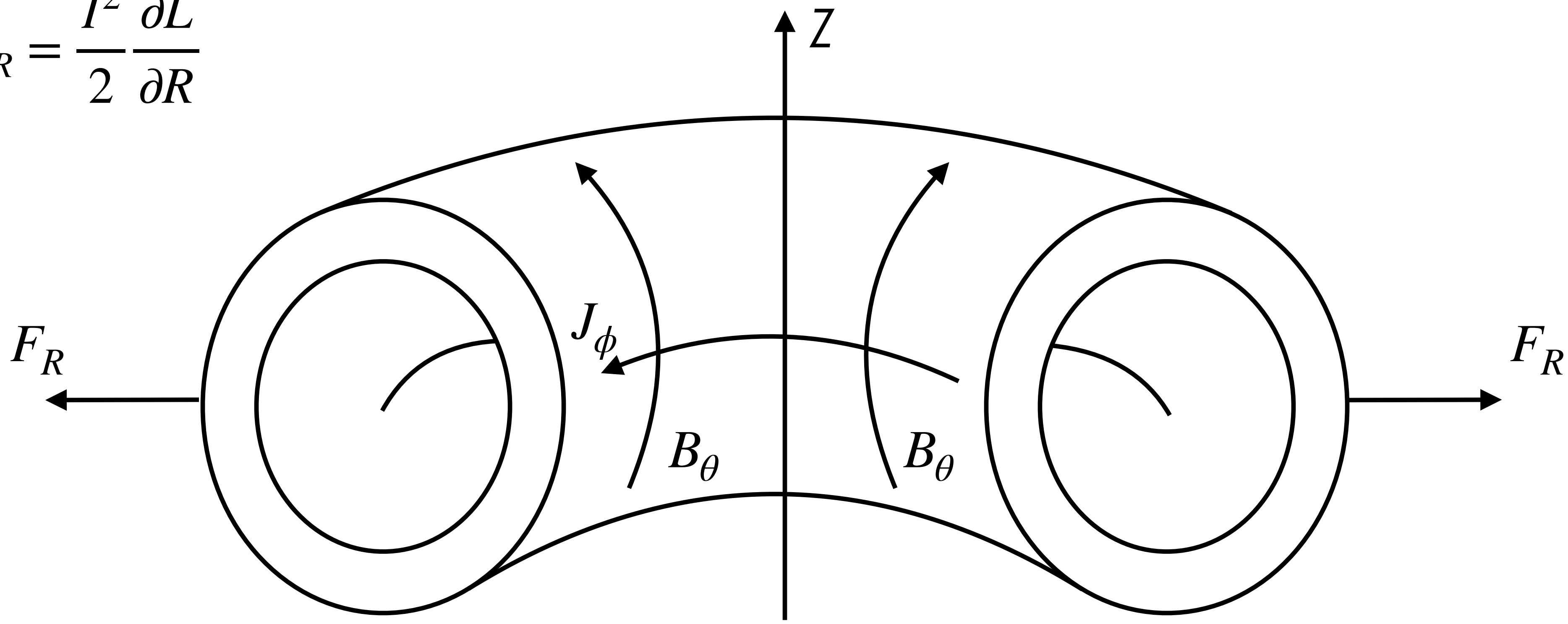
Tire tube



The hoop force

- **The hoop force is the EM equivalent of the tire tube force**
  - *This force tends to decrease the magnetic energy of the system by increasing the major radius of the plasma*

$$F_R = \frac{I^2}{2}\frac{\partial L}{\partial R}$$



The 1/R force

- **A radial force exists due to the interaction between a poloidal current and the toroidal field $B_\phi = R_0 B_0 / R$**

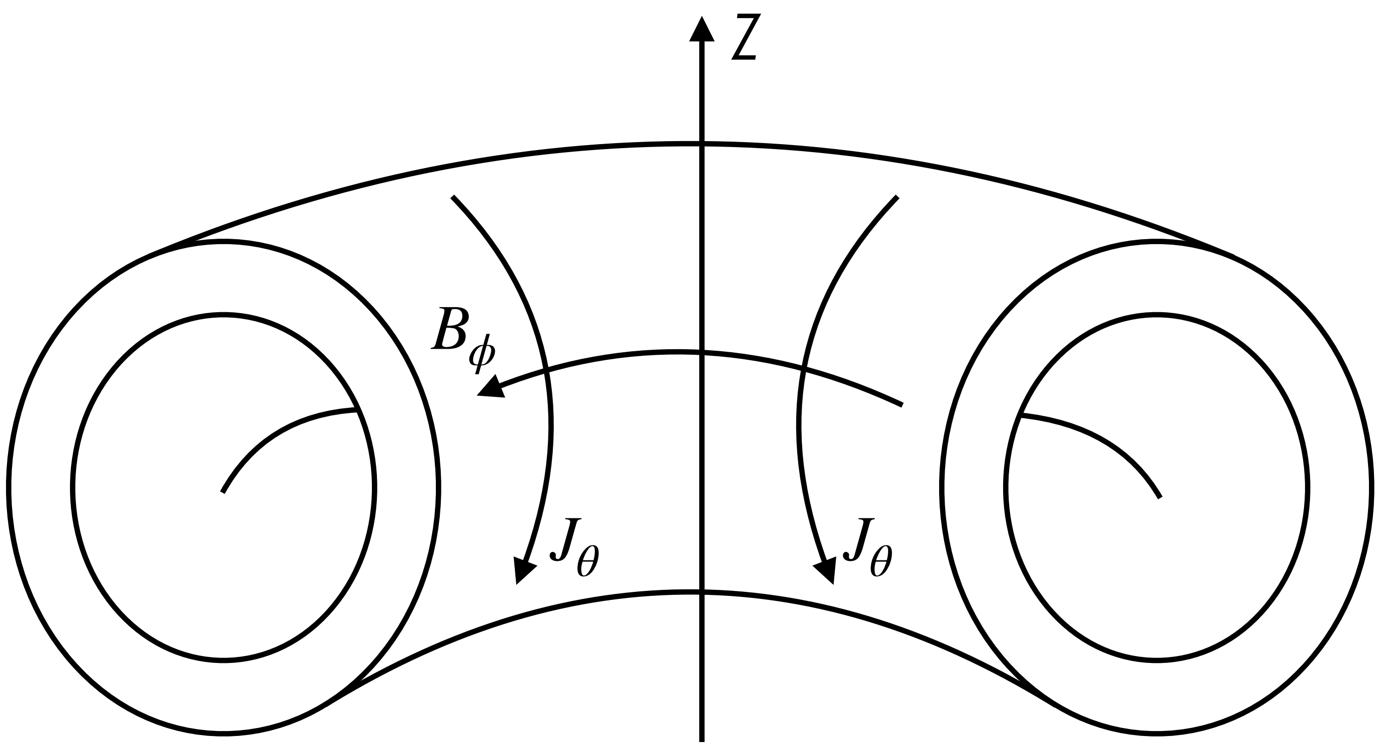
- **Depending on the conditions, the plasma can be diamagnetic or paramagnetic**
  - *The radial force points outwards if the plasma is diamagnetic*
  - *The radial force points inwards if the plasma is paramagnetic*

