PGF5112 - Plasma Physics I

By

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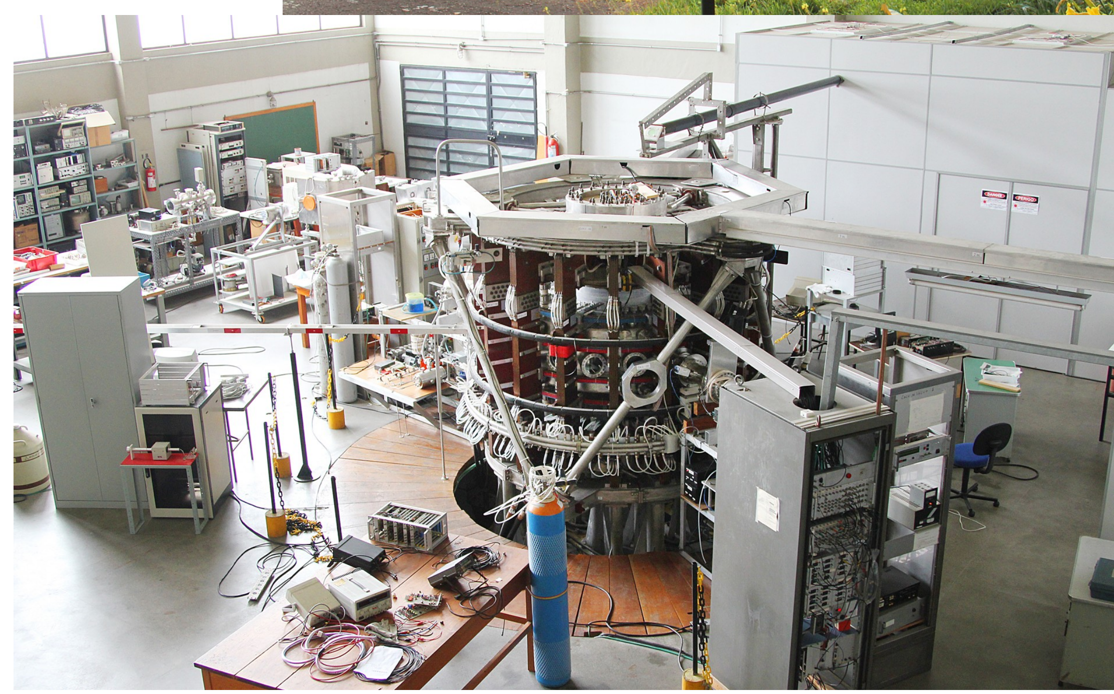

Postgraduate course ministered remotely from the

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- **Single particle orbits: the motion of charged particles in electromagnetic fields**
  - *Introduction*
  - *Uniform and static electric field*
  - *Uniform and static magnetic field*
  - *Uniform and static electric and magnetic fields*
  - *Non-uniform and static magnetic field (physical insight)*
  - *Non-uniform and static electric field (physical insight)*
  - *Non-uniform and time-dependent electric and magnetic fields*

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The single particle orbit theory

- **Knowing the trajectory of charged particles in special field configurations is important as it provides a good physical insight into some dynamic processes**

- **Here, we are interested in the motion of charged particles in the presence of electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) fields, which are known as functions of $\mathbf{r}$ and t**
  - Therefore, the fields are not affected by the charged particles

- **The relativistic equation of motion for a charged particle under the action of the Lorentz force due to $\mathbf{E}$ and $\mathbf{B}$ fields is**

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

  - Here, $\mathbf{p} = \gamma m \mathbf{v}$ is the relativistic particle momentum, with $\gamma = \dfrac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

    q and m are the particle charge and rest mass, respectively

    $c = 2.99 \times 10^8$ m/s is the speed of light in vacuum

The classical/non-relativistic single particle orbit theory

- **In many situations of practical interest, the term $v^2/c^2 \ll 1$**
  - Therefore, $\gamma \approx 1$ and m can be considered constant (independent of v)

- **Relativistic effects are important only for highly energetic particles**
  - A 1 MeV proton has $v = 1.4 \times 10^7$ m/s, i.e. $v^2/c^2 = 0.002 \ll 1$
  - Radiative effects, which are relativistic effects, will also be neglected here

- **In such a situations, the motion equation reduces to the non-relativistic equation**

$$m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

- **If the velocity obtained from this equation does not satisfy the condition $v^2/c^2 \ll 1$, then the relativistic equation of motion must be used instead**

The classical/non-relativistic single particle orbit theory

- **Let's consider the particle kinetic energy**

$$W = \frac{1}{2}mv^2 = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} \quad \rightarrow \quad \frac{dW}{dt} = \frac{d}{dt}\left(\frac{1}{2}m\mathbf{v}\cdot\mathbf{v}\right) = m\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}$$

