PGF5112 - Plasma Physics I

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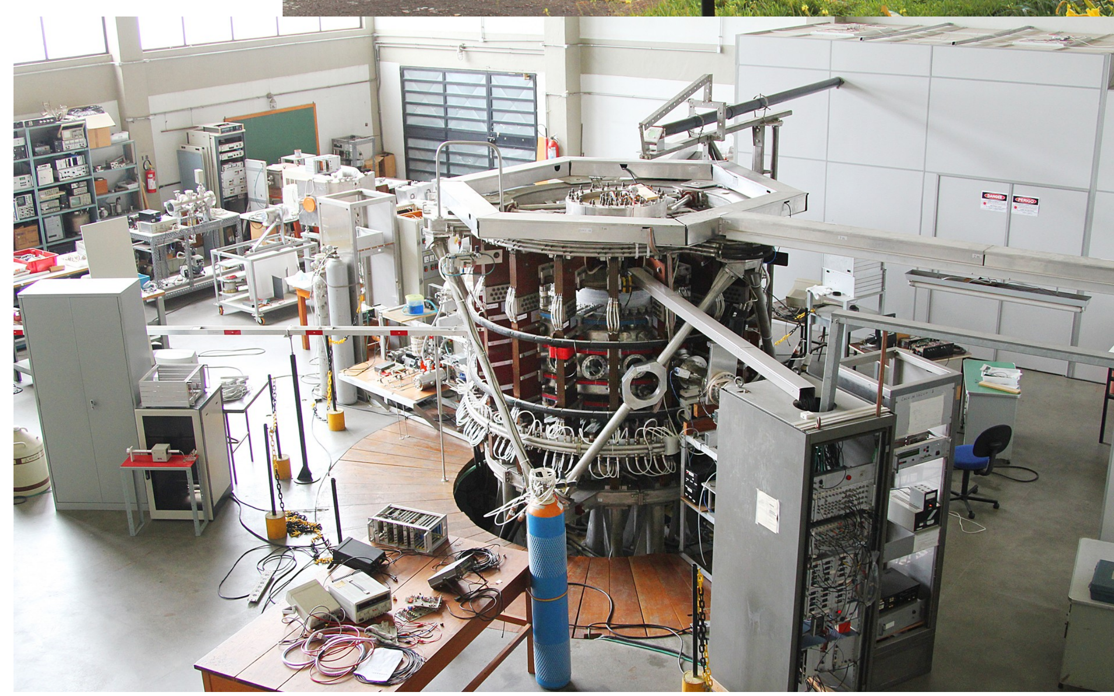

Postgraduate course ministered remotely from the

Institute of Physics of the University of São Paulo

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São Paulo - SP, 08 April 2021

SCIENTIA VINCES

- **Single particle orbits: the motion of charged particles in electromagnetic fields**
  - *Introduction (previous lecture)*
  - *Uniform and static electric field (previous lecture)*
  - *Uniform and static magnetic field (previous lecture)*
  - *Uniform and static electric and magnetic fields (previous lecture)*
  - *Non-uniform and static magnetic field (physical insight) - (previous lecture)*
  - *Non-uniform and static electric field (physical insight) - (previous lecture)*
  - *Non-uniform and time-dependent electric and magnetic fields*

- ***Particle orbits in a tokamak***
  - *Physical description of a tokamak*
  - *Trapped and passing particles*

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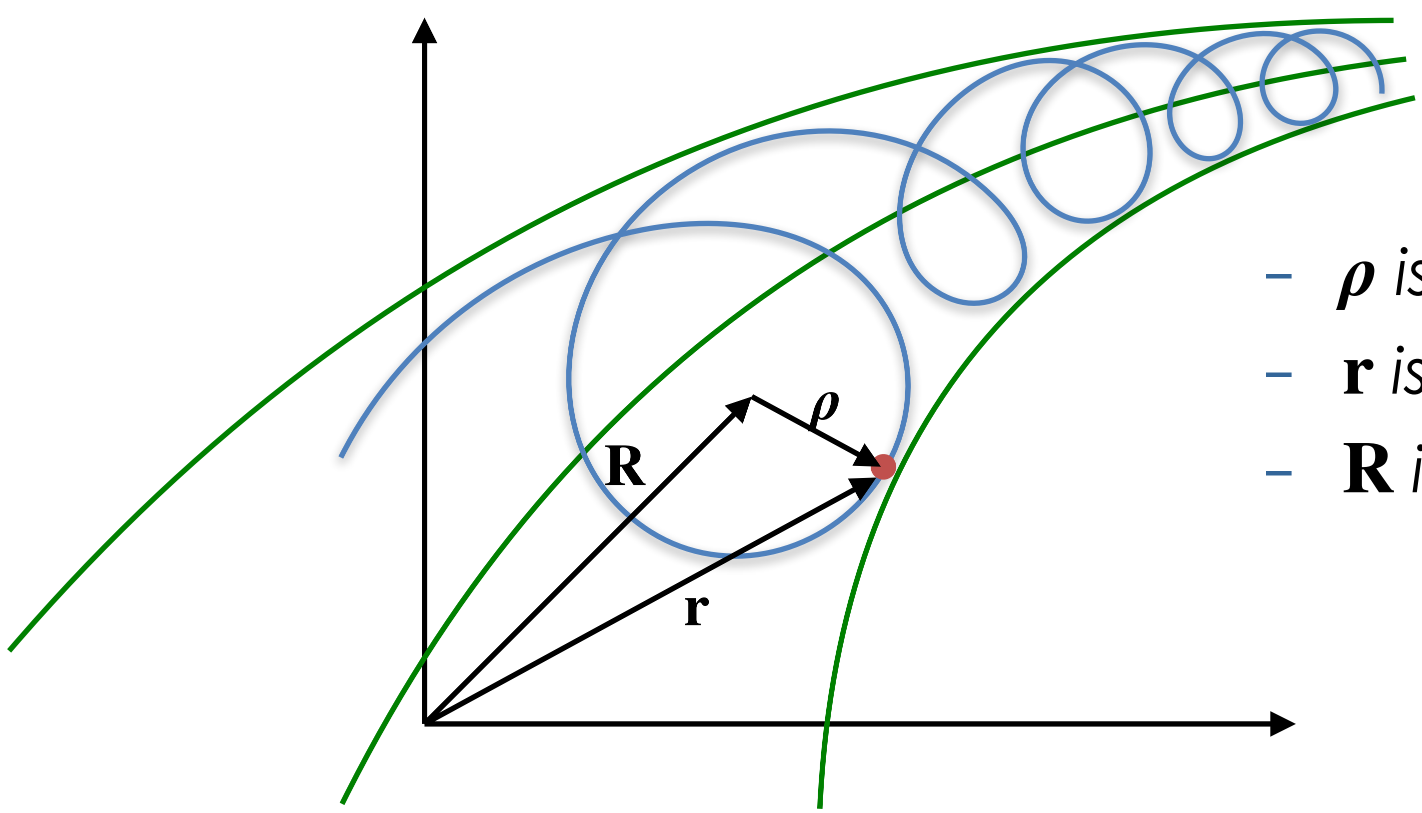
The trajectories of charged particles in non-uniform and time-dependent electric and magnetic fields

- To study the trajectory of charged particles in non-uniform and time-dependent electric and magnetic fields, let's expand the fields around a position $\mathbf{R}$, which is the guiding center position of the particle

$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}(\mathbf{R},t) + \left[(\mathbf{r}-\mathbf{R})\cdot\nabla\right]\mathbf{B}(\mathbf{r},t)\Big|_{\mathbf{r}=\mathbf{R}} + \frac{1}{2}\left[(\mathbf{r}-\mathbf{R})\cdot\nabla\right]^2\mathbf{B}(\mathbf{r},t)\Big|_{\mathbf{r}=\mathbf{R}} + O^3$$

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{R},t) + \left[(\mathbf{r}-\mathbf{R})\cdot\nabla\right]\mathbf{E}(\mathbf{r},t)\Big|_{\mathbf{r}=\mathbf{R}} + \frac{1}{2}\left[(\mathbf{r}-\mathbf{R})\cdot\nabla\right]^2\mathbf{E}(\mathbf{r},t)\Big|_{\mathbf{r}=\mathbf{R}} + O^3$$



- *$\boldsymbol{\rho}$ is the cyclotron/Larmor radius*
- *$\mathbf{r}$ is the instantaneous particle position*
- *$\mathbf{R}$ is the guiding center position*

$$\mathbf{r}(t) = \mathbf{R}(t) + \boldsymbol{\rho}(t)$$

The trajectories of charged particles in non-uniform and time-dependent electric and magnetic fields

- **To study the trajectory of charged particles in non-uniform and time-dependent electric and magnetic fields, let's expand the fields around a position $\mathbf{R}$, which is the guiding center position of the particle**

$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}(\mathbf{R},t) + \left[(\mathbf{r}-\mathbf{R})\cdot\nabla\right]\mathbf{B}(\mathbf{r},t)\Big|_{\mathbf{r}=\mathbf{R}} + \frac{1}{2}\left[(\mathbf{r}-\mathbf{R})\cdot\nabla\right]^2\mathbf{B}(\mathbf{r},t)\Big|_{\mathbf{r}=\mathbf{R}} + O^3$$

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{R},t) + \left[(\mathbf{r}-\mathbf{R})\cdot\nabla\right]\mathbf{E}(\mathbf{r},t)\Big|_{\mathbf{r}=\mathbf{R}} + \frac{1}{2}\left[(\mathbf{r}-\mathbf{R})\cdot\nabla\right]^2\mathbf{E}(\mathbf{r},t)\Big|_{\mathbf{r}=\mathbf{R}} + O^3$$

- **Using the definition of the instantaneous particle position: $\mathbf{r}(t) = \mathbf{R}(t) + \epsilon\boldsymbol{\rho}(t)$**
  - Here, ϵ is a parameter introduced to explicit the order of the expansion
  - Therefore, the fields become (in a simplified notation)

$$\mathbf{B}(\mathbf{r},t) = \mathbf{B_0} + \epsilon(\boldsymbol{\rho}\cdot\nabla)\mathbf{B_0} + \frac{\epsilon^2}{2}(\boldsymbol{\rho}\cdot\nabla)^2\mathbf{B_0}$$

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E_0} + \epsilon(\boldsymbol{\rho}\cdot\nabla)\mathbf{E_0} + \frac{\epsilon^2}{2}(\boldsymbol{\rho}\cdot\nabla)^2\mathbf{E_0}$$

  - Note that $\mathbf{E_0} = \mathbf{E}(\mathbf{R},t)$ and $\mathbf{B_0} = \mathbf{B}(\mathbf{R},t)$ still depend on time

Intermezzo matematico: the method of averaging

- **Consider the equation of motion**

$$\frac{d\mathbf{z}}{dt} = \mathbf{f}(\mathbf{z}, t, \tau)$$

  - Here, $\mathbf{f}$ is a periodic function of its last argument, with period 2π, and $\tau = \dfrac{t}{\epsilon}$
  - The small parameter ϵ characterizes the separation between the short oscillation period and the timescale for the slow secular evolution of $\mathbf{z}(t, \tau)$

- **The idea of the method of averaging is to treat t and τ as independent variables, and to look for solutions of the form $\mathbf{z}(t, \tau)$ that are periodic in τ. Thus, we replace the equation of motion above by the modified equation of motion below**

$$\frac{\partial\mathbf{z}}{\partial t} + \frac{1}{\epsilon}\frac{\partial\mathbf{z}}{\partial\tau} = \mathbf{f}(\mathbf{z}, t, \tau)$$

Intermezzo matematico: the method of averaging

- **Let's denote the τ-average of $\mathbf{z}(t,\tau)$ by $\mathbf{Z}(t)$, and seek a change of variables of the form**

$$\mathbf{z}(t,\tau) = \mathbf{Z}(t) + \epsilon \boldsymbol{\zeta}(\mathbf{Z},t,\tau)$$

  - Here, $\boldsymbol{\zeta}$ is a periodic function of τ with vanishing mean and $\mathbf{Z}(t)$ is a function free of oscillations

$$\langle \boldsymbol{\zeta}(\mathbf{Z},t,\tau)\rangle = \frac{1}{2\pi}\oint \boldsymbol{\zeta}(\mathbf{Z},t,\tau)d\tau = 0$$

- **Inserting the expression for $\mathbf{z}(t,\tau)$ into the motion equation (up to 2nd order) yields**

$$\frac{\partial}{\partial t}\left(\mathbf{Z}+\epsilon\boldsymbol{\zeta}\right) + \frac{1}{\epsilon}\frac{\partial}{\partial \tau}\left(\mathbf{Z}+\epsilon\boldsymbol{\zeta}\right) = \mathbf{f}(\mathbf{Z},t,\tau) + \epsilon(\boldsymbol{\zeta}\cdot\nabla)\mathbf{f}(\mathbf{Z},t,\tau) + \frac{\epsilon^2}{2}(\boldsymbol{\zeta}\cdot\nabla)^2\mathbf{f}(\mathbf{Z},t,\tau)$$

  - Since $\boldsymbol{\zeta}(\mathbf{Z},t,\tau)$ depends on time explicitly, but also through $\mathbf{Z}=\mathbf{Z}(t)$, then

$$\frac{d\mathbf{Z}}{dt} + \epsilon\left[\frac{\partial}{\partial t} + \left(\frac{d\mathbf{Z}}{dt}\cdot\nabla\right)\right]\boldsymbol{\zeta} + \frac{\partial \boldsymbol{\zeta}}{\partial \tau} = \mathbf{f}(\mathbf{Z},t,\tau) + \epsilon(\boldsymbol{\zeta}\cdot\nabla)\mathbf{f}(\mathbf{Z},t,\tau) + \frac{\epsilon^2}{2}(\boldsymbol{\zeta}\cdot\nabla)^2\mathbf{f}(\mathbf{Z},t,\tau)$$

Intermezzo matematico: the method of averaging

- **The evolution of $\mathbf{Z}(t)$ is determined by substituting the expansions below into the previous equation of motion:**

$$\zeta = \zeta_0(\mathbf{Z}, t, \tau) + \epsilon\zeta_1(\mathbf{Z}, t, \tau) + \epsilon^2\zeta_2(\mathbf{Z}, t, \tau) + \ldots$$

$$\mathbf{Z} = \mathbf{Z}_0(t) + \epsilon\mathbf{Z}_1(t) + \epsilon^2\mathbf{Z}_2(t) + \ldots$$

$$\frac{d\mathbf{Z}}{dt} = \mathbf{F}_0(\mathbf{Z}, t) + \epsilon\mathbf{F}_1(\mathbf{Z}, t) + \epsilon^2\mathbf{F}_2(\mathbf{Z}, t) + \ldots$$