$$F_R \propto (J_\theta B_\phi)_{inner} - (J_\theta B_\phi)_{outer}$$

B_ϕ is decreases with R

J_θ is is larger in the inner parte due to the smaller area

$$F_R \propto B_0 - B_\phi(r)$$

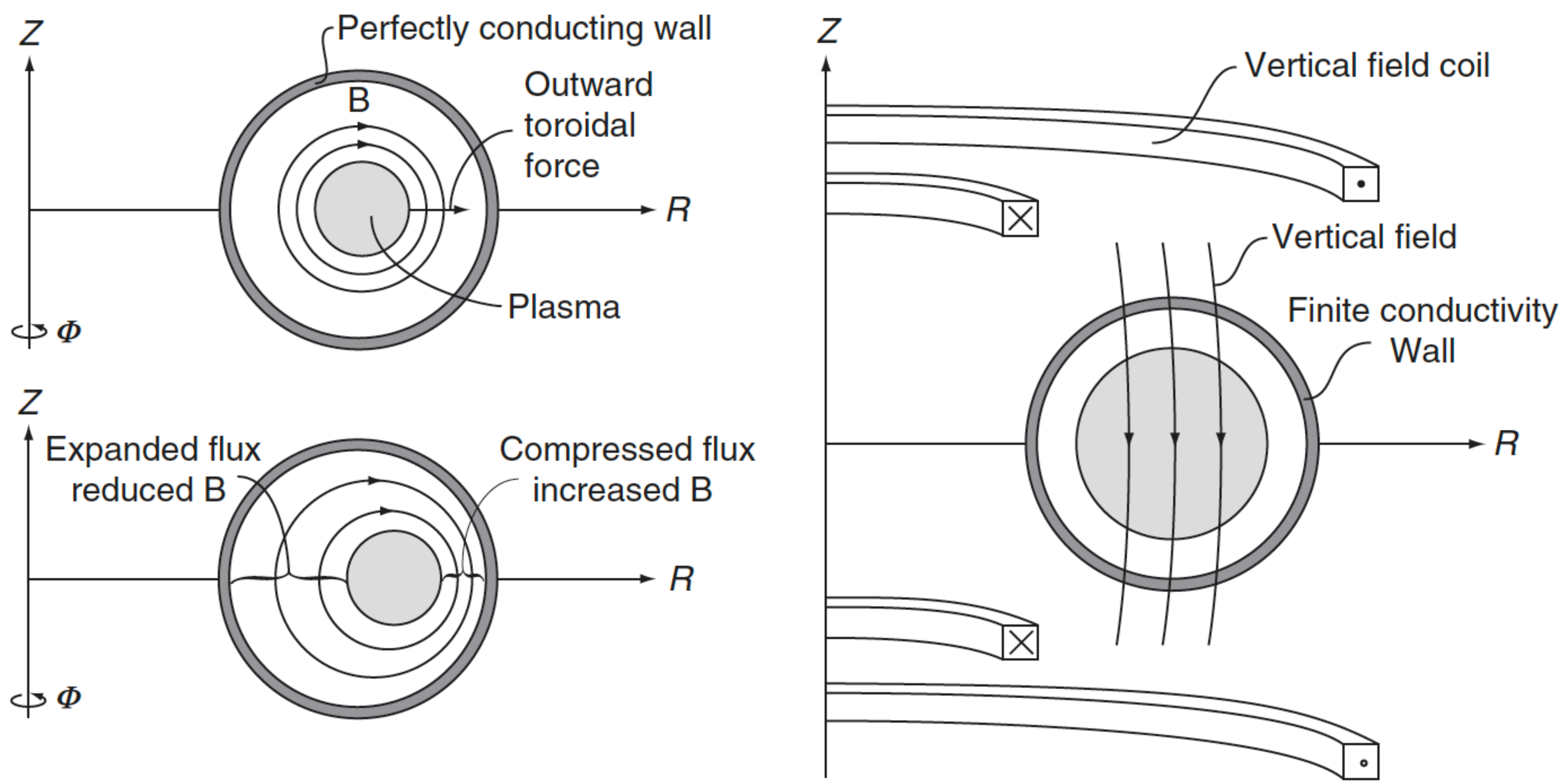


Diamagnetic Plasma

The radial force balance

- **In a configuration maintained by external coil currents, a vertical magnetic field in needed to balance the kinetic and EM forces in the radial direction,**
  - *The Lorentz force (LHS) must balance the hoop force (RHS 1st term), the tire tube force (RHS 2nd term) and the 1/R force (RHS 3rd term)*

$$2\pi R_0 I_P B_Z = \frac{I_P^2}{2}\frac{dL}{dR_0} + 4\pi^2 \int_0^a \left(p(r) + \frac{B_0}{\mu_0}\left[B_0 - B_\phi(r)\right] \right) r dr$$