- **Using the motion equation, this equation becomes**

$$\frac{dW}{dt} = m\frac{d\mathbf{v}}{dt}\cdot\mathbf{v} = q\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)\cdot\mathbf{v} = q\left(\mathbf{E}\cdot\mathbf{v}\right) + \cancel{q(\mathbf{v}\times\mathbf{B})\cdot\mathbf{v}} = q\left(\mathbf{E}\cdot\mathbf{v}\right)$$

- **From the equation above, one concludes that**
  - Any change in the particle kinetic energy is done by electric fields
  - Magnetic fields do no work on charged particles, i.e. the particle kinetic energy is conserved when there is only a magnetic field

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Single particle orbits: the trajectories of charged particles in an uniform and static electric field

- **Charged particles in externally applied electromagnetic fields are subject to the Lorentz force**

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}\left(\mathbf{E}_{\text{ext}} + \mathbf{v}\times\mathbf{B}_{\text{ext}}\right)$$

- **For the case in which $\mathbf{B}_{\text{ext}} = 0$ and $\mathbf{E}_{\text{ext}} = \mathbf{E_0}$ is uniform and static, one has**

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}\mathbf{E_0}$$

- **The solution of this equation is obtained by direct integration**

$$\int_0^t \frac{d\mathbf{v}}{dt}\,dt = \int_0^t \frac{q}{m}\mathbf{E_0}\,dt \quad \rightarrow \quad \int_{\mathbf{v}(0)}^{\mathbf{v}(t)} d\mathbf{v} = \frac{q}{m}\mathbf{E_0}\int_0^t dt \quad \rightarrow \quad \mathbf{v}(t) = \mathbf{v}(0) + \frac{q}{m}\mathbf{E_0}\,t$$

(Uniforme Rectilinear Motion)

$$\int_0^t \frac{d\mathbf{r}}{dt}\,dt = \int_0^t \mathbf{v}(0)\,dt + \frac{q}{m}\mathbf{E_0}\int_0^t t\,dt \quad \rightarrow \quad \mathbf{r}(t) = \mathbf{r}(0) + \mathbf{v}(0)t + \frac{q}{2m}\mathbf{E_0}\,t^2$$

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Single particle orbits: the trajectories of charged particles in an uniform and static magnetic field

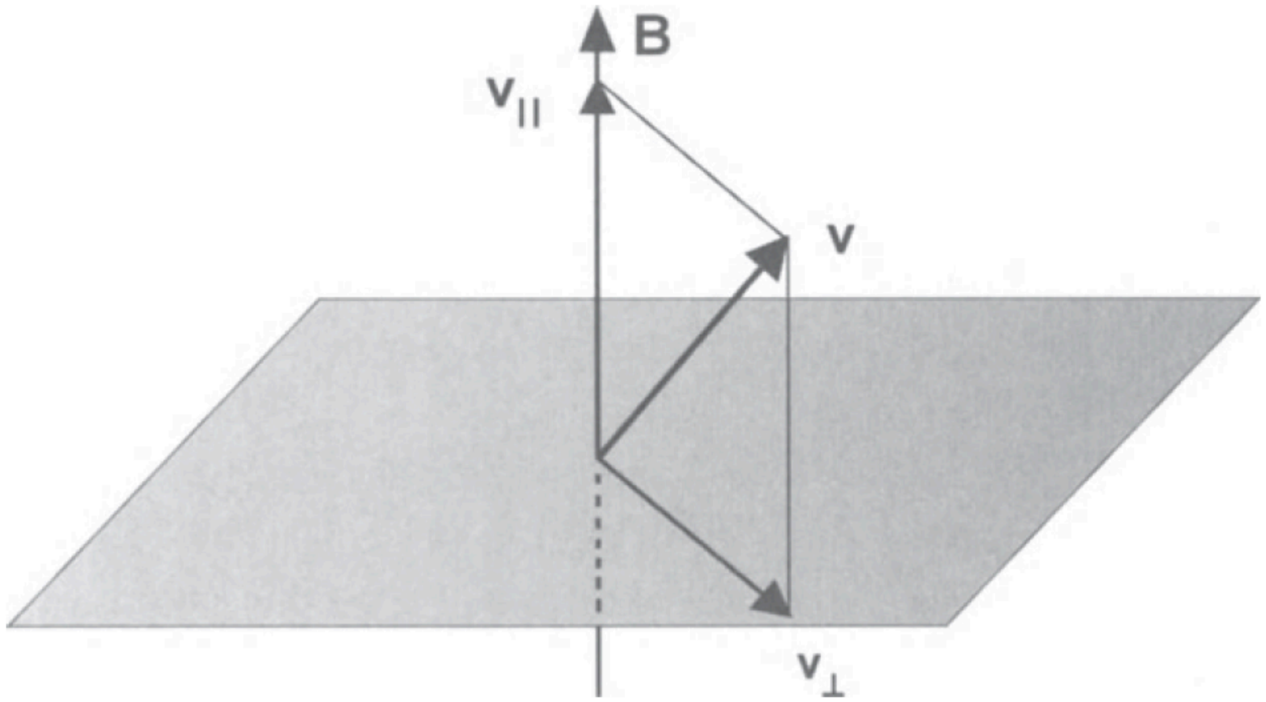
- For the case in which $\mathbf{E}_{\text{ext}} = 0$ and $\mathbf{B}_{\text{ext}} = \mathbf{B_0}$ is uniform and static, one has