  - The solution is then obtained by solving the motion equation order by order

- **To lowest order, we obtain** $\mathbf{F}_0(\mathbf{Z}, t) + \dfrac{\partial\zeta_0}{\partial\tau} = \mathbf{f}(\mathbf{Z}, t, \tau)$
  - Taking the τ-average of this equation yields

$$\mathbf{F}_0(\mathbf{Z}, t) = \langle \mathbf{f}(\mathbf{Z}, t, \tau) \rangle \equiv \langle \mathbf{f} \rangle(\mathbf{Z}, t)$$

  - Integrating the oscillating component of the lowest order equation yields

$$\zeta_0(\mathbf{Z}, t, \tau) = \int_0^\tau \left[\mathbf{f}(\mathbf{Z}, t, \tau') - \langle \mathbf{f} \rangle(\mathbf{Z}, t)\right] d\tau'$$

Intermezzo matematico: the method of averaging

- **To first order, we obtain** $\mathbf{F}_1 + \dfrac{\partial \zeta_0}{\partial t} + (\mathbf{F}_0 \cdot \nabla)\zeta_0 + \dfrac{\partial \zeta_1}{\partial \tau} = (\zeta_0 \cdot \nabla)\mathbf{f}(\mathbf{Z}, t, \tau)$
  - Taking the τ-average of this equation yields

    $$\mathbf{F}_1(\mathbf{Z}, t) = \langle [\zeta_0(\mathbf{Z}, t, \tau) \cdot \nabla]\mathbf{f}(\mathbf{Z}, t, \tau)\rangle \equiv \langle(\zeta_0 \cdot \nabla)\mathbf{f}\rangle(\mathbf{Z}, t)$$

  - Integrating the oscillating component of the first order equation yields

    $$\zeta_1(\mathbf{Z}, t, \tau) = \int_0^{\tau} \left[(\zeta_0 \cdot \nabla)\mathbf{f}(\mathbf{Z}, t, \tau') - \langle(\zeta_0 \cdot \nabla)\mathbf{f}\rangle(\mathbf{Z}, t) - \frac{\partial \zeta_0(\mathbf{Z}, t, \tau')}{\partial t} - (\mathbf{F}_0 \cdot \nabla)\zeta_0(\mathbf{Z}, t, \tau')\right] d\tau'$$

- **To second order, we obtain** $\mathbf{F}_2 + \dfrac{\partial \zeta_1}{\partial t} + (\mathbf{F}_0 \cdot \nabla)\zeta_1 + \dfrac{\partial \zeta_1}{\partial \tau} = (\zeta_1 \cdot \nabla)\mathbf{f} + \dfrac{1}{2}(\zeta_0 \cdot \nabla)^2\mathbf{f}$
  - Taking the τ-average of this equation yield

    $$\mathbf{F}_2(\mathbf{Z}, t) = \langle(\zeta_1 \cdot \nabla)\mathbf{f}\rangle(\mathbf{Z}, t) + \frac{1}{2}\langle(\zeta_0 \cdot \nabla)^2\mathbf{f}\rangle(\mathbf{Z}, t)$$

- **The evolution of $\mathbf{Z}(t)$ up to second order only is, therefore, given by**

  $$\frac{d\mathbf{Z}}{dt} = \mathbf{F}_0 + \mathbf{F}_1 + \mathbf{F}_2 = \langle\mathbf{f}\rangle(\mathbf{Z}, t) + \langle(\zeta_0 \cdot \nabla)\mathbf{f}\rangle(\mathbf{Z}, t) + \langle(\zeta_1 \cdot \nabla)\mathbf{f}\rangle(\mathbf{Z}, t) + \frac{1}{2}\langle(\zeta_0 \cdot \nabla)^2\mathbf{f}\rangle(\mathbf{Z}, t)$$

  - Note that, at the end, the parameter ϵ is set to unity

Guiding center motion

- **To use the method of averaging, the equations of motion are written in the form of first-order differential equations**

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}$$

$$m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

- **Let's denote the τ-average of $\mathbf{r}(t,\tau)$ by $\mathbf{R}(t)$ and the τ-average of $\mathbf{v}(t,\tau)$ by $\mathbf{U}(t)$, and seek a change of variables of the form**

$$\mathbf{r}(t,\tau) = \mathbf{R}(t) + \epsilon\boldsymbol{\rho}(\mathbf{R},\mathbf{U},t,\tau)$$

$$\mathbf{v}(t,\tau) = \mathbf{U}(t) + \mathbf{u}(\mathbf{R},\mathbf{U},t,\tau)$$

  - Here, $\langle\boldsymbol{\rho}(\mathbf{R},\mathbf{U},t,\tau)\rangle = 0$ and $\langle\mathbf{u}(\mathbf{R},\mathbf{U},t,\tau)\rangle = 0$
  - Note that $\boldsymbol{\rho} \ll \mathbf{R}$ (first order) while $\mathbf{u}$ can be of the same order than $\mathbf{U}$

Guiding center motion

- **Since we know that $u_\perp = \rho\Omega_c$, and we have made $\rho \to \epsilon\rho$, then $\Omega_c \to \epsilon^{-1}\Omega_c$ and, consequently, $B \to \epsilon^{-1}B$. In addition, since we also know that the magnitude of the ExB drift is $w_{ExB} = E/B$, we must also have $E \to \epsilon^{-1}E$**

- **Therefore, the modified equations of motion become ($(\mathbf{E}, \mathbf{B}, \mathbf{\Omega}_c) \to \epsilon^{-1}(\mathbf{E}, \mathbf{B}, \mathbf{\Omega}_c)$):**

$$\frac{\partial \mathbf{r}}{\partial t} + \frac{1}{\epsilon}\frac{\partial \mathbf{r}}{\partial \tau} = \mathbf{v}$$

$$m\frac{\partial \mathbf{v}}{\partial t} + \frac{m}{\epsilon}\frac{\partial \mathbf{v}}{\partial \tau} = \frac{q}{\epsilon}(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

- **In addition, here we consider the motion of a charged particle in the limit in which the EM fields experienced by the particle do not vary much in a gyroperiod, so that**

$$|(\boldsymbol{\rho} \cdot \nabla)\mathbf{E}| \ll |\mathbf{E}| \qquad \frac{1}{|\mathbf{E}|}\left|\frac{\partial \mathbf{E}}{\partial t}\right| \ll \frac{|\mathbf{\Omega_c}|}{2\pi} \qquad \frac{1}{|\mathbf{B}|}\left|\frac{\partial \mathbf{B}}{\partial t}\right| \ll \frac{|\mathbf{\Omega_c}|}{2\pi}$$

$$|(\boldsymbol{\rho} \cdot \nabla)\mathbf{B}| \ll |\mathbf{B}|$$

Guiding center motion

- The evolution of $\mathbf{R}(t)$ and $\mathbf{U}(t)$ are determined by substituting the expansions below into the modified equation of motion, and solve order by order:

$$\boldsymbol{\rho}(\mathbf{R},\mathbf{U},t,\tau) = \boldsymbol{\rho}_{\mathbf{0}}(\mathbf{R},\mathbf{U},t,\tau) + \epsilon\boldsymbol{\rho}_{\mathbf{1}}(\mathbf{R},\mathbf{U},t,\tau) + \epsilon^2\boldsymbol{\rho}_{\mathbf{2}}(\mathbf{R},\mathbf{U},t,\tau) + \ldots$$

$$\mathbf{R}(t) = \mathbf{R}_{\mathbf{0}}(t) + \epsilon\mathbf{R}_{\mathbf{1}}(t) + \epsilon^2\mathbf{R}_{\mathbf{2}}(t) + \ldots$$

$$\mathbf{u}(\mathbf{R},\mathbf{U},t,\tau) = \mathbf{u}_{\mathbf{0}}(\mathbf{R},\mathbf{U},t,\tau) + \epsilon\mathbf{u}_{\mathbf{1}}(\mathbf{R},\mathbf{U},t,\tau) + \epsilon^2\mathbf{u}_{\mathbf{2}}(\mathbf{R},\mathbf{U},t,\tau) + \ldots$$

$$\mathbf{U}(t) = \mathbf{U}_{\mathbf{0}}(t) + \epsilon\mathbf{U}_{\mathbf{1}}(t) + \epsilon^2\mathbf{U}_{\mathbf{2}}(t) + \ldots$$

- The dynamical equation for the gyrophase (γ) is likewise expanded

$$\frac{d\gamma}{dt} = \frac{1}{\epsilon}\left[\omega_0(\mathbf{R},\mathbf{U},t) + \epsilon\omega_1(\mathbf{R},\mathbf{U},t) + \epsilon^2\omega_2(\mathbf{R},\mathbf{U},t) + \ldots\right]$$

  - Here, again, $\Omega_c \to \epsilon^{-1}\Omega_c$

Guiding center motion

- Since the equation $\dfrac{d\mathbf{r}}{dt} = \mathbf{v}$ is linear, it follows that $\dfrac{d\mathbf{R}}{dt} = \mathbf{U}$ to all orders in ϵ, i.e.:

$$\frac{d\mathbf{R_0}}{dt} = \mathbf{U_0} \qquad \frac{d\mathbf{R_1}}{dt} = \mathbf{U_1} \qquad \frac{d\mathbf{R_2}}{dt} = \mathbf{U_2} \qquad \ldots$$

- The modified momentum equation, up to 2nd order, becomes

$$m\frac{\partial}{\partial t}(\mathbf{U}+\boldsymbol{u}) + \frac{m}{\epsilon}\frac{\partial}{\partial \tau}(\mathbf{U}+\boldsymbol{u}) = \frac{1}{\epsilon}\left[\mathbf{F_0}(\mathbf{R},\mathbf{U},t,\tau) + \epsilon\mathbf{F_1}(\mathbf{R},\mathbf{U},t,\tau) + \epsilon^2\mathbf{F_2}(\mathbf{R},\mathbf{U},t,\tau)\right]$$

  - Note that $\mathbf{u}(\mathbf{R},\mathbf{U},t,\tau)$ depends on t explicitly, but also through $\mathbf{R}(t)$ and $\mathbf{U}(t)$

- Therefore

$$m\frac{d\mathbf{U}}{dt} + m\left[\frac{\partial}{\partial t} + \left(\frac{d\mathbf{R}}{dt}\cdot\nabla\right) + \left(\frac{d\mathbf{U}}{dt}\cdot\nabla_U\right)\right]\boldsymbol{u} + \frac{1}{\epsilon}\frac{\partial \boldsymbol{u}}{\partial \tau} =$$

$$= \frac{\mathbf{F_0}(\mathbf{R},\mathbf{U},t,\tau)}{\epsilon} + \mathbf{F_1}(\mathbf{R},\mathbf{U},t,\tau) + \epsilon\mathbf{F_2}(\mathbf{R},\mathbf{U},t,\tau)$$

Guiding center motion

- **Substitution of the expansions into the modified equation of motion yields**

$$m\frac{d\mathbf{U_0}}{dt}+\epsilon m\frac{d\mathbf{U_1}}{dt}+\epsilon^2 m\frac{d\mathbf{U_2}}{dt}+m\frac{\partial \boldsymbol{u_0}}{\partial t}+m\left(\frac{d\mathbf{R}}{dt}\cdot\nabla\right)\boldsymbol{u_0}+m\left(\frac{d\mathbf{U}}{dt}\cdot\nabla_U\right)\boldsymbol{u_0}+$$

$$+\epsilon^2 m\left(\frac{d\mathbf{R}}{dt}\cdot\nabla\right)\boldsymbol{u_2}+\epsilon m\frac{\partial \boldsymbol{u_1}}{\partial t}+\epsilon m\left(\frac{d\mathbf{R}}{dt}\cdot\nabla\right)\boldsymbol{u_1}+\epsilon m\left(\frac{d\mathbf{U}}{dt}\cdot\nabla_U\right)\boldsymbol{u_1}+\epsilon^2 m\frac{\partial \boldsymbol{u_2}}{\partial t}+$$

$$+\epsilon^2 m\left(\frac{d\mathbf{U}}{dt}\cdot\nabla_U\right)\boldsymbol{u_2}+\frac{m}{\epsilon}\left(\frac{\partial \boldsymbol{u_0}}{\partial \tau}+\epsilon\frac{\partial \boldsymbol{u_1}}{\partial \tau}+\epsilon^2\frac{\partial \boldsymbol{u_0}}{\partial \tau}\right)=$$

$$=\frac{q}{\epsilon}\left(\mathbf{E_0}+\mathbf{U_0}\times\mathbf{B_0}+\mathbf{u_0}\times\mathbf{B_0}\right)+$$

$$+q\left(\mathbf{E_1}+\mathbf{U_1}\times\mathbf{B_0}+\mathbf{U_0}\times\mathbf{B_1}+\mathbf{u_1}\times\mathbf{B_0}+\mathbf{u_0}\times\mathbf{B_1}\right)+$$

$$+\epsilon q\left(\mathbf{E_2}+\mathbf{U_2}\times\mathbf{B_0}+\mathbf{u_2}\times\mathbf{B_0}+\mathbf{U_1}\times\mathbf{B_1}+\mathbf{u_1}\times\mathbf{B_1}+\mathbf{U_0}\times\mathbf{B_2}+\mathbf{u_0}\times\mathbf{B_2}\right)$$

Guiding center motion: 0th order terms

- **To lowest order ($O(\epsilon^{-1})$), the momentum equation is**

$$\frac{\partial \mathbf{u_0}}{\partial \tau} = \frac{q}{m}\left(\mathbf{E_0} + \mathbf{U_0} \times \mathbf{B_0} + \mathbf{u_0} \times \mathbf{B_0}\right) \quad \rightarrow \quad \frac{\partial \mathbf{u_0}}{\partial \tau} + \mathbf{u_0} \times \mathbf{\Omega_{c0}} = \frac{q}{m}\left(\mathbf{E_0} + \mathbf{U_0} \times \mathbf{B_0}\right)$$

  - Here, one has defined $\mathbf{\Omega_{c0}}(\mathbf{R}, t) = -q\mathbf{B_0}/m$

- **Taking the τ-average of this equation yields:**

$$\left\langle \frac{\partial \mathbf{u_0}}{\partial \tau} \right\rangle + \langle \mathbf{u_0} \rangle \times \mathbf{\Omega_{c0}} = \frac{q}{m}\left(\mathbf{E_0} + \mathbf{U_0} \times \mathbf{B_0}\right) \quad \rightarrow \quad \mathbf{E_0} + \mathbf{U_0} \times \mathbf{B_0} = 0$$