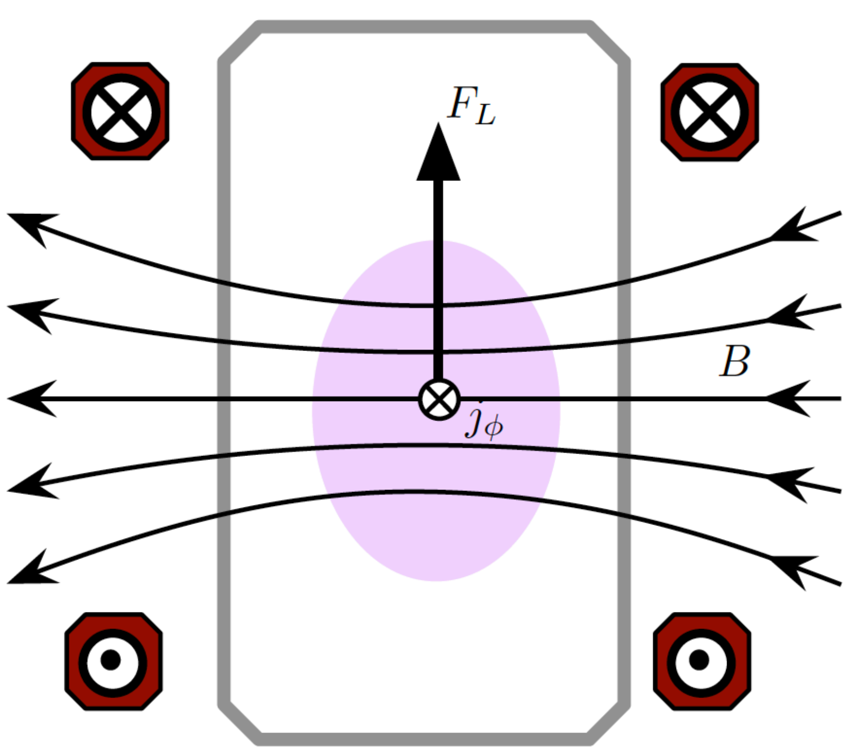


The poloidal field coils are used to control the plasma position

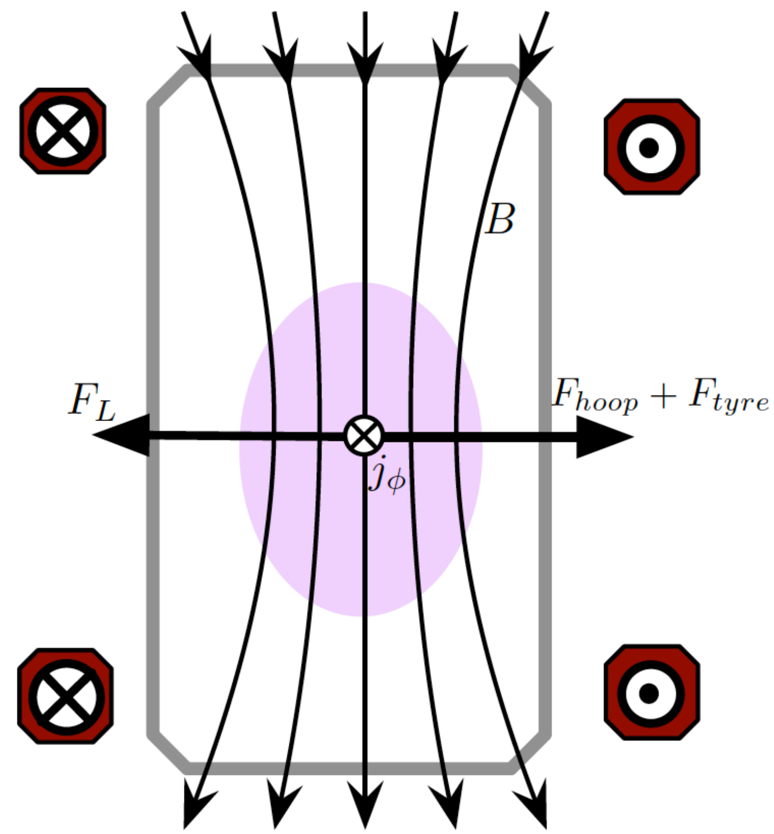
- **There is no natural force acting on the plasma in the vertical direction**
  - *A horizontal field is applied only to correct the plasma position*

- **To balance the naturally occurring radial forces in the plasma, a vertical field must be applied by the PF coils**