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}\mathbf{v} \times \mathbf{B_0}$$

- Decompose $\mathbf{v}$ in its parallel and perpendicular (to $\mathbf{B_0}$) components: $\mathbf{v} = \mathbf{v}_{||} + \mathbf{v}_{\perp}$

$$\frac{d\mathbf{v}_{\|}}{dt} + \frac{d\mathbf{v}_{\perp}}{dt} = \frac{q}{m}\mathbf{v}_{\perp} \times \mathbf{B_0}$$

(The term $\mathbf{v}_{\|} \times \mathbf{B_0} = 0$ because $\mathbf{v}_{\|} \parallel \mathbf{B_0}$)



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- **Decompose $\mathbf{v}$ in its parallel and perpendicular (to $\mathbf{B_0}$) components: $\mathbf{v} = \mathbf{v}_{||} + \mathbf{v}_{\perp}$**

$$\frac{d\mathbf{v}_{||}}{dt} + \frac{d\mathbf{v}_{\perp}}{dt} = \frac{q}{m}\mathbf{v}_{\perp} \times \mathbf{B_0}$$

(The term $\mathbf{v}_{||} \times \mathbf{B_0} = 0$ because $\mathbf{v}_{||} \parallel \mathbf{B_0}$)

  - In the parallel direction: uniforme rectilinear motion

$$\frac{d\mathbf{v}_{||}}{dt} = 0 \quad \rightarrow \quad \mathbf{v}_{||} = \mathbf{v_0}$$

  - In the perpendicular direction: cyclotron motion

$$\frac{d\mathbf{v}_{\perp}}{dt} = \frac{q}{m}\mathbf{v}_{\perp} \times \mathbf{B_0}$$

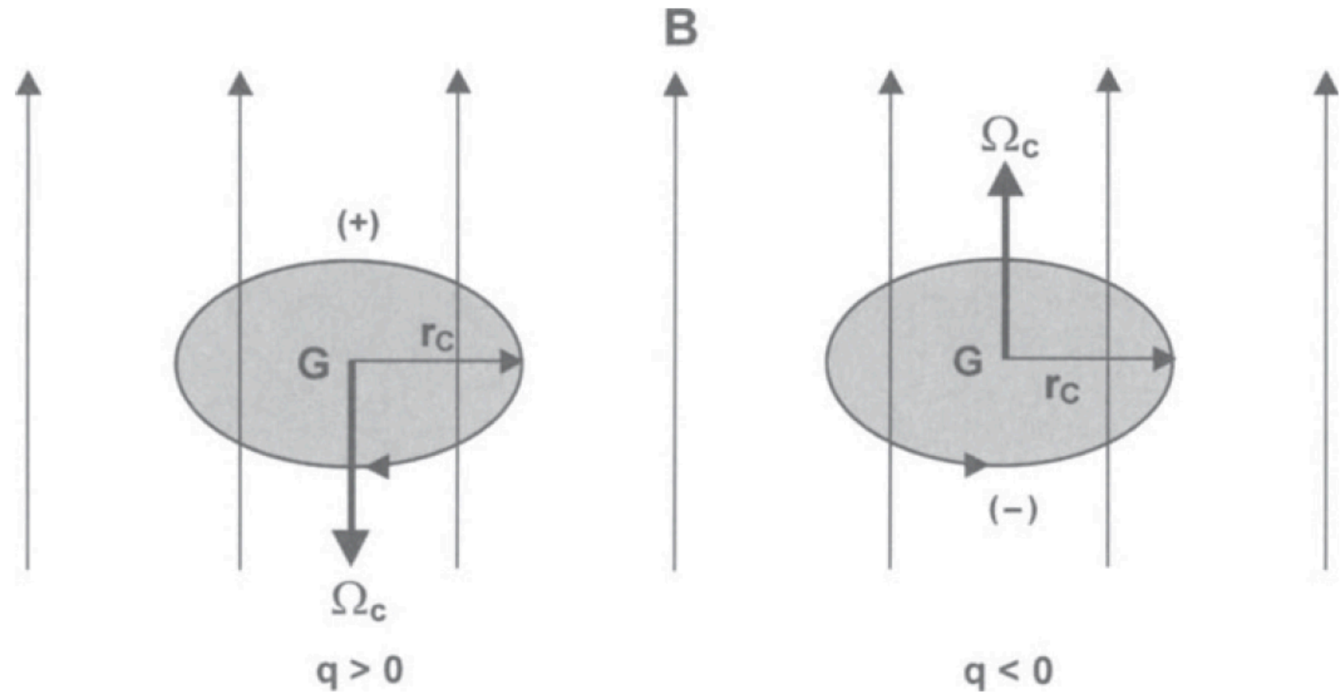
  - If one defines $\mathbf{\Omega_c} = -\dfrac{q}{m}\mathbf{B_0}$, the motion equation becomes: $\dfrac{d\mathbf{v}_{\perp}}{dt} = \mathbf{\Omega_c} \times \mathbf{v}_{\perp}$