  - The most general solution to this 0th order equation is $\mathbf{U_0} = U_{0,\parallel}\hat{\mathbf{b}} + \mathbf{w_{ExB}}$, where is the so-called ExB drift:

$$\mathbf{w_{ExB}} = \frac{\mathbf{E_0} \times \mathbf{B_0}}{B_0^2}$$

  - Here, $\hat{\mathbf{b}} = \mathbf{B_0}/B_0$ is a unit vector pointing along $\mathbf{B_0}$
  - Note that the equation $\mathbf{E_0} + \mathbf{U_0} \times \mathbf{B_0} = 0$ is satisfied only if $E_{0,\parallel} = \epsilon|\mathbf{E_0}|$, i.e. the parallel component of the 0th order electric field must be included in $\mathbf{E_1}$

Guiding center motion: 0th order terms

- **Using the equation for the gyrophase, the momentum equation can be written as**

$$\frac{\partial \mathbf{u_0}}{\partial \tau} + \mathbf{u_0} \times \mathbf{\Omega_{c0}} = \frac{d\gamma}{d\tau}\frac{\partial \mathbf{u_0}}{\partial \gamma} + \mathbf{u_0} \times \mathbf{\Omega_{c0}} = \epsilon \frac{d\gamma}{dt}\frac{\partial \mathbf{u_0}}{\partial \gamma} + \mathbf{u_0} \times \mathbf{\Omega_{c0}} = \frac{q}{m}\left(\mathbf{E_0} + \mathbf{U_0} \times \mathbf{B_0}\right) = 0$$

$$\epsilon \frac{d\gamma}{dt}\frac{\partial \mathbf{u_0}}{\partial \gamma} + \mathbf{u_0} \times \mathbf{\Omega_{c0}} = \left(\omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \ldots\right)\frac{\partial \mathbf{u_0}}{\partial \gamma} + \mathbf{u_0} \times \mathbf{\Omega_{c0}} = 0$$

$$\omega_0 \frac{\partial \mathbf{u_0}}{\partial \gamma} + \mathbf{u_0} \times \mathbf{\Omega_{c0}} = 0$$

  - Here, we used the solution of the τ-averaged 0th order equation of motion

- **Integration of the momentum equation, with $\Omega_{c0} = -qB_0/m$, yields**

$$\mathbf{u_0} = \mathbf{c} + \mathrm{u}_{0,\perp}\left[-\hat{\mathbf{e}}_\mathbf{1}\sin\left(\frac{\Omega_{c0}}{\omega_0}\gamma\right) + \hat{\mathbf{e}}_\mathbf{2}\cos\left(\frac{\Omega_{c0}}{\omega_0}\gamma\right)\right]$$

  - Here, $\hat{\mathbf{e}}_\mathbf{1}$ and $\hat{\mathbf{e}}_\mathbf{2}$ are unit vectors such that $\hat{\mathbf{e}}_\mathbf{1} \times \hat{\mathbf{e}}_\mathbf{2} = \hat{\mathbf{b}}$ and $\mathbf{c}$ is a constant

Guiding center motion: 0th order terms

- **Periodicity constraint requires that $\mathbf{c} = 0$ and** $\omega_0 = \Omega_{c0}(\mathbf{R}, t) = -\,qB_0(\mathbf{R}, t)/m$
  - Therefore, the gyration velocity becomes

$$\mathbf{u_0} = \mathrm{u}_{0,\perp}\left[-\hat{\mathbf{e}}_\mathbf{1}\sin\gamma + \hat{\mathbf{e}}_\mathbf{2}\cos\gamma\right] \text{ with } \gamma = \gamma_0 + \Omega_{c0}t$$

- **Keeping only 0th order terms in the velocity equation** $\dfrac{\partial \mathbf{r}}{\partial t} + \dfrac{1}{\epsilon}\dfrac{\partial \mathbf{r}}{\partial \tau} = \mathbf{v}$ **, and using that** $d\mathbf{R_0}/dt = \mathbf{U_0}$ **, yields** $d\boldsymbol{\rho}_\mathbf{0}/d\tau = \mathbf{u_0}$ **, which can be written as**

$$\Omega_{c0}\frac{\partial \boldsymbol{\rho}_\mathbf{0}}{\partial \gamma} = \mathbf{u_0}$$

- **Integration of this equation yields** $\boldsymbol{\rho}_\mathbf{0} = \rho_0\left[\hat{\mathbf{e}}_\mathbf{1}\cos\gamma + \hat{\mathbf{e}}_\mathbf{2}\sin\gamma\right]$ **with** $\rho_0 = u_{0,\perp}/\Omega_{c0}$
  - Sometimes, it is convenient to write $\boldsymbol{\rho}_\mathbf{0} = \mathbf{u_0} \times \hat{\mathbf{b}}/\Omega_{c0}$ or $\boldsymbol{u}_\mathbf{0} = \boldsymbol{\Omega}_\mathbf{c0} \times \boldsymbol{\rho}_\mathbf{0}$

Guiding center motion: 1th order terms

- To first order ($O(\epsilon^0)$), the modified momentum equation is

$$m\frac{d\mathbf{U_0}}{dt} + m\frac{\partial \mathbf{u_0}}{\partial t} + m\left(\frac{d\mathbf{R_0}}{dt}\cdot\nabla\right)\boldsymbol{u_0} + m\left(\frac{d\mathbf{U_0}}{dt}\cdot\nabla_{U_0}\right)\boldsymbol{u_0} + m\frac{\partial \mathbf{u_1}}{\partial \tau} =$$

$$= q\left(\mathbf{E_1} + \mathbf{U_1}\times\mathbf{B_0} + \mathbf{U_0}\times\mathbf{B_1} + \mathbf{u_1}\times\mathbf{B_0} + \mathbf{u_0}\times\mathbf{B_1}\right)$$

- Taking the τ-average of this equation yields

$$\frac{d\mathbf{U_0}}{dt} = \frac{q}{m}\left(\langle\mathbf{E_1}\rangle + \mathbf{U_1}\times\mathbf{B_0} + \mathbf{U_0}\times\langle\mathbf{B_1}\rangle + \langle\mathbf{u_0}\times\mathbf{B_1}\rangle\right)$$

  - Let's calculate the τ-average of each term separately

$$\langle\mathbf{E_1}\rangle = \langle\mathbf{E_{1,\parallel}} + (\boldsymbol{\rho_0}\cdot\nabla)\mathbf{E_0}\rangle = \langle\mathbf{E_{1,\parallel}}\rangle + (\langle\boldsymbol{\rho_0}\rangle\cdot\nabla)\mathbf{E_0} = E_{1,\parallel}\hat{\mathbf{b}}$$

$$\langle\mathbf{B_1}\rangle = \langle(\boldsymbol{\rho_0}\cdot\nabla)\mathbf{B_0}\rangle = (\langle\boldsymbol{\rho_0}\rangle\cdot\nabla)\mathbf{B_0} = 0$$

$$\langle\mathbf{u_0}\times\mathbf{B_1}\rangle = \left\langle\mathbf{u_0}\times\left[(\boldsymbol{\rho_0}\cdot\nabla)\mathbf{B_0}\right]\right\rangle = \left\langle\left(\boldsymbol{\Omega_{c0}}\times\boldsymbol{\rho_0}\right)\times\left[(\boldsymbol{\rho_0}\cdot\nabla)\mathbf{B_0}\right]\right\rangle$$

Guiding center motion: 1th order terms

- **Substitution into the τ-averaged 1th order momentum equation yields**

$$\frac{d}{dt}\left(\mathrm{U}_{0_{\|}}\hat{\mathbf{b}}+\mathbf{w}_{\mathbf{ExB}}\right)=\frac{q}{m}\left\{\mathbf{E}_{1,\|}+\mathbf{U}_{1}\times\mathbf{B}_{0}+\left\langle\left(\boldsymbol{\Omega}_{c0}\times\boldsymbol{\rho}_{0}\right)\times\left[\left(\boldsymbol{\rho}_{0}\cdot\nabla\right)\mathbf{B}_{0}\right]\right\rangle\right\}$$

  - The last term on the RHS can be written as

$$\left(\boldsymbol{\Omega}_{c0}\times\boldsymbol{\rho}_{0}\right)\times\left[\left(\boldsymbol{\rho}_{0}\cdot\nabla\right)\mathbf{B}_{0}\right]=\left\{\boldsymbol{\Omega}_{c0}\cdot\left[\left(\boldsymbol{\rho}_{0}\cdot\nabla\right)\mathbf{B}_{0}\right]\right\}\boldsymbol{\rho}_{0}-\left\{\boldsymbol{\rho}_{0}\cdot\left[\left(\boldsymbol{\rho}_{0}\cdot\nabla\right)\mathbf{B}_{0}\right]\right\}\boldsymbol{\Omega}_{c0}$$

- **Exercise: using the Einstein notation, show that**

$$\left\{\boldsymbol{\Omega}_{c0}\cdot\left[\left(\boldsymbol{\rho}_{0}\cdot\nabla\right)\mathbf{B}_{0}\right]\right\}\boldsymbol{\rho}_{0}=\Omega_{c0}\left[\left(\boldsymbol{\rho}_{0}\boldsymbol{\rho}_{0}\right)\cdot\nabla B_{0}\right]$$

$$\left\{\boldsymbol{\rho}_{0}\cdot\left[\left(\boldsymbol{\rho}_{0}\cdot\nabla\right)\mathbf{B}_{0}\right]\right\}\boldsymbol{\Omega}_{c0}=\left[\left(\boldsymbol{\rho}_{0}\boldsymbol{\rho}_{0}\right):\nabla\mathbf{B}_{0}\right]\boldsymbol{\Omega}_{c0}$$

$$\left\langle\boldsymbol{\rho}_{0}\boldsymbol{\rho}_{0}\right\rangle=\frac{\rho_{0}^{2}}{2}\left(\mathbf{I}-\hat{\mathbf{b}}\hat{\mathbf{b}}\right)$$

Guiding center motion: 1th order terms

- **Using the results from the previous exercise, we have**

$$\left\langle (\boldsymbol{\Omega}_{c0} \times \boldsymbol{\rho}_0) \times \left[(\boldsymbol{\rho}_0 \cdot \nabla)\, \mathbf{B}_0 \right] \right\rangle = \Omega_{c0} \left[\frac{\rho_0^2}{2} \left(\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}} \right) \cdot \nabla B_0 \right] - \left[\frac{\rho_0^2}{2} \left(\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}} \right) : \nabla \mathbf{B}_0 \right] \boldsymbol{\Omega}_{c0}$$

$$\left\langle (\boldsymbol{\Omega}_{c0} \times \boldsymbol{\rho}_0) \times \left[(\boldsymbol{\rho}_0 \cdot \nabla)\, \mathbf{B}_0 \right] \right\rangle = -\frac{m u_{0,\perp}^2}{2 q B_0} \nabla B_0 = = -\frac{\mu}{q} \nabla B_0$$

  - Here, $\mathbf{I}$ is the identity tensor, and we used that $\mathbf{I} : \nabla \mathbf{B}_0 = \nabla \cdot \mathbf{B}_0 = 0$, that $\hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla B_0 = \hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla \mathbf{B}_0$ and that $\mu = m u_{0,\perp}^2 / 2B_0$ is the magnitude of the magnetic moment associated to the gyromotion

- **Therefore, the τ-averaged 1th order momentum equation becomes**

$$m \frac{d}{dt} \left(\mathrm{U}_{0,\parallel} \hat{\mathbf{b}} + \mathbf{w}_{\mathbf{ExB}} \right) = q \mathbf{E}_{\mathbf{1},\parallel} + q \mathbf{U}_{\mathbf{1}} \times \mathbf{B}_{\mathbf{0}} - \mu \nabla B_0$$

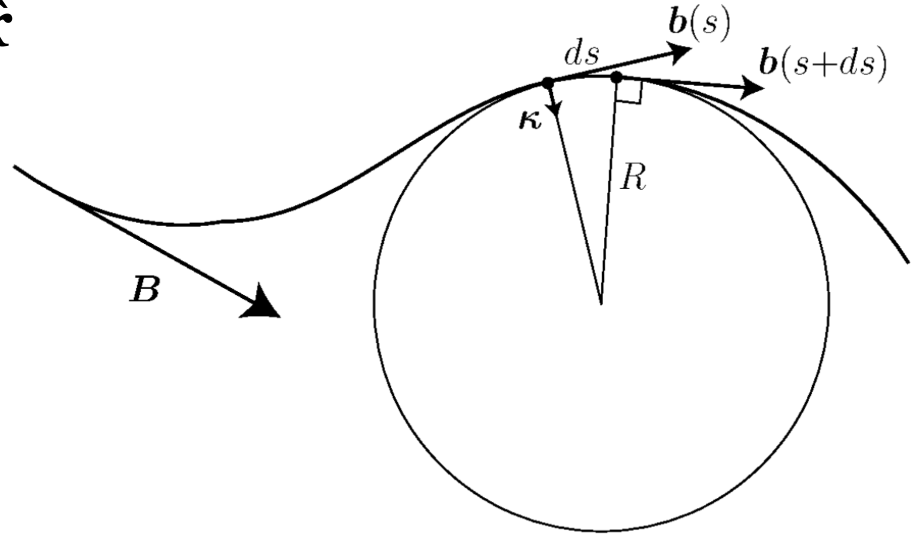
Guiding center motion: 1th order terms

- **Let's now separate the momentum equation in its parallel and perpendicular components**
  - Parallel component

$$m\frac{d\mathrm{U}_{0,\|}}{dt} + m\mathrm{U}_{0,\|}\hat{\mathbf{b}}\cdot\frac{d\hat{\mathbf{b}}}{dt} + m\hat{\mathbf{b}}\cdot\frac{d\mathbf{w}_{\mathbf{ExB}}}{dt} = qE_{1,\|} - \mu\nabla_{\|}B_0$$

$$\frac{d\hat{\mathbf{b}}}{dt} = \frac{\partial\hat{\mathbf{b}}}{\partial t} + (\mathbf{U_0}\cdot\nabla)\hat{\mathbf{b}} = \frac{\partial\hat{\mathbf{b}}}{\partial t} + (\mathbf{w}_{\mathbf{ExB}}\cdot\nabla)\hat{\mathbf{b}} + \mathrm{U}_{0,\|}\hat{\boldsymbol{\kappa}}$$