Vertical position control



Radial position control

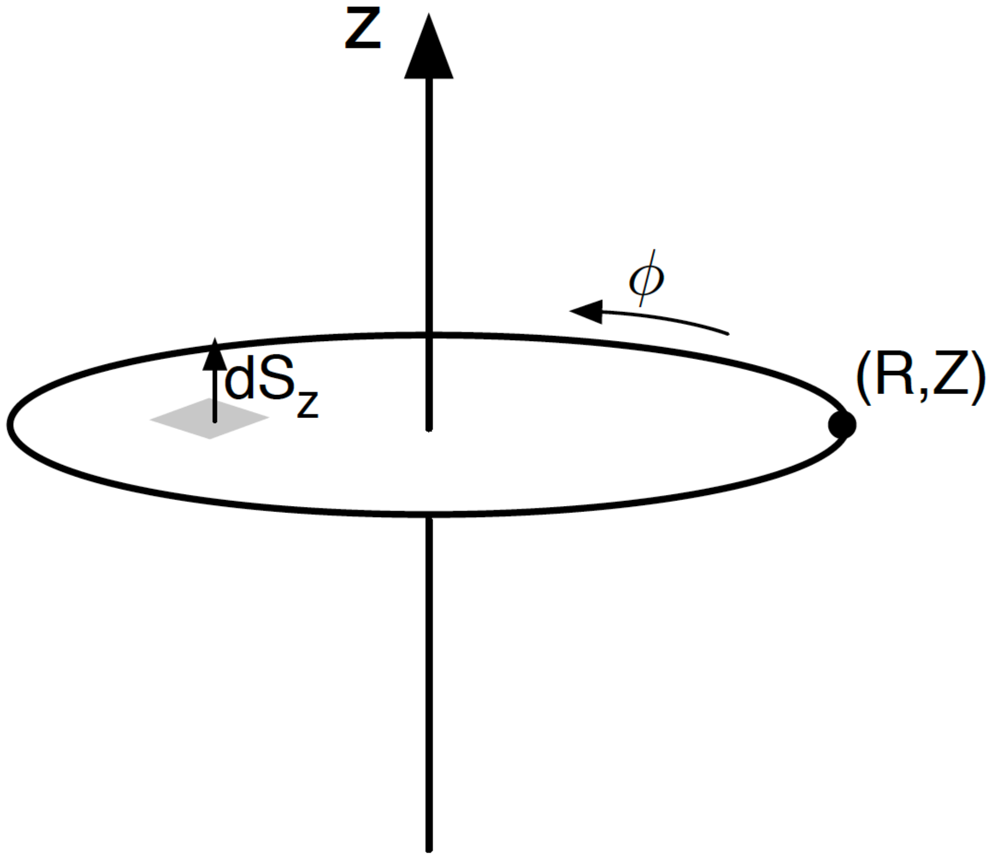


Poloidal field coils generate poloidal flux

- **The poloidal flux is defined as**

$$\psi(R,Z) = \int_S \mathbf{B} \cdot d\mathbf{S_Z}$$

$$\psi(R,Z) = 2\pi \int_0^R B_Z(R',Z)\, R'\, dR'$$



  – *Here, S is the surface area defined by a disk, centered on the Z-axis, passing through a point with coordinates (R,Z), and $d\mathbf{S_Z}$ is an element of surface whose normal points in the Z-direction*

Magnetic field from the poloidal flux

- **From the definition of the poloidal flux** $\psi(R,Z) = 2\pi \int_0^R B_Z(R',Z)\, R'\, dR'$

$$\frac{\partial \psi}{\partial R} = 2\pi R B_Z \qquad \frac{\partial \psi}{\partial Z} = 2\pi \int_0^R \frac{\partial B_Z}{\partial Z} R'\, dR'$$

- **Using that**

$$\nabla \cdot \mathbf{B} = \frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \frac{1}{R}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_Z}{\partial Z} = 0$$

combined with axisymmetry, one have that

$$\frac{\partial B_Z}{\partial Z} = -\frac{1}{R}\frac{\partial}{\partial R}(RB_R)$$

and thus

$$\frac{\partial \psi}{\partial Z} = 2\pi \int_0^R \frac{\partial B_Z}{\partial Z} R'\, dR' = -2\pi \int_0^R \frac{1}{R'}\frac{\partial}{\partial R'}(R'B_R)\, R'\, dR' = -2\pi R B_R$$

- **Therefore, the magnetic field can be calculated from the poloidal flux** $\psi(R,Z)$

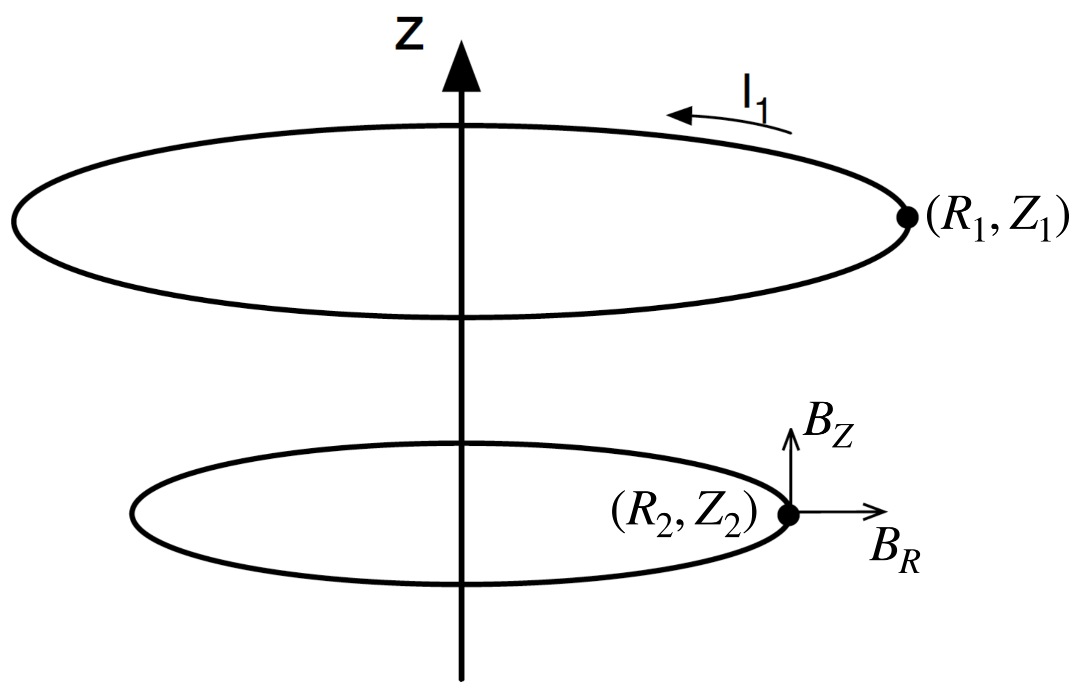
$$B_R = -\frac{1}{2\pi R}\frac{\partial \psi}{\partial Z} \qquad B_Z = \frac{1}{2\pi R}\frac{\partial \psi}{\partial R}$$