The quantity $\hat{\boldsymbol{\kappa}} = (\hat{\mathbf{b}}\cdot\nabla)\hat{\mathbf{b}}$ is termed the curvature vector and it points towards the center of the circle that most closely approximates the magnetic field line at a particular point



- **Exercise: show that** $\hat{\mathbf{b}}\cdot\dfrac{d\hat{\mathbf{b}}}{dt} = 0$

- **Therefore, the momentum equation becomes**

$$m\frac{d\mathrm{U}_{0,\|}}{dt} + m\hat{\mathbf{b}}\cdot\frac{d\mathbf{w}_{\mathbf{ExB}}}{dt} = qE_{1,\|} - \mu\nabla_{\|}B_0$$

Guiding center motion: 1th order terms

- **The parallel and perpendicular components of the momentum equation are**
  - Parallel component

$$m\frac{dU_{0,\parallel}}{dt} = qE_{1,\parallel} - \mu\nabla_{\parallel}B_0 - m\hat{\mathbf{b}}\cdot\frac{d\mathbf{w}_{\mathbf{ExB}}}{dt}$$

  - Perpendicular component

$$\mathbf{U}_{\mathbf{1},\perp} = \mathbf{B_0}\times\left[\frac{m}{qB_0^2}\frac{d\mathbf{U_0}}{dt} + \frac{\mu}{qB_0^2}\nabla B_0\right]$$

- **Comments**
  - The 0th order parallel drift ($U_{0,\parallel}$) is determined at 1th order
  - The 1th order correction to the parallel drift is underdetermined at this order, which implies that $U_{1,\parallel} = \epsilon^2|\mathbf{U_1}|$ and, at this order, we have $\mathbf{U_1} = \mathbf{U}_{\mathbf{1},\perp}$

Guiding center motion: 1th order terms

- Making use of the τ-averaged 1th order ($O(\epsilon^0)$) momentum equation allow us to write the oscillating component of the first order modified momentum equation

$$\frac{\partial \mathbf{u_0}}{\partial t} + \left(\frac{d\mathbf{R_0}}{dt} \cdot \nabla\right) \boldsymbol{u_0} + \left(\frac{d\mathbf{U_0}}{dt} \cdot \nabla_{U_0}\right) \boldsymbol{u_0} + \frac{\partial \mathbf{u_1}}{\partial \tau} = \frac{q}{m}\left(\mathbf{U_0} \times \mathbf{B_1} + \mathbf{u_1} \times \mathbf{B_0} + \mathbf{u_0} \times \mathbf{B_1}\right)$$

- This equation must be integrated in order to find $\mathbf{u_1} = \mathbf{u_1}(\mathbf{R}, \mathbf{U}, t, \tau)$

- Then, keeping only 1th order terms in the velocity equation $\frac{\partial \mathbf{r}}{\partial t} + \frac{1}{\epsilon}\frac{\partial \mathbf{r}}{\partial \tau} = \mathbf{v}$, and using that $d\mathbf{R_1}/dt = \mathbf{U_1}$, yields

$$\omega_1 \frac{\partial \boldsymbol{\rho_1}}{\partial \gamma} = \mathbf{u_1} - \frac{\partial \boldsymbol{\rho_0}}{\partial t} - \left(\frac{d\mathbf{R_0}}{dt} \cdot \nabla\right) \boldsymbol{\rho_0} + \left(\frac{d\mathbf{U_0}}{dt} \cdot \nabla_{U_0}\right) \boldsymbol{\rho_0}$$

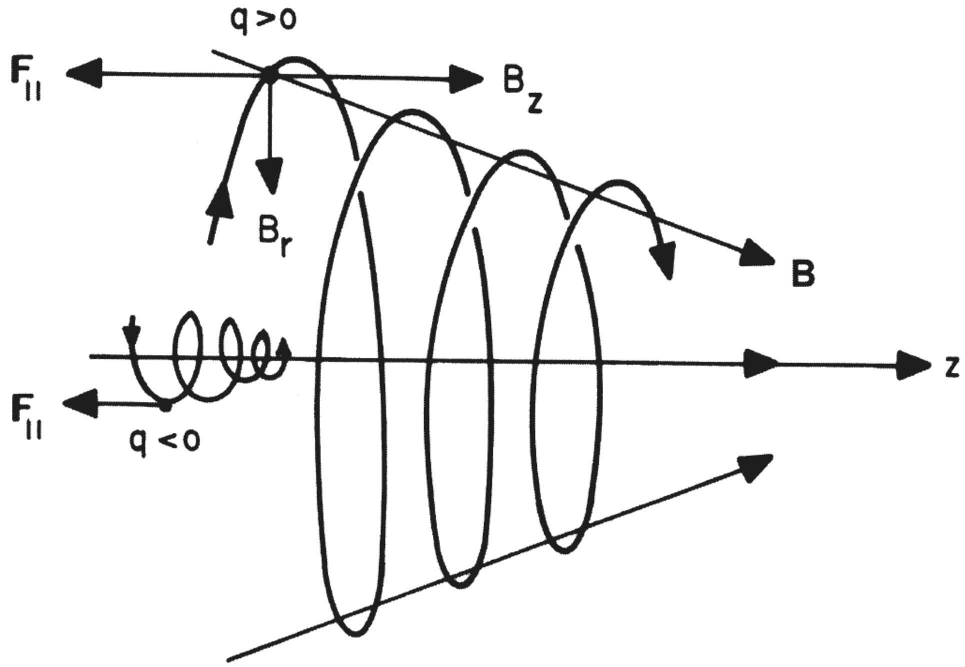
- This equation must then be integrated for $\boldsymbol{\rho_1} = \boldsymbol{\rho_1}(\mathbf{R}, \mathbf{U}, t, \tau)$ to be found. During this integration, the first order correction to the Larmor frequency (ω_1) is also found

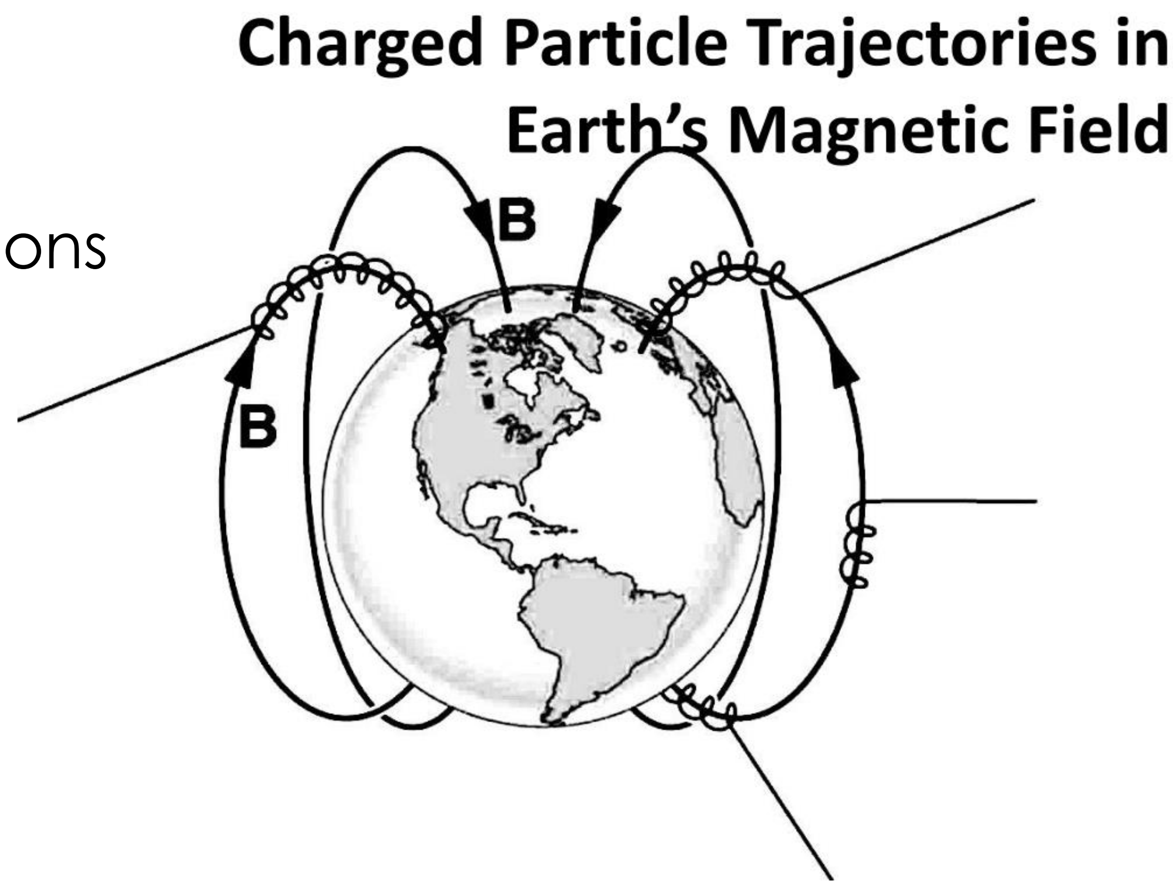
Guiding center motion: 1th order terms

- **In the absence of an $\mathbf{E}_0$-field, and for a static $\mathbf{B}_0$-field, the parallel drift velocity reduces to**

$$m\frac{d\mathrm{U}_{0,\|}}{dt} = -\,\mu\nabla_{\|}B_0$$

  - Particles tend to move away from regions with stronger $\mathbf{B}_0$-field





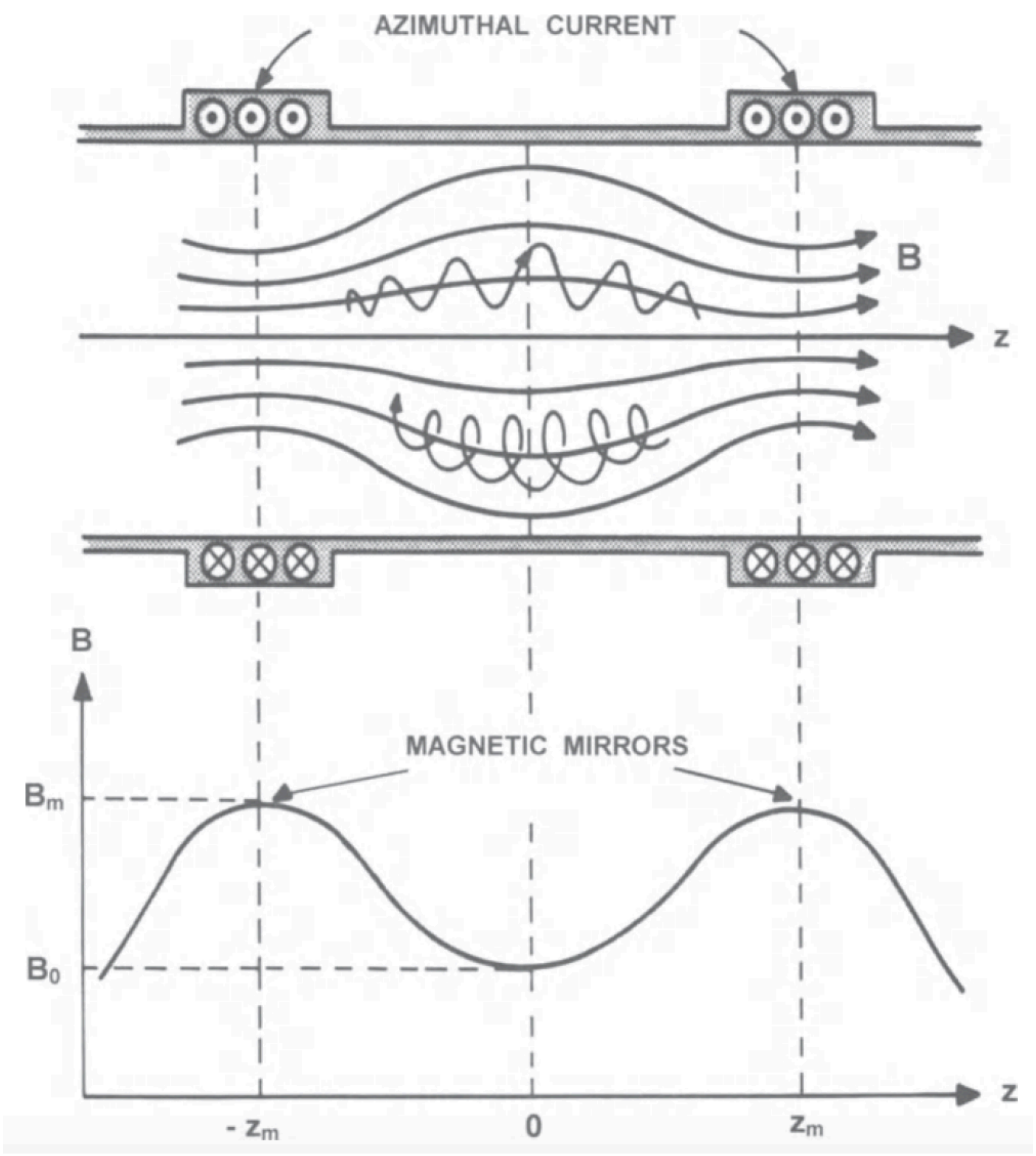
Guiding center motion: 1th order terms

- **In the absence of an $\mathbf{E}_0$-field, and for a static $\mathbf{B}_0$-field, the parallel drift velocity reduces to**

$$m\frac{d\mathrm{U}_{0,\parallel}}{dt} = -\mu\nabla_{\parallel}B_0$$

  - Particles tend to move away from regions with stronger $\mathbf{B}_0$-field
  - First magnetic confinement devices used this effect to trap particles in localized regions of space (magnetic bottles)

Mirror Machines

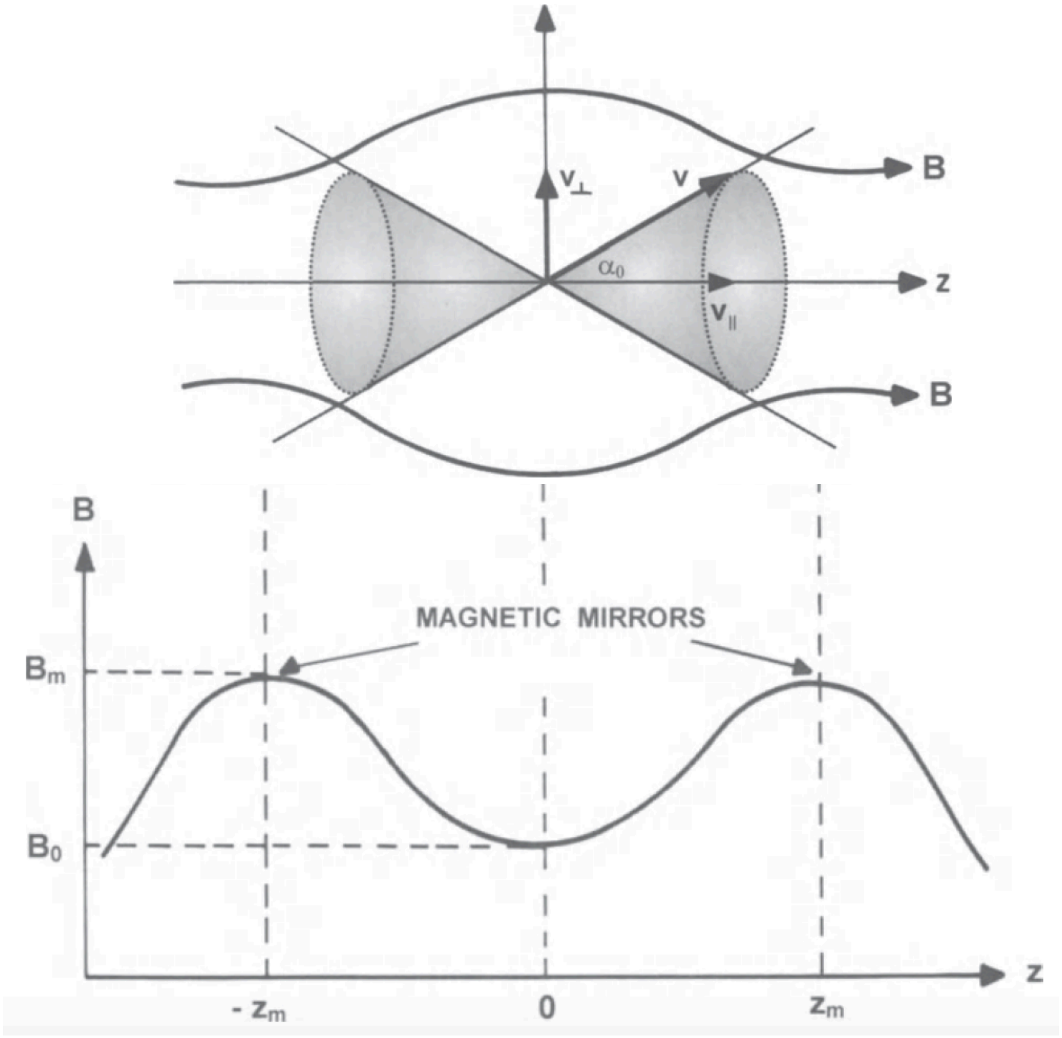


Guiding center motion: 1th order terms

- **In the absence of an $\mathbf{E}_0$-field, and for a static $\mathbf{B}_0$-field, the parallel drift velocity reduces to**

$$m\frac{d\mathrm{U}_{0,\parallel}}{dt} = -\mu\nabla_{\parallel}B_0$$

  - Particles tend to move away from regions with stronger $\mathbf{B}_0$-field
  - First magnetic confinement devices used this effect to trap particles in localized regions of space (magnetic bottles)



- **Exercise: show that particles can scape from the magnetic bottle through the "throats" of the bottle if the pitch angle**

$$\alpha_0 < \sin^{-1}\left[\left(\frac{B_0}{B_m}\right)^{1/2}\right] = \sin^{-1}\left(\frac{v_\perp}{v}\right)\Big|_{z=0}$$

Guiding center motion: 1th order terms

- There exists drifts perpendicular to the $\mathbf{B_0}$-field due to inertial force and due to magnetic field gradient

$$\mathbf{U}_{\mathbf{1},\perp} = \mathbf{B_0} \times \left[\frac{m}{qB_0^2} \frac{d\mathbf{U_0}}{dt} + \frac{\mu}{qB_0^2} \nabla B_0 \right]$$

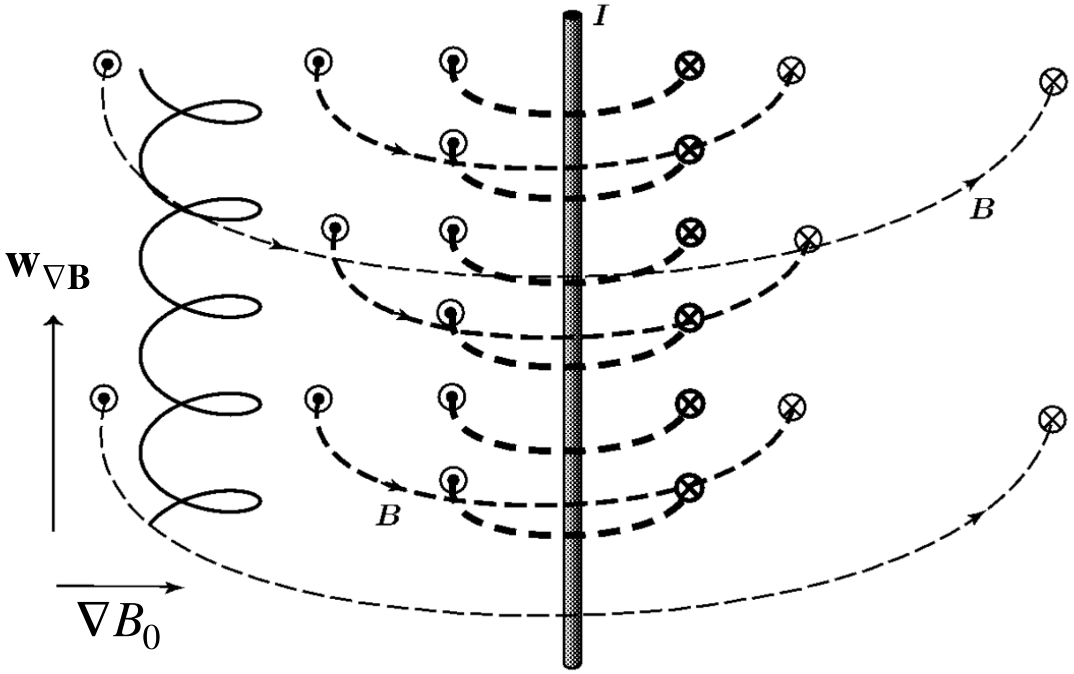
- The perpendicular drift due to magnetic field gradient

$$\mathbf{w}_{\nabla \mathbf{B}} = \frac{\mu}{qB_0^2} \mathbf{B_0} \times \nabla B_0$$

- Exercise: Given the magnetic field of a vertical infinite wire with constant current ($\mathbf{I}$),

$$\mathbf{B_0} = \frac{\mu_0 I}{2\pi R} \hat{\mathbf{e}}_\theta$$

calculate $\mathbf{w}_{\nabla \mathbf{B}}$ for an electron and a proton and the associated electric current density



Guiding center motion: 1th order terms

- **The perpendicular drift due to inertial force**

$$\mathbf{U}_{1,\perp} = \frac{m}{qB_0^2}\mathbf{B_0} \times \frac{d\mathbf{U_0}}{dt} = \frac{m}{qB_0^2}\mathbf{B_0} \times \frac{d}{dt}(U_{0,\parallel}\hat{\mathbf{b}}) + \frac{m}{qB_0^2}\mathbf{B_0} \times \frac{d\mathbf{w}_{\mathbf{ExB}}}{dt}$$

  - The drift due to magnetic field curvature

$$\mathbf{w}_{\mathbf{curv}} = \frac{mU_{0,\parallel}}{qB_0^2}\mathbf{B_0} \times \frac{d\hat{\mathbf{b}}}{dt}$$

Using the relation $\frac{d\hat{\mathbf{b}}}{dt} = \frac{\partial\hat{\mathbf{b}}}{\partial t} + (\mathbf{w}_{\mathbf{ExB}} \cdot \nabla)\hat{\mathbf{b}} + \mathrm{U}_{0,\parallel}\hat{\boldsymbol{\kappa}}$ this drift becomes

$$\mathbf{w}_{\mathbf{curv}} = \frac{mU_{0,\parallel}}{qB_0^2}\mathbf{B_0} \times \left(\frac{\partial\hat{\mathbf{b}}}{\partial t} + (\mathbf{w}_{\mathbf{ExB}} \cdot \nabla)\hat{\mathbf{b}} + U_{0,\parallel}\hat{\boldsymbol{\kappa}}\right)$$

In the absence of $\mathbf{E_0}$-field, and for static $\mathbf{B_0}$-field, the curvature drift reduces to

$$\mathbf{w}_{\mathbf{curv}} = \frac{2W_\parallel}{qB_0^4}\mathbf{B_0} \times \left[(\mathbf{B_0} \cdot \nabla)\mathbf{B_0}\right]$$

- **Exercise: calculate $\mathbf{w}_{\mathbf{curv}}$, and the associated current density, for the $\mathbf{B_0}$-field configuration of the previous exercise**

Exercises: Earth's ring current

- **Exercise: show that in the absence of $\mathbf{E_0}$-field, and for static $\mathbf{B_0}$-field, the curvature and the gradient drifts can be combined as (what assumption must be made?)**

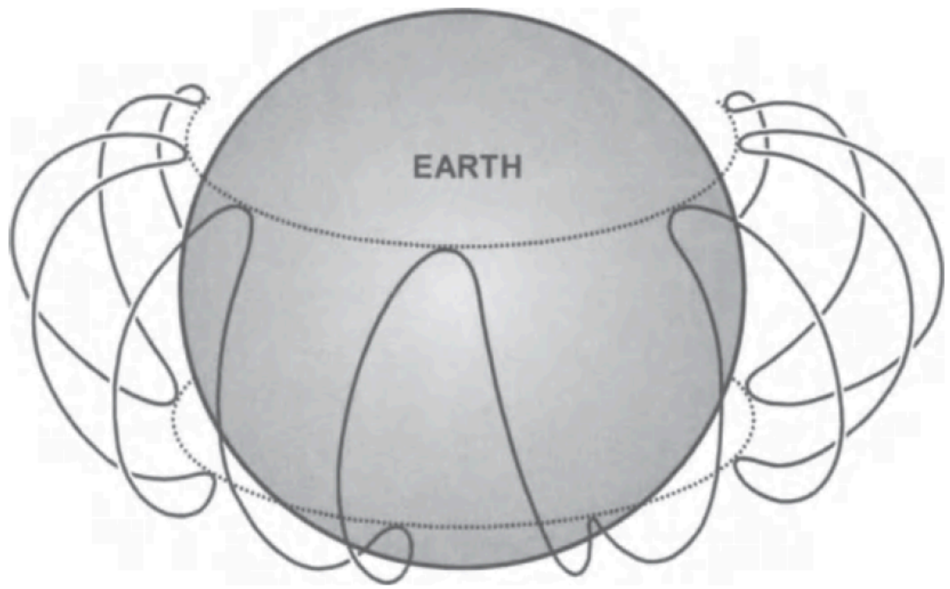
$$\mathbf{w_{CG}} = -\frac{m}{qB_0^3}\left(U_{0,\|}^2 + \frac{1}{2}U_{0,\perp}^2\right)(\nabla B_0 \times \mathbf{B_0})$$

- **Exercise: suppose that the magnetic field of the Earth can be approximated by the field of a magnetic dipole with** $B_0 = 3.12 \times 10^{-5}\ T$**:**

$$B_r = -2B_0\left(\frac{R_E}{R_E + h}\right)^3 \cos\theta$$

$$B_\theta = -B_0\left(\frac{R_E}{R_E + h}\right)^3 \sin\theta$$

$$R_E = 6370\ km \qquad \text{(Earth's Radius)}$$



Describe the trajectory of charged particles at $h = 300\ km$**, as shown in the figure above, and calculate the associated electron and ion current densities. Suppose that** $n(h = 300\ km) = 1 \times 10^9\ m^{-3}$ **and** $\rho_m(h = 300\ km) = 2.67 \times 10^{-17}\ kg/m^{-3}$ (Oxigen)

Guiding center motion: 1th order terms

- **The perpendicular drift due to inertial force**

$$\mathbf{U}_{\mathbf{1},\perp} = \frac{m}{qB_0^2}\mathbf{B_0} \times \frac{d\mathbf{U_0}}{dt} = \frac{m}{qB_0^2}\mathbf{B_0} \times \frac{d}{dt}(U_{0,\parallel}\hat{\mathbf{b}}) + \frac{m}{qB_0^2}\mathbf{B_0} \times \frac{d\mathbf{w}_{\mathbf{ExB}}}{dt}$$

  - The polarization drift

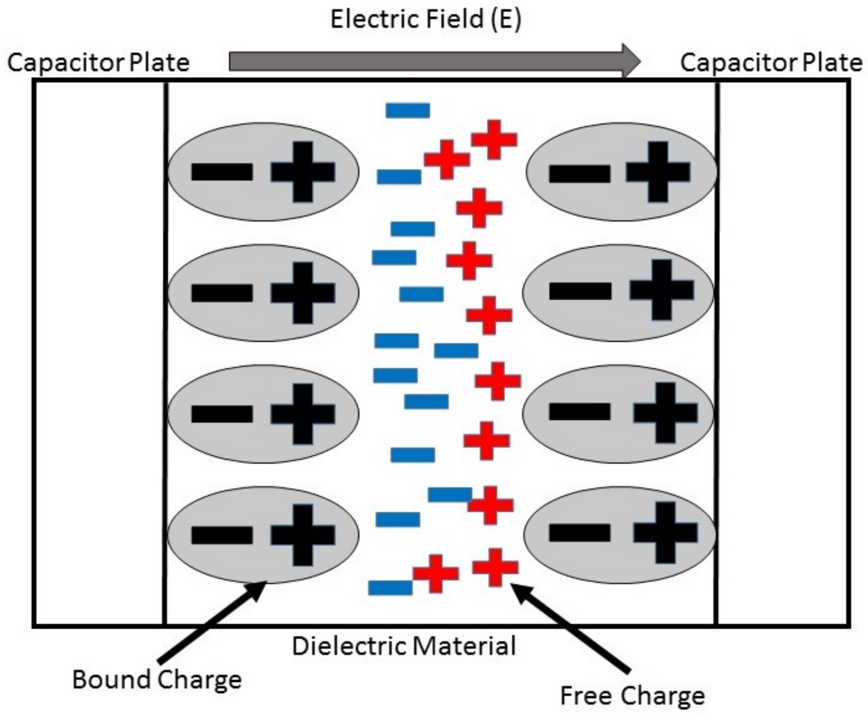
$$\mathbf{w}_{\mathbf{pol}} = \frac{m}{qB_0^2}\mathbf{B_0} \times \frac{d\mathbf{w}_{\mathbf{ExB}}}{dt}$$