Magnetic field from mutual inductances

- **From the definition of mutual inductance**

$$\psi(R_2, Z_2) = M(R_2, Z_2, R_1, Z_1)\, I_1$$

$$\psi_2 = M_{21}\, I_1$$



- **We have already saw that the mutual inductance between two coaxial circular current filament loops is**

$$M = \mu_0 \sqrt{a_1 a_2} \left[\left(\frac{2}{k} - k \right) K(k^2) - \frac{2}{k} E(k^2) \right]$$

Therefore, the magnetic field at (R_2, Z_2), created by the current I_1, is

$$B_R(R_2, Z_2) = -\frac{I_1}{2\pi R} \frac{\partial M_{21}}{\partial Z} \qquad B_Z(R_2, Z_2) = \frac{I_1}{2\pi R} \frac{\partial M_{21}}{\partial R}$$

Voltage induced by a change in poloidal flux

- **Taking the time derivative of $\psi(R, Z)$ through a fixed disk yields**

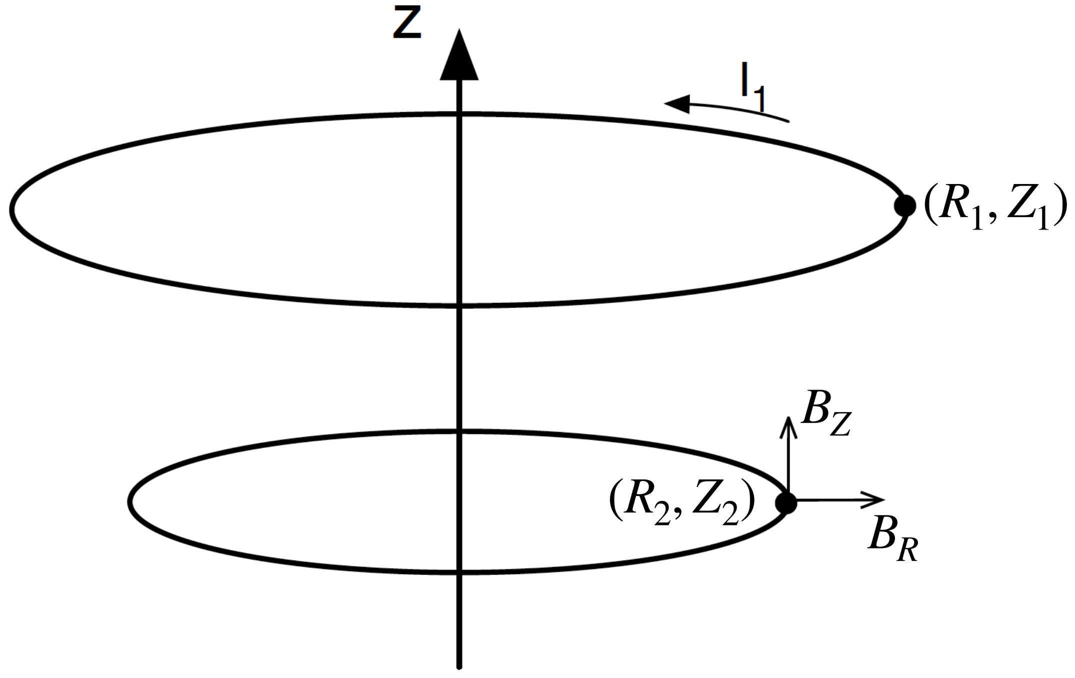
$$\frac{\partial \psi}{\partial t} = \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S_Z}$$

- **From Faraday's induction law, and axisymmetry,**

$$\frac{\partial \psi(R, Z)}{\partial t} = -\oint \mathbf{E} \cdot d\mathbf{l} = -2\pi R E_\phi(R, Z) = V_{\text{loop}}(R, Z)$$

Therefore, a change in the current I_1 induces a voltage at (R_2, Z_2)

$$\frac{\partial \psi(R_2, Z_2)}{\partial t} = M_{21} \frac{dI_1}{dt} = V_{\text{loop}}(R_2, Z_2)$$



Measuring magnetic fields and poloidal fluxes

- **Local measurements of the magnetic field at (R_p, Z_p) can be made by integrating the signal of magnetic probes (of area A_p and N turns)**

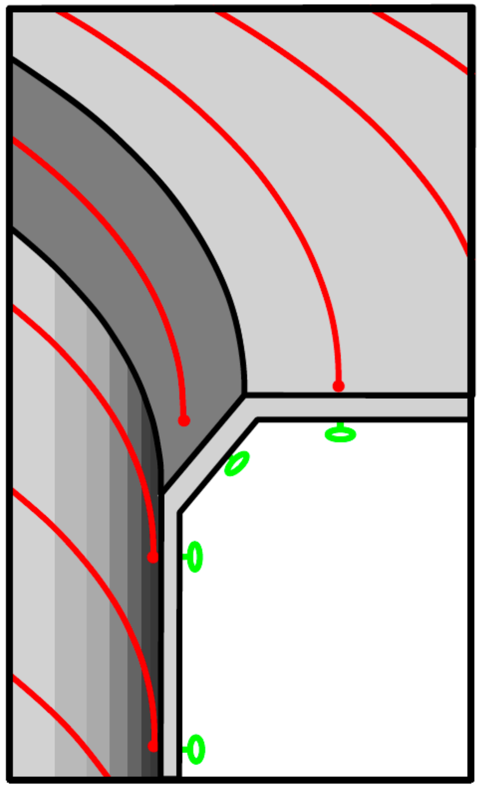
$$V_p = -\frac{d}{dt}\int_{A_p} \mathbf{B}\cdot d\mathbf{S_Z} = -A_p N \frac{dB}{dt} \qquad B_p(t; R_p, Z_p) = -\frac{1}{NA_p}\int_0^t V_p(t')dt'$$

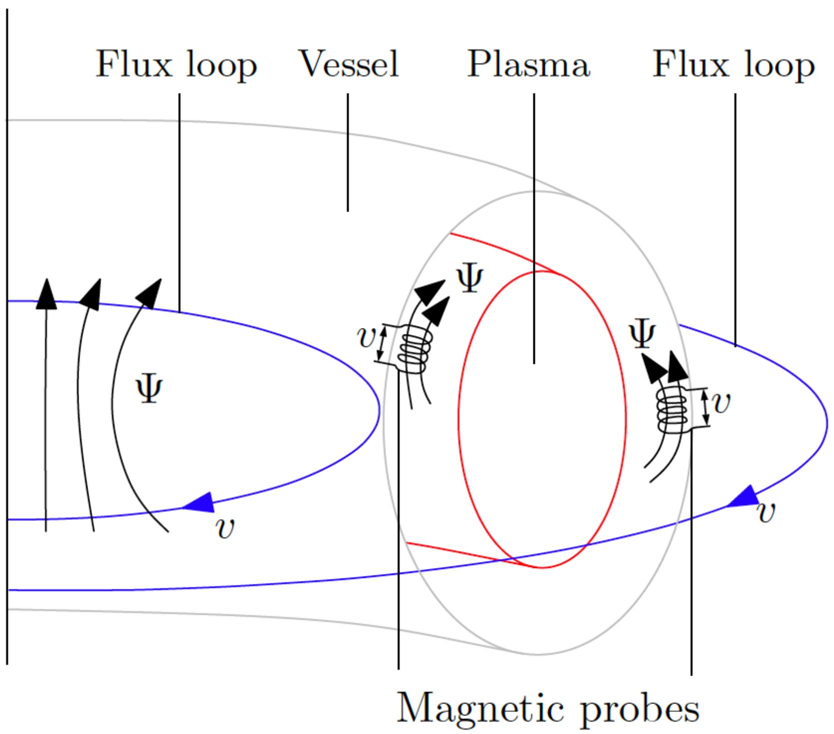
(If A_p is small enough, $\mathbf{B}$ is constant)

- **Measurements of the poloidal flux function at (R_{fl}, Z_{fl}) can be made by integrating the signal from flux loops**

$$V_{fl} = -\frac{\partial \psi}{\partial t}$$

$$\psi(t; R_{fl}, Z_{fl}) = -\int_0^t V_{fl}(t')dt'$$



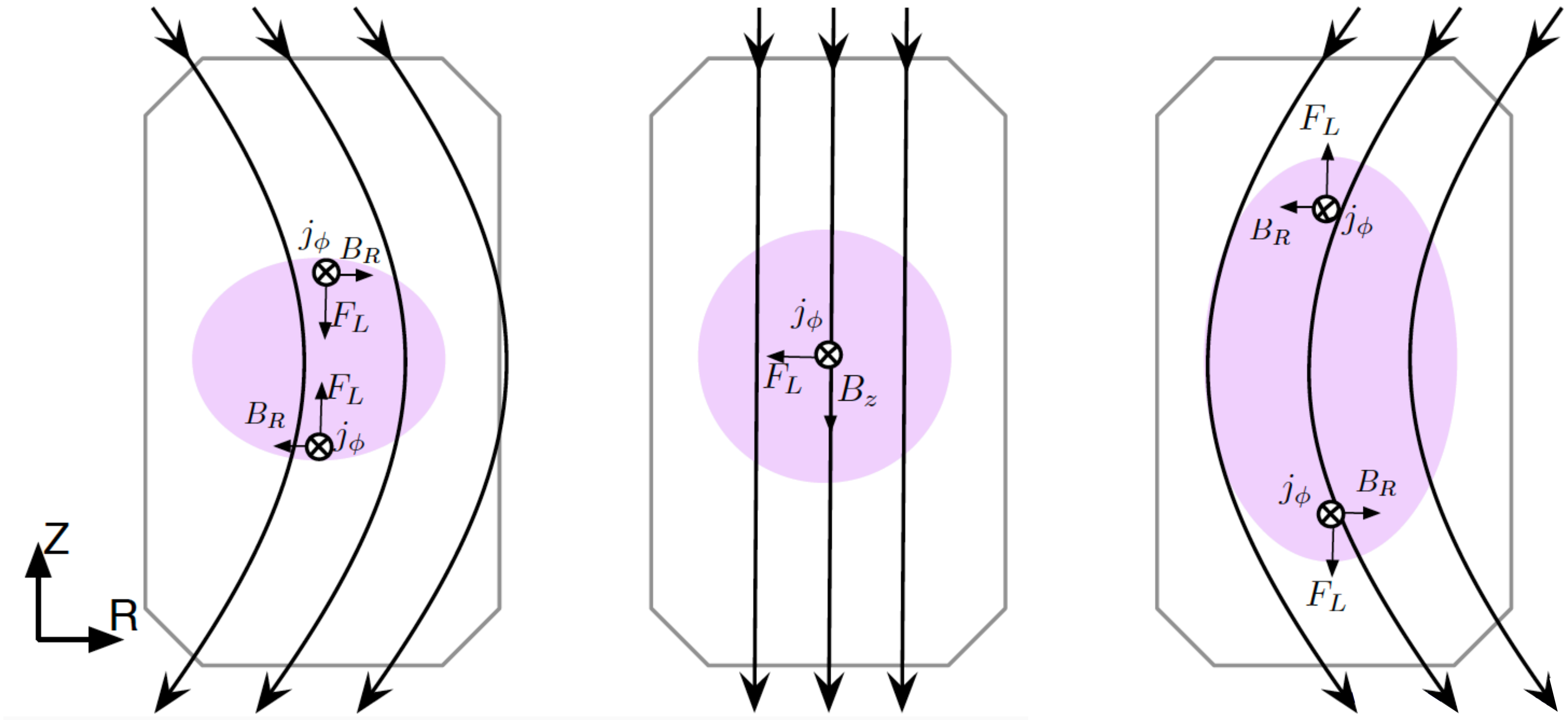


PGF5112 - Plasma Physics I

- **Tokamak engineering**
  - *Magnetic forces*
  - *Toroidal field coils*
  - *The central solenoid*
  - *Poloidal field coils*
  - *The vertical plasma instability and the RZIP model*