For a static $\mathbf{B_0}$-field, the polarization drift reduces to

$$\mathbf{w}_{\mathbf{pol}} = \frac{m}{qB_0^2}\frac{d\mathbf{E}_{\mathbf{0},\perp}}{dt}$$

The polarization current density

- **Since the polarization drift is charge-dependent, a time-dependent electric field (perpendicular to B_0) will produce a net polarization current in a neutral plasma, so that the plasma medium behaves like a dielectric**



  - *The polarization current density is given by*

$$\mathbf{J_P} = \frac{1}{\delta V}\sum_j q_j\,\mathbf{w_{pol,j}} = \frac{1}{\delta V}\left(\sum_j m_j\right)\frac{1}{B_0^2}\frac{d\mathbf{E_{0,\perp}}}{dt} = \frac{\rho_m}{B_0^2}\frac{d\mathbf{E_{0,\perp}}}{dt}$$

  - *A static $\mathbf{E_0}$-field does not produce a polarization field since the ions and electrons will move around to preserve quasi-neutrality*

The plasma dielectric constant

- **To calculate the plasma dielectric constant, let's insert the polarization current in the Ampère-Maxwell equation**
  - *Since* $\mathbf{E_0} = \mathbf{E_0}(\mathbf{r_0}, t)$, *the partial time derivatives become total time derivatives*

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J_P} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) = \mu_0 \left(\frac{\rho_m}{B_0^2} \frac{\partial \mathbf{E_{0,\perp}}}{\partial t} + \epsilon_0 \frac{\partial \mathbf{E_0}}{\partial t} \right) = \mu_0 \epsilon_0 \left(1 + \frac{\rho_m}{\epsilon_0 B_0^2} \right) \frac{d\mathbf{E_{0,\perp}}}{dt} + \epsilon_0 \frac{d\mathbf{E_{0,\parallel}}}{dt}$$

- **Therefore, the plasma perpendicular dielectric current is**

$$\nabla \times \mathbf{B} = \mu_0 \epsilon \frac{d\mathbf{E_0}}{dt} \quad \textit{where} \quad \epsilon_\parallel = \epsilon_0 \quad \textit{and} \quad \epsilon_\perp = \epsilon_0 \left(1 + \frac{\rho_m}{\epsilon_0 B_0^2} \right)$$

- **The resulting charge density that accumulates due to the polarization drift must satisfy the charge continuity equation**

$$\frac{\partial \rho_P}{\partial t} + \nabla \cdot \mathbf{J_P} = 0 \quad \rightarrow \quad \frac{\partial \rho_P}{\partial t} + \nabla \cdot \left(\frac{\rho_m}{B_0^2} \frac{d\mathbf{E_{0,\perp}}}{dt} \right) = 0 \quad \rightarrow \quad \rho_P = -\frac{\rho_m}{B_0^2} \nabla \cdot \mathbf{E_{0,\perp}}$$

- **Writing the total charge density as** $\rho_{total} = \rho + \rho_P$ **yields**

$$\nabla \cdot \mathbf{E_{0,\parallel}} + \nabla \cdot \mathbf{E_{0,\perp}} = \frac{\rho}{\epsilon_0} - \frac{\rho_m}{\epsilon_0 B_0^2} \nabla \cdot \mathbf{E_{0,\perp}} \quad \rightarrow \quad \nabla \cdot \left[\epsilon_0 \mathbf{E_{0,\parallel}} + \epsilon_0 \left(1 + \frac{\rho_m}{\epsilon_0 B_0^2} \right) \mathbf{E_{0,\perp}} \right] = \rho \quad \rightarrow \quad \nabla \cdot \mathbf{E_0} = \frac{\rho}{\epsilon}$$

Plasma as an electric and magnetic medium

- **Let's estimate the magnitude of the electric permittivity and magnetic permeability of a hydrogen fusion plasma with parameters:**
  - *Plasma density: 1 x 10^{20} m^{-3}*
  - *Plasma temperature: 1 x 10^{8} K ($W_\perp = 1/2 m v_\perp^2 \approx k_B T/2$ = 7 x 10^{-16} J)*
  - *Magnetic field: 1 T*
  - *Physical constants: m_i =1.67x10^{-27} kg, ϵ_0 = 8.85x10^{-12} F/m and μ_0 = 4π x10^{-7} H/m*

- **Plasma perpendicular electric permittivity**

$$\epsilon_\perp/\epsilon_0 = 1 + \frac{1.67 \times 10^{-27} \times 1 \times 10^{20}}{8.85 \times 10^{-12} \times 1^2} = 1 + 1.89 \times 10^4 \approx 1.89 \times 10^4 \gg 1$$

- **Plasma magnetic permeability: let's combine $\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})$ with $\mathbf{M} = -nW_\perp \mathbf{B}/B^2$**

$$\mathbf{B} = \mu \mathbf{H} \quad \text{with} \quad \mu = \mu_0 \Big/ \left(1 + \frac{\mu_0 n W_\perp}{B^2}\right). \text{ Therefore, } \mu/\mu_0 = 1 \Big/ \left(1 + \frac{\mu_0 n W_\perp}{B^2}\right)$$

$$\mu/\mu_0 = 1 \Big/ \left(1 + \frac{4\pi \times 10^{-7} \times 1 \times 10^{20} \times 7 \times 10^{-16}}{1^2}\right) = 1 \Big/ \left(1 + 8.8 \times 10^{-2}\right) \approx 1$$

Conservation of the magnetic flux (Bittencourt's, Ch. 4, sec. 4.1)

- **Exercise: suppose there exists a time-dependent magnetic field $\mathbf{B_0} = B_0(t)\hat{\mathbf{k}}$**
  - *Use Faraday's law to show that, in cylindrical coordinates,* $\mathbf{E_0} = -\dfrac{\mathbf{r}}{2} \times \dfrac{d\mathbf{B_0}}{dt}$
  - *Calculate the corresponding ExB drift*
  - *The force acting on a charge due to the electric field is q$\mathbf{E_0}$ and, therefore, the increase in the transverse kinetic energy over one cyclotron period is*

$$\delta\left(\frac{1}{2}mv_\perp^2\right) = q\oint \mathbf{E_0}\cdot d\mathbf{r}$$

From this result, show that the magnetic flux through a Larmor orbit $\Phi_m = B_0\pi r_c^2$ *is conserved:*

$$\delta\Phi_m = \delta\left(B_0\pi r_c^2\right) = 0$$

and, as a consequence, the particle magnetic moment is also conserved

Plasma heating through magnetic pumping

- **Exercise: using the $\mathbf{E_0}$ and $\mathbf{B_0}$-fields from previous exercise for a group of particles**
  - *Suppose that, at $t = t_0$, the average kinetic energy of each particle is*

$$E_{kin} = \frac{1}{2}m\langle v_{\|}^2\rangle + \frac{1}{2}m\langle v_{\perp}^2\rangle = \frac{1}{2}k_B T_{\|} + k_B T_{\perp}$$

    and that $T_{\|}(t_0) = T_{\perp}(t_0) = T_0$. In addition, suppose that, from $t = t_0$ up to $t = t_1$, the $\mathbf{B_0}$-field varies adiabatically: $\mathbf{B_0} = B_0\left[1 + (t - t_0)/(t_1 - t_0)\right]\hat{\mathbf{k}}$, however, there is not enough time for the temperatures to equilibrate. What are the values of $T_{\|}(t_1)$ and $T_{\perp}(t_1)$?
  - *From $t = t_1$ up to $t = t_2$, the magnetic field is kept constant until $T_{\|}(t_2) = T_{\perp}(t_2) = T_2$. What is the value of T_2?*
  - *From $t = t_2$ up to $t = t_3$, the $\mathbf{B_0}$-field is brought, again adiabatically, to its initial value: $\mathbf{B_0} = B_0\left[2 - (t - t_2)/(t_3 - t_2)\right]\hat{\mathbf{k}}$. However, there is not enough time for the temperatures to equilibrate. What are the values of $T_{\|}(t_3)$ and $T_{\perp}(t_3)$?*
  - *From $t = t_3$ up to $t = t_f$, the $\mathbf{B_0}$-field is kept constant until $T_{\|}(t_f) = T_{\perp}(t_f) = T_f$. What is the final temperature of the plasma?*

Answer: $T_f = 10\,T_0/9$ (for one single loop of $\mathbf{B_0}$-field sweep)

Guiding center motion: 2th order terms

- **To second order ($O(\epsilon^1)$), the momentum equation is**

$$\frac{d\mathbf{U}_1}{dt} + \left(\frac{d\mathbf{R}_1}{dt}\cdot\nabla\right)\boldsymbol{u}_0 + \left(\frac{d\mathbf{U}_1}{dt}\cdot\nabla_{U_1}\right)\boldsymbol{u}_0 + \frac{\partial \mathbf{u}_1}{\partial t} + \left(\frac{d\mathbf{R}_0}{dt}\cdot\nabla\right)\boldsymbol{u}_1 + \left(\frac{d\mathbf{U}_0}{dt}\cdot\nabla_{U_0}\right)\boldsymbol{u}_1 + \frac{\partial \mathbf{u}_2}{\partial \tau} =$$

$$= \frac{q}{m}\left(\mathbf{E}_2 + \mathbf{U}_2\times\mathbf{B}_0 + \mathbf{u}_2\times\mathbf{B}_0 + \mathbf{U}_1\times\mathbf{B}_1 + \mathbf{u}_1\times\mathbf{B}_1 + \mathbf{U}_0\times\mathbf{B}_2 + \mathbf{u}_0\times\mathbf{B}_2\right)$$

- **Taking the τ-average of this equation yields**

$$\frac{d\mathbf{U}_1}{dt} = \frac{q}{m}\left(\langle\mathbf{E}_2\rangle + \mathbf{U}_2\times\mathbf{B}_0 + \mathbf{U}_1\times\mathbf{B}_1 + \langle\mathbf{u}_1\times\mathbf{B}_1\rangle + \mathbf{U}_0\times\langle\mathbf{B}_2\rangle + \langle\mathbf{u}_0\times\mathbf{B}_2\rangle\right)$$

  - Let's calculate each τ-average term separately

$$\langle\mathbf{E}_2\rangle = \langle(\boldsymbol{\rho}_1\rangle\cdot\nabla)\mathbf{E}_0 + \frac{1}{2}\langle(\boldsymbol{\rho}_0\cdot\nabla)^2\mathbf{E}_0\rangle = \frac{1}{2}\langle(\boldsymbol{\rho}_0\cdot\nabla)^2\mathbf{E}_0\rangle$$

$$\langle\mathbf{B}_2\rangle = \langle(\boldsymbol{\rho}_1\rangle\cdot\nabla)\mathbf{B}_0 + \frac{1}{2}\langle(\boldsymbol{\rho}_0\cdot\nabla)^2\mathbf{B}_0\rangle = \frac{1}{2}\langle(\boldsymbol{\rho}_0\cdot\nabla)^2\mathbf{B}_0\rangle$$

$$\langle\mathbf{u}_1\times\mathbf{B}_1\rangle = \left\langle\mathbf{u}_1\times\left[\left(\boldsymbol{\rho}_1\cdot\nabla\right)\mathbf{B}_0\right]\right\rangle \qquad \langle\mathbf{u}_0\times\mathbf{B}_2\rangle = \frac{1}{2}\left\langle\mathbf{u}_0\times\left[(\boldsymbol{\rho}_1\cdot\nabla)\mathbf{B}_0 + \frac{1}{2}(\boldsymbol{\rho}_0\cdot\nabla)^2\mathbf{B}_0\right]\right\rangle$$

Guiding center motion: 2th order terms

- Neglecting 1st and 2nd order corrections to the magnetic field yields

$$\frac{d\mathbf{U_1}}{dt} = \frac{q}{m}\left[\frac{1}{2}\langle(\boldsymbol{\rho_0}\cdot\nabla)^2\mathbf{E_0}\rangle + \mathbf{U_2}\times\mathbf{B_0}\right]$$

  - The solution of this equation gives

$$\mathbf{U}_{\mathbf{2},\perp} = \frac{\rho_0^2}{4}\frac{\nabla_\perp^2\mathbf{E_0}\times\mathbf{B_0}}{B_0^2}$$

Summary of particle drifts

- **The 0th order drift**
  - $U_{0,\|}\hat{\mathbf{b}}$: Parallel drift
  - $\mathbf{w_{ExB}}$: ExB drift

$$\mathbf{U_0} = U_{0,\|}\hat{\mathbf{b}} + \mathbf{w_{ExB}}$$

- **The 1th order drift**
  - $\mathbf{w_{\nabla B}}$: magnetic field gradient drift
  - $\mathbf{w_{ExB}}$: magnetic field curvature drift
  - $\mathbf{w_{pol}}$: polarization drift

$$\mathbf{U_1} = \mathbf{w_{\nabla B}} + \mathbf{w_{curv}} + \mathbf{w_{pol}}$$

- **The 2th order drift**
  - $\mathbf{w_{\nabla^2 E}}$: second order $\mathbf{E}$-drift

$$\mathbf{U_2} = \mathbf{w_{\nabla^2 E}}$$

Exercises

- The cyclotron resonance: Show that when a circularly polarized electric field rotates in the counterclockwise direction, looking along $\mathbf{B}_0$, a positive particle is able to absorb energy from the electric field, so that its speed increases continuously in time (see Bittencourt's Ch. 4, Sec. 3.4). What about a negative particle?