Elongated plasmas are naturally unstable

- **A robust control system is needed to control elongated plasma due to the upper and lower competing forces**



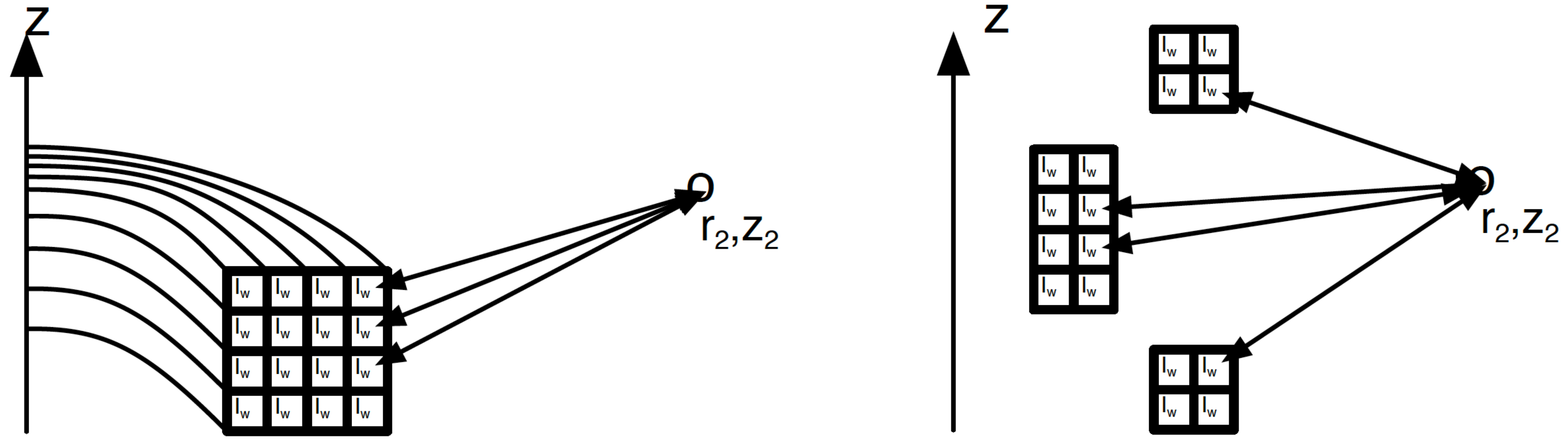
- **Modeling of the entire machine, including all the inductive coupling is required to control the plasma**

Modeling coils with multiple windings

- **Let's model a magnetic coils with N_w windings as**

$$\psi(R_2, Z_2) = \sum_{i=1}^{N_w} M(R_2, Z_2, R_i, Z_i)\, I_i = \sum_{i=1}^{N_w} M(R_2, Z_2, R_i, Z_i)\, I_w = M_c(R_2, Z_2)\, I_w$$

- **Filaments connected in series can be treated as a single coil**



The coupling of coils with multiple windings

- **The circuit equation for coil a (with current I_a and N_{wa} windings) with mutual coupling with coil b (with N_{wb})**

$$V_a = \sum_{i=1}^{N_{wa}} \left(R_i I_a + L_i \frac{dI_a}{dt} + M_{ib} \frac{dI_b}{dt} \right) = R_a I_a + L_a \frac{dI_a}{dt} + M_{ab} \frac{dI_b}{dt}$$

with

- $M_{ab} = \sum_{i=1}^{N_{wa}} M_{ib}$ *and* M_{ib} *is the mutual inductance between coil* b *and the ith filament of coil* a*, which is located at* (R_i, Z_i)*. Therefore,* $M_{ab} = \sum_{i=1}^{N_{wa}} \sum_{j=1}^{N_{wb}} M_{ij}$
- $L_a = \sum_{i=1}^{N_{wa}} L_i$ *and* L_i *is the self-inductance of the ith filament of coil* a
- $R_a = \sum_{i=1}^{N_{wa}} R_i$ *and* R_i *is the resistance of the ith filament of coil* a

The coupling of coils with multiple windings

- **The circuit equations of a set of magnetic coils can be combined as**

$$V_1 = \sum_{i=2}^{N_c} R_1 I_1 + L_1 \frac{dI_1}{dt} + M_{1i} \frac{dI_i}{dt} \qquad \ldots \qquad V_{N_c} = \sum_{i=1}^{N_c-1} R_{N_c} I_{N_c} + L_{N_c} \frac{dI_{N_c}}{dt} + M_{N_c i} \frac{dI_i}{dt}$$

or in matrix form as

$$\mathbf{V_a} = \mathbf{R_a I_a} + \mathbf{M_{aa} \dot{I}_a}$$

where $\mathbf{V_a} = [V_1 \quad V_2 \quad V_3 \ldots \quad V_{N_c}]^T$ **and** $\mathbf{I_a} = [I_1 \quad I_2 \quad I_3 \ldots \quad I_{N_c}]^T$

$$\mathbf{M_{aa}} = \begin{pmatrix} L_1 & M_{12} & \ldots & M_{1N_c} \\ M_{21} & L_2 & \ldots & M_{2N_c} \\ \vdots & \vdots & \ddots & \vdots \\ M_{N_c1} & M_{N_c2} & \ldots & L_{N_c} \end{pmatrix} \qquad \mathbf{R_a} = \begin{pmatrix} R_1 & 0 & \ldots & 0 \\ 0 & R_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & R_{Nc} \end{pmatrix}$$

  - *Here, the subindex a stands for* ***a****ctive coils with actively controlled voltage* V_a

Modeling the vacuum vessel

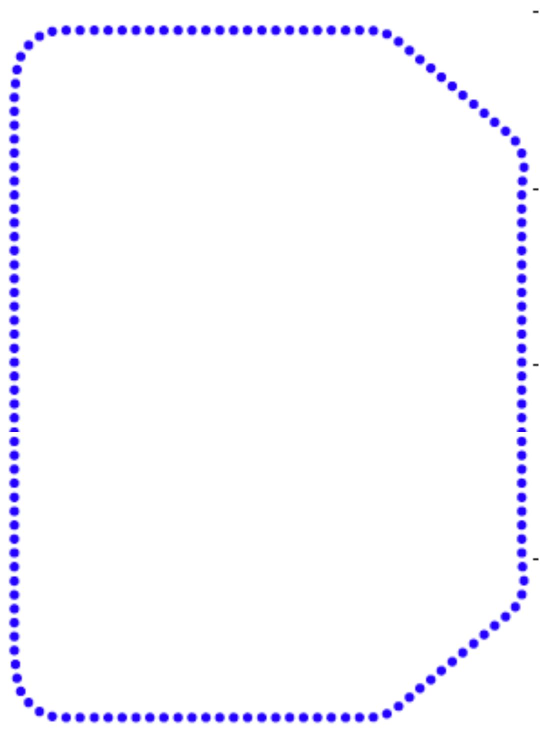
- **The vacuum vessel is usually a complex 3D structure due to access ports, etc.**
  - *To simplify the model, the vacuum vessel is assumed to be axisymmetric*
  - *The vacuum vessel walls are discretized into toroidal filaments*

- **Since the vacuum vessel filaments are short-circuited the total voltage around the torus must be zero. Therefore, the circuit equation can also be modeled as**

$$0 = \mathbf{R_v I_v} + \mathbf{M_{vv}} \dot{\mathbf{I}}_\mathbf{v} + \mathbf{M_{va}} \dot{\mathbf{I}}_\mathbf{a}$$

  - *Here,* $\mathbf{I_v} = [I_{v1} \quad I_{v2} \quad I_{v3} \ldots \quad I_{vN_v}]^T$
  - $\mathbf{R_v} \in \mathbb{R}^{N_v \times N_v}$ *contains the resistances of each filament*
  - $\mathbf{M_{vv}} \in \mathbb{R}^{N_v \times N_v}$ *contains the self-inductance of each filament and the mutual inductances between filaments*
  - $\mathbf{M_{va}} \in \mathbb{R}^{N_v \times N_a}$ *contains the mutual inductances between filaments and between filaments and active coils*

TCABR Vacuum Vessel



Combined model for vessel + active coils

- The circuit equation for the active coils is now also modified to include the voltage induced by changing vessel currents

$$\mathbf{V_a} = \mathbf{R_a I_a} + \mathbf{M_{aa}\dot{I}_a} + \mathbf{M_{av}\dot{I}_v}$$

- Combining these two sets of equations (active coils and vacuum vessel) yields

$$\begin{pmatrix} \mathbf{V_a} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{M_{aa}} & \mathbf{M_{av}} \\ \mathbf{M_{va}} & \mathbf{M_{vv}} \end{pmatrix} \begin{pmatrix} \mathbf{\dot{I}_a} \\ \mathbf{\dot{I}_v} \end{pmatrix} + \begin{pmatrix} \mathbf{R_a} & 0 \\ 0 & \mathbf{R_v} \end{pmatrix} \begin{pmatrix} \mathbf{I_a} \\ \mathbf{I_v} \end{pmatrix}$$

or more compactly, just as

$$\mathbf{V} = \mathbf{RI} + \mathbf{M\dot{I}}$$

Modeling the plasma as made by several filaments

- **To model the plasma, let's discretize the plasma current density into several current filaments**
  - *As we have done for the vacuum vessel, the total voltage around the torus must be zero (note that the loop voltage is driven by $\dot{\mathbf{I}}_{\mathbf{a}}$)*

$$0 = R_P I_P + L_P \frac{dI_P}{dt} + \mathbf{M}_{\mathbf{pa}} \dot{\mathbf{I}}_{\mathbf{a}} + \mathbf{M}_{\mathbf{pv}} \dot{\mathbf{I}}_{\mathbf{v}}$$

- **Combining this equation with the other two sets of equations yields a more complete model**

$$\begin{pmatrix} \mathbf{V_a} \\ \mathbf{0} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{M_{aa}} & \mathbf{M_{av}} & \mathbf{M_{ap}} \\ \mathbf{M_{va}} & \mathbf{M_{vv}} & \mathbf{M_{vp}} \\ \mathbf{M_{pa}} & \mathbf{M_{pv}} & L_p \end{pmatrix} \begin{pmatrix} \dot{\mathbf{I}}_{\mathbf{a}} \\ \dot{\mathbf{I}}_{\mathbf{v}} \\ \dot{I}_p \end{pmatrix} + \begin{pmatrix} \mathbf{R_a} & 0 & 0 \\ 0 & \mathbf{R_v} & 0 \\ 0 & 0 & R_p \end{pmatrix} \begin{pmatrix} \mathbf{I_a} \\ \mathbf{I_v} \\ I_p \end{pmatrix}$$

and, again, this system of equations can be written in a more compact form

$$\mathbf{V} = \mathbf{RI} + \mathbf{M}\dot{\mathbf{I}}$$

- **When the linearized momentum equations (radial and vertical components) of the plasma are inserted into these matrices, this model is called the RZIP model**
  - *The RZIP model is widely used for tuning plasma control systems*

References

- **Tokamak engineering**
  - *Magnetic forces and the central solenoid*
    - *Magnetic Fields: Ch. 6, section 2*
  - *Toroidal field coils*
    - *Fundamentals of Magnetic Thermonuclear Reactor Design: Ch. 12, section 2*
    - *Magnetic Fields: Ch. 7, section 3*
  - *Poloidal field coils, vertical plasma instability and the RZIP model*
    - *Magnetic modeling and control of tokamaks, by F. Felici, Eindhoven University of Technology*