- Solve exercises 4.4, 4.6, 4.7 and 4.11 from Bittencourt's Ch. 4

PGF5112 - Plasma Physics I

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  - *Physical description of a tokamak*
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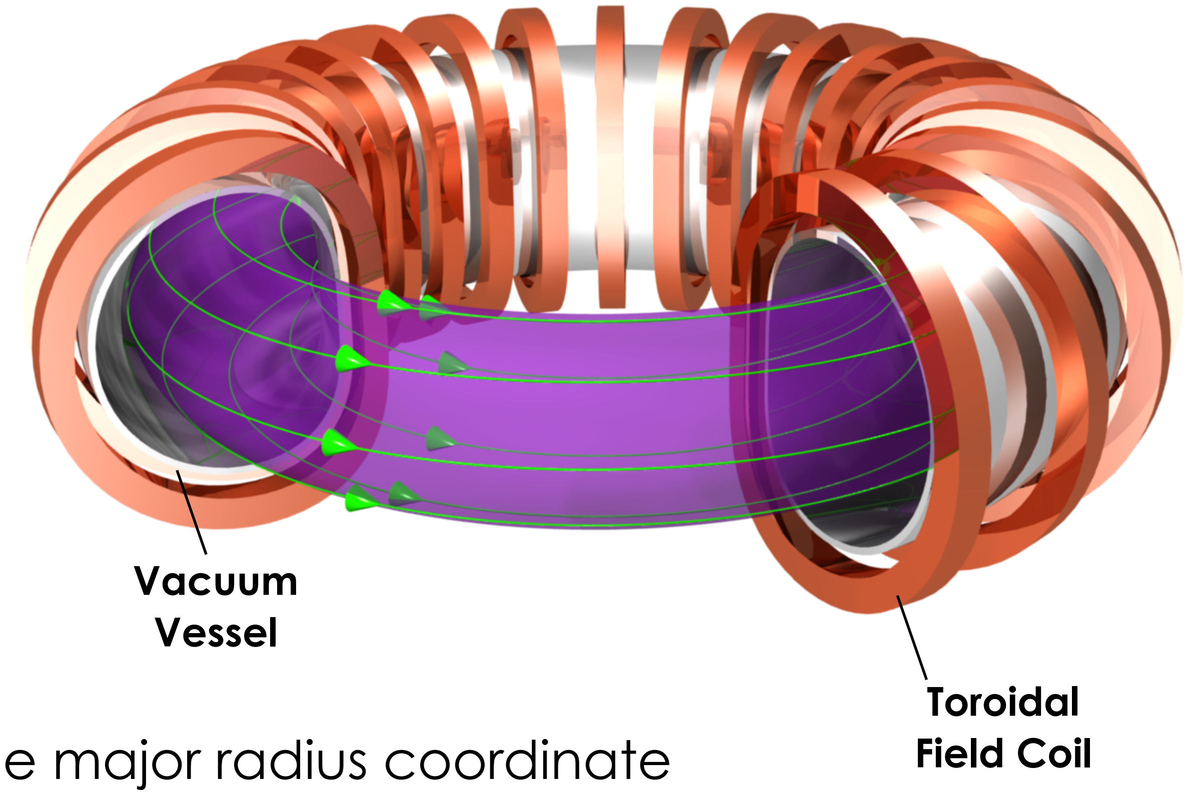
Description of the magnetic fields in a tokamak

- **Tokamaks machines are symmetric with respect to the vertical axis in the center of the machine (axisymmetric)**

- **The word tokamak is a Russian acronym (***toroidalnaja kamera s magnitnymi katushkami***) that can be translated as** toroidal chamber with magnetic coils

- **The main components of a tokamak are**
  - The vacuum vessel (VV)
    + The pressure must be optimized to facilitate the plasma breakdown
  - The toroidal field (TF) coils
    + These coils are responsible for confining the particles
    + The toroidal field intensity decreases with the major radius coordinate

$$B_\phi = \frac{R_0 B_{T0}}{R}$$



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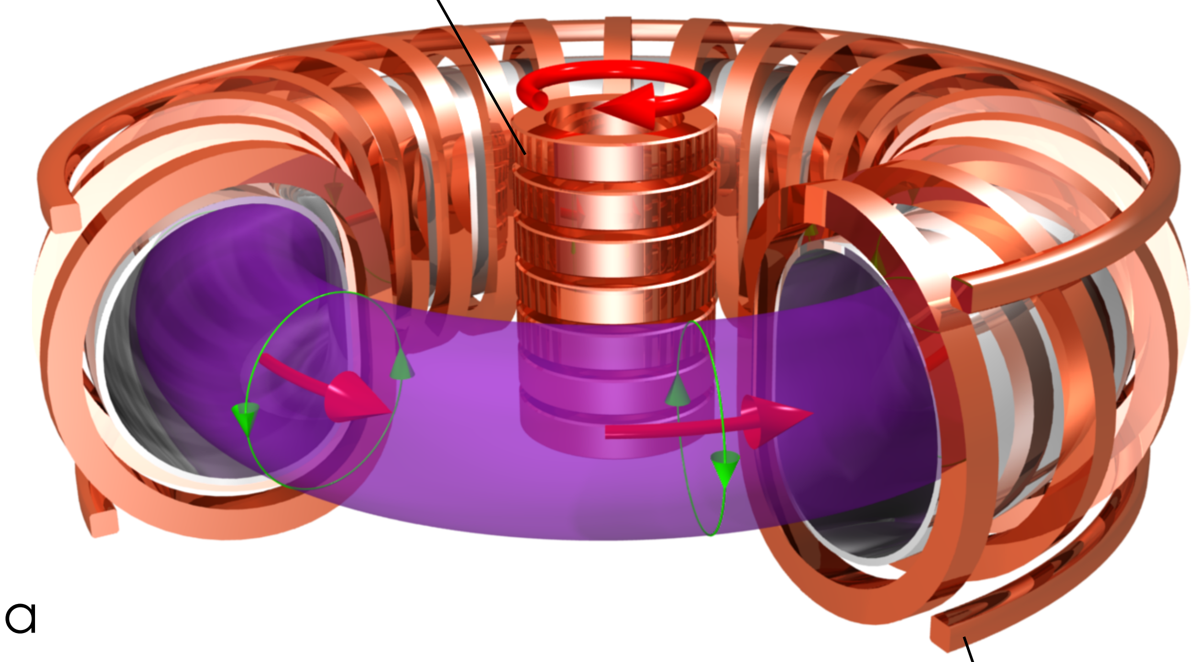
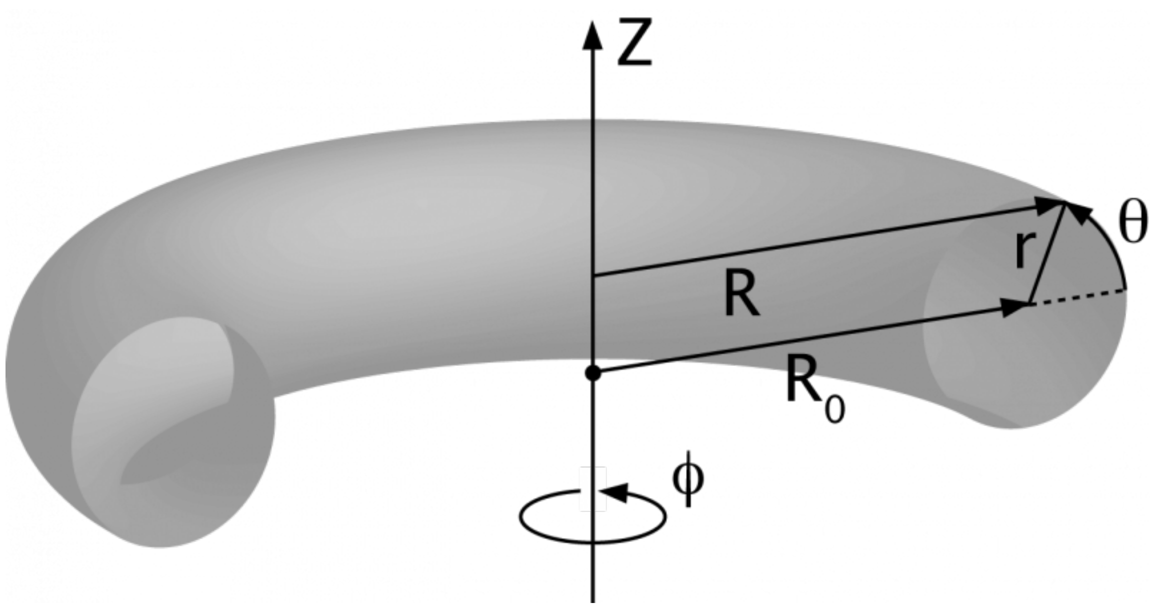
- **The main components of a tokamak are**
  - The central solenoide (CS)
    - The CS is responsible for driving the plasma current by induction (transformer action)
  - The poloidal field (PF) coils
    - These coils are needed to shape the plasma boundary and to control the plasma position



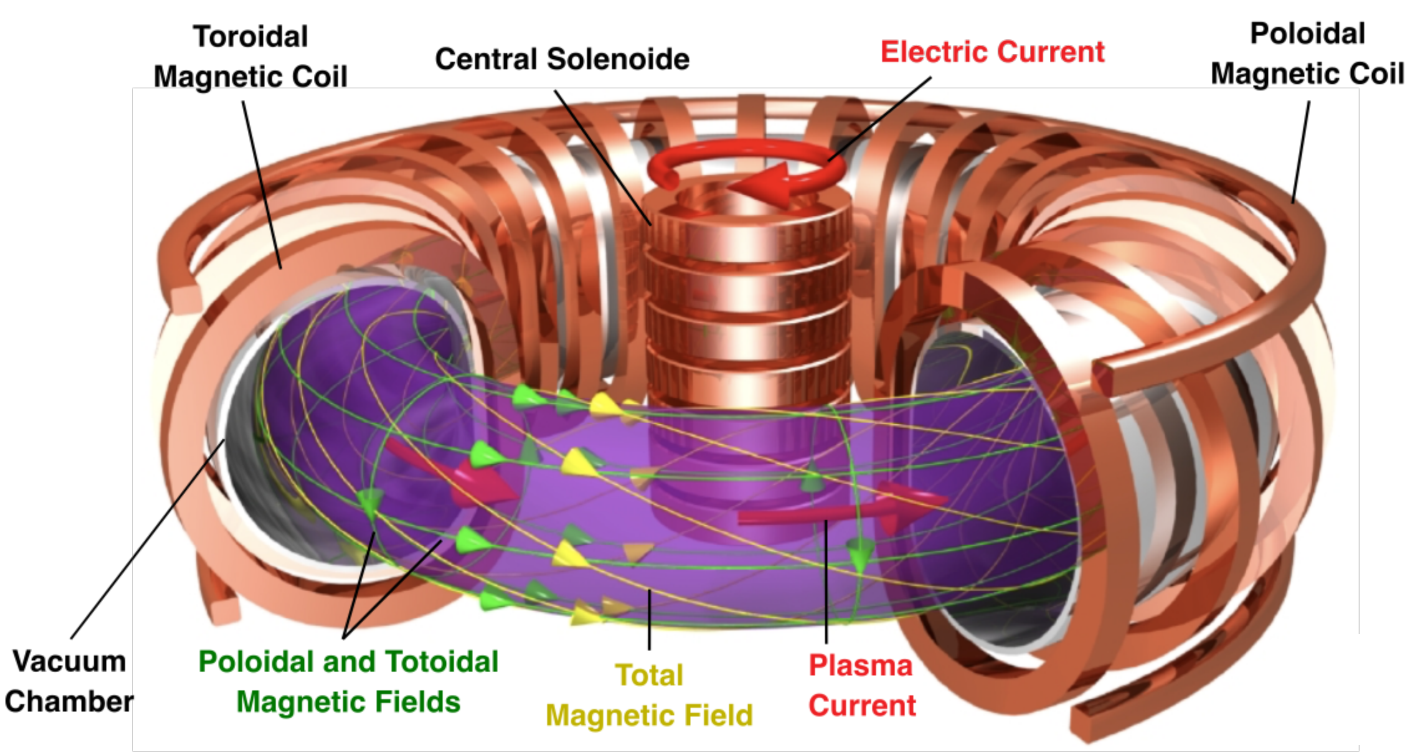
Description of the magnetic fields in a tokamak

- **The total magnetic field in a tokamak is helicoidal**

- **Important parameters that can be used to characterize a tokamak is**
  - Major radius: R_0
  - (Horizontal) Minor radius: a
  - The aspect ratio: $A = R_0/a$



$r = a$ (Plasma Boundary)



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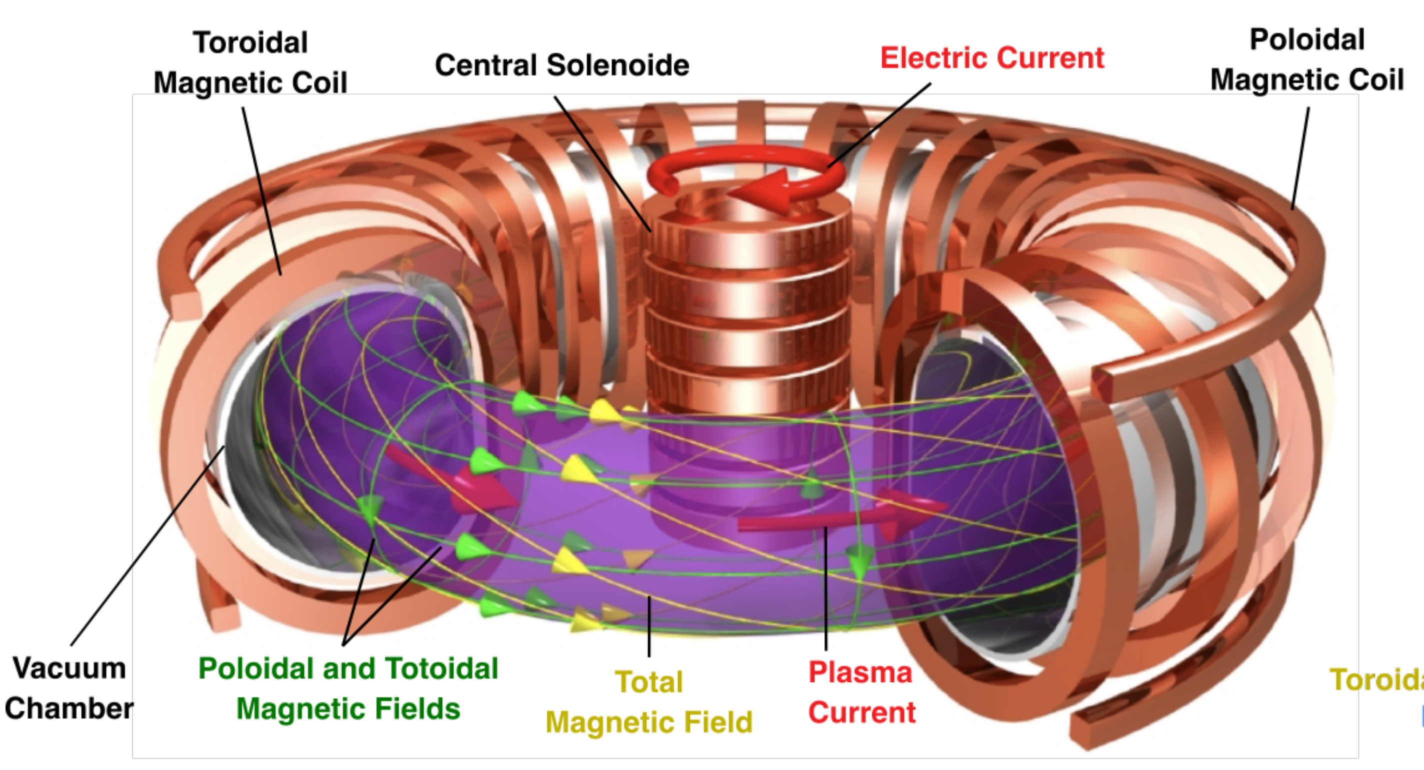
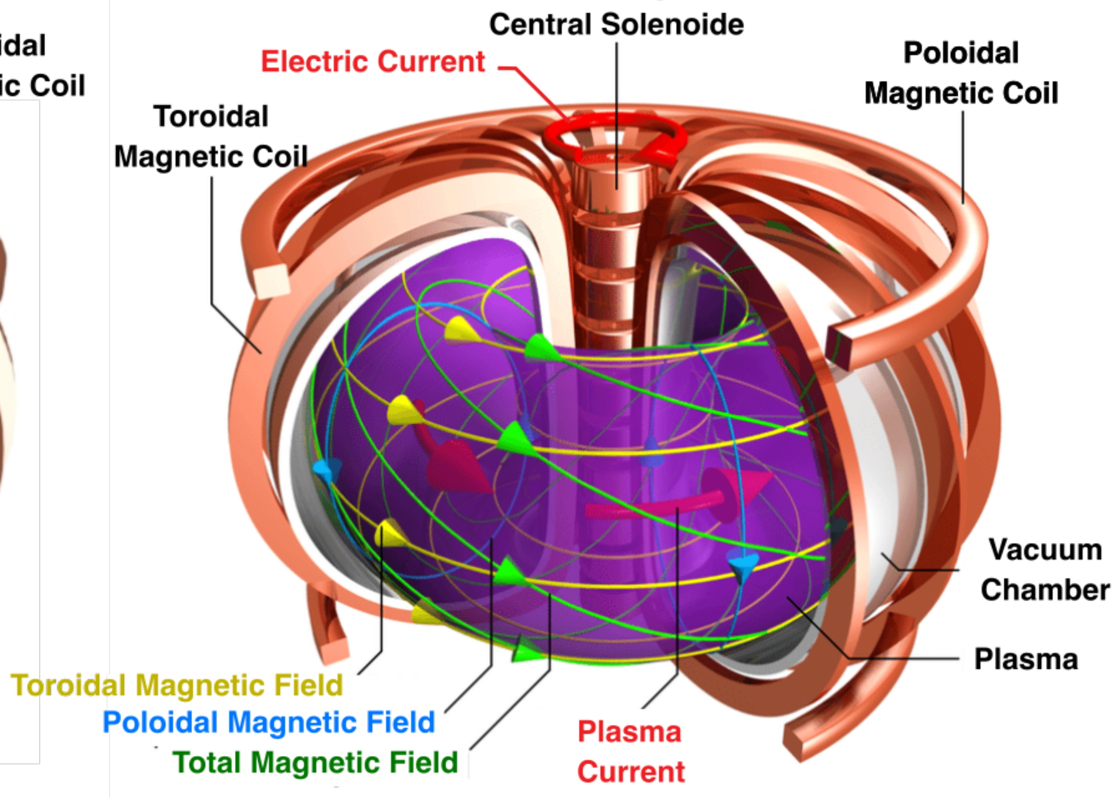
(a) Tokamak of conventional aspect ratio (A ≈ 3)



(b) Tokamak of low aspect ratio (A < 2)



Particle drifts in a tokamak

- **Let's calculate the trajectory of charged particles in a tokamak using**

$$\mathbf{U} = U_{0,\|}\hat{\mathbf{b}} + \mathbf{w_{ExB}} + \mathbf{w_{\nabla B}} + \mathbf{w_{curv}} + \mathbf{w_{pol}} + \mathbf{w_{\nabla^2 E}}$$

$$\mathbf{U} \approx \mathbf{w_{CG}} = -\frac{m}{qB_0^3}\left(w_{0,\|}^2 + \frac{1}{2}w_{0,\perp}^2\right)(\nabla B_0 \times \mathbf{B_0})$$

  - Note that to use the equation above we must impose that $\nabla \times \mathbf{B} = 0$ and also to neglect the induced (toroidal) electric field

- **The magnetic field in a tokamak can be written as the sum of the poloidal and toroidal fields**

$$\mathbf{B} = \mathbf{B_P} + \mathbf{B_T} = \mathbf{B_P} + \frac{R_0 B_{T0}}{R}\hat{\mathbf{e}}_\phi$$

  - In tokamaks, $\mathbf{B_P} \ll \mathbf{B_T}$. Therefore, taking $\nabla \times \mathbf{B} = \nabla \times \mathbf{B_P} + \nabla \times \mathbf{B_T} \approx \nabla \times \mathbf{B_T} = 0$ is somewhat justified. In addition, we will assume that the field gradient is dominated by the toroidal field:

$$\nabla B = \nabla\left(B_T\sqrt{1 + \frac{B_P^2}{B_T^2}}\right) \approx \nabla B_T$$

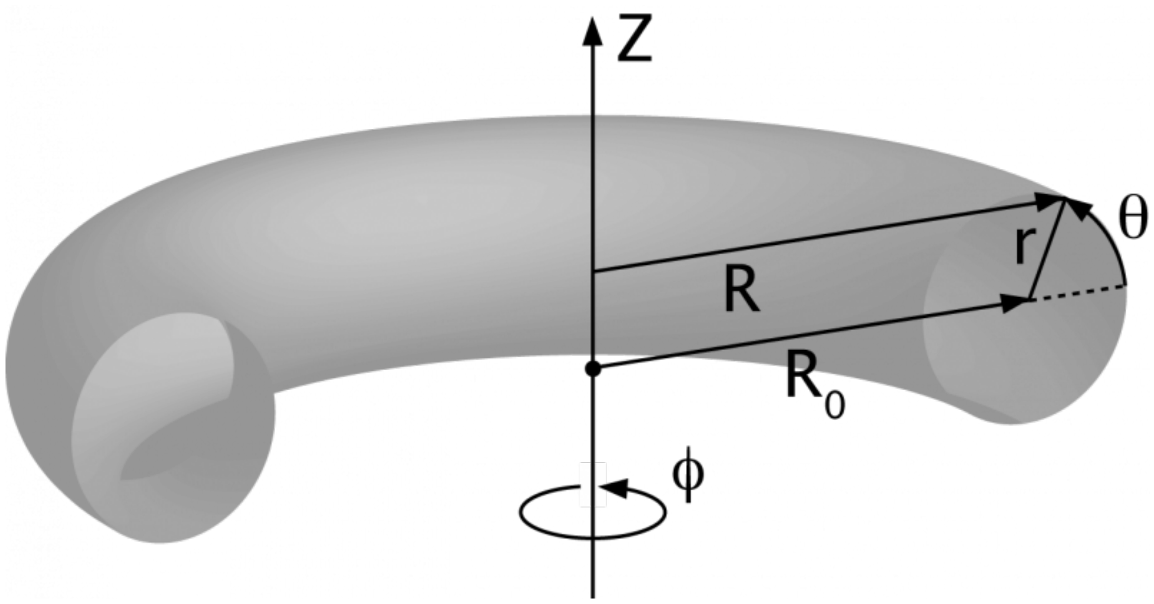
Particle drifts in a tokamak

- **In a $\{R, \phi, Z\}$ coordinate system, we have that**

$$\nabla B_T = \nabla\left(\frac{R_0 B_{T0}}{R}\right) = -\frac{R_0 B_{T0}}{R^2}\hat{\mathbf{e}}_{\mathbf{R}}$$

$$\nabla B_0 \times \mathbf{B_0} = -\frac{R_0 B_0}{R^2}\hat{\mathbf{e}}_{\mathbf{R}} \times \left(\mathbf{B_P} + \frac{R_0 B_0}{R}\hat{\mathbf{e}}_\phi\right) = \frac{B_{T0}}{R}\left(B_{T0}\hat{\mathbf{e}}_{\mathbf{Z}} + B_{P0}\cos\theta\,\hat{\mathbf{e}}_\phi\right)$$

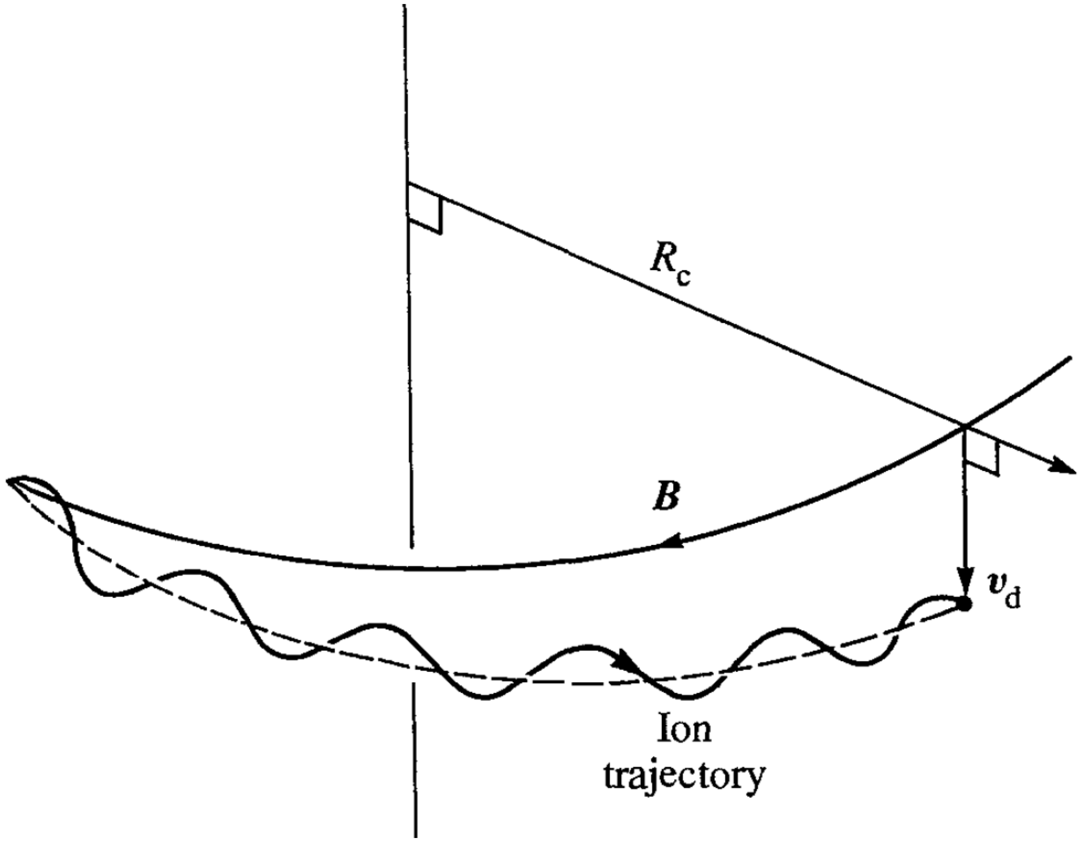
$$\mathbf{w}_{\mathbf{CG}} = -\frac{m}{qB_{T0}^2 R}\left(w_{0,\parallel}^2 + \frac{1}{2}w_{0,\perp}^2\right)\left(B_{T0}\hat{\mathbf{e}}_{\mathbf{Z}} + B_{P0}\cos\theta\,\hat{\mathbf{e}}_\phi\right)$$



Particle drifts in a tokamak

- **In a tokamak, charged particles drift in two directions (charge/mass dependent)**
  - In the vertical direction: constant drift
  - In the toroidal direction: the magnitude depends on the poloidal angle

$$\mathbf{w}_{\mathbf{CG}} = -\frac{m}{qB_{T0}^2 R}\left(w_{0,\parallel}^2 + \frac{1}{2}w_{0,\perp}^2\right)\left(B_{T0}\hat{\mathbf{e}}_{\mathbf{Z}} + B_{P0}\cos\theta\,\hat{\mathbf{e}}_{\phi}\right)$$



Electrons drift in the opposite direction

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Trapped and passing particles

- **In addition to the drift calculated in the previous topic, the particles also have a parallel velocity along the field lines**

- **Since the field lines in a tokamak is helicoidal, the particles would access the high toroidal field side (HFS) region and the low toroidal field side (LFS) region**
  - Depending on their ratio $v_\perp/v$, particles could be reflected, in a similar way as in mirror machines, and be trapped in the LFS region

- **The total kinetic energy of a particle is conserved and is given by**

$$K = \frac{1}{2}mw_\parallel^2 + \frac{1}{2}mw_\perp^2 = \frac{1}{2}mw_\parallel^2 + \mu B$$

  - Where μ is the particle magnetic moment (first adiabatic constant) and

$$B \approx B_T = \frac{R_0 B_{T0}}{R} = \frac{B_{T0}}{1 + r/R_0 \cos\theta} = \approx B_{T0}\left(1 - \frac{r}{R_0}\cos\theta\right) = B_{T0}\left[1 - \epsilon + 2\epsilon\sin^2\left(\frac{\theta}{2}\right)\right]$$

Trapped and passing particles

- **Therefore, the energy equation of a particle in a tokamak field becomes**

$$\frac{1}{2}mw_\parallel^2 + \mu \Delta B \sin^2\left(\frac{\theta}{2}\right) = K - \mu B_{\min}$$

  - Where

$$B_{\min} = B(r,\theta)\,|_{\min} = \frac{B_{T0}}{1+\epsilon} = B_{T0}(1-\epsilon)$$

$$B_{\max} = B(r,\theta)\,|_{\max} = \frac{B_{T0}}{1-\epsilon} = B_{T0}(1+\epsilon)$$

$$\Delta B = B_{\max} - B_{\min}$$

- **Exercise: show that particles are trapped in the LFS region if** $\frac{w_\parallel^2}{w^2} < \epsilon = \frac{r}{R_0}$, **otherwise, they are passing particles**

References

- **The single particle orbit theory**
  - *Bittencourt: Ch. 2, 3 and 4*

- **Particle orbits in tokamaks**
  - *Fitzpatrick: Ch. 2*