The cyclotron frequency (or the gyrofrequency)

- **The quantity $\mathbf{\Omega_c} = -\frac{q}{m}\mathbf{B_0}$ is called the cyclotron frequency or gyrofrequency**

$$\frac{d\mathbf{v}_\perp}{dt} = \mathbf{\Omega_c} \times \mathbf{v}_\perp$$

(Equation of Motion)



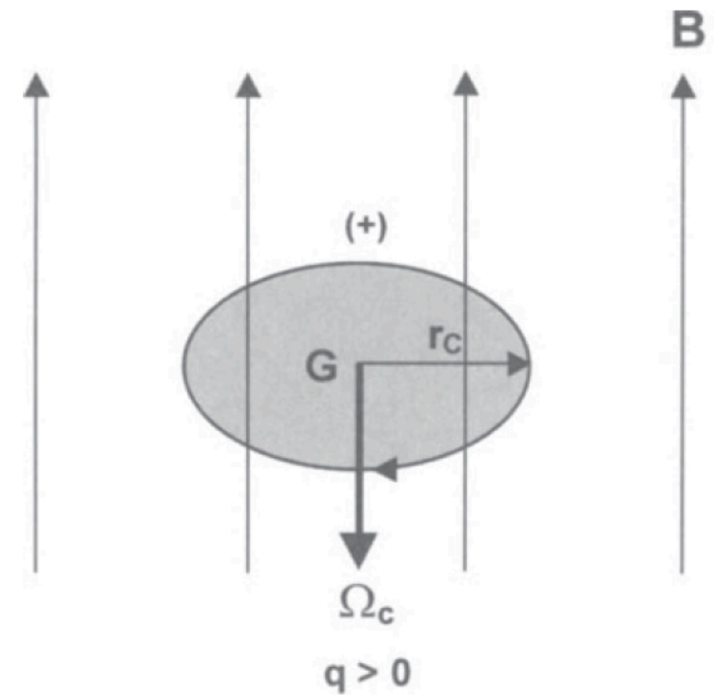
- **Its direction is chosen accordingly to the diamagnetic direction**
  - $\mathbf{\Omega_c}$ is opposite to $\mathbf{B_0}$ for a positive charge (q > 0), which moves such that the magnetic field created by it is opposite to $\mathbf{B_0}$
  - $\mathbf{\Omega_c}$ points in the direction of $\mathbf{B_0}$ for a negative charge (q < 0), which also moves such that the magnetic field created by it is opposite to $\mathbf{B_0}$
  - Note that $\mathbf{\Omega_c}$ always points in the direction of the particle angular momentum

The Larmor radius (or the gyroradius)

- **Since Ω_c is constant and, from kinetic energy conservation, $|\mathbf{v}_\perp|$ is also constant, the equation of motion implies that**
  - The particle acceleration is constant in magnitude
  - Its direction is perpendicular to both $\mathbf{v}_\perp$ and $\mathbf{B_0}$

- **Therefore, the equation of motion can be integrated directly**

$$\frac{d\mathbf{v}_\perp}{dt} = \mathbf{\Omega_c} \times \mathbf{v}_\perp = \mathbf{\Omega_c} \times \frac{d\mathbf{r_c}}{dt} \qquad \rightarrow \qquad \mathbf{v}_\perp = \mathbf{\Omega_c} \times \mathbf{r_c}$$



  - Here, $\mathbf{r_c}$ is the particle vector position measured with respect to a point (G in the figure) in the plane perpendicular to $\mathbf{B_0}$ which contains the particle
  - Since $|\mathbf{v}_\perp|$ is constant, $|\mathbf{r_c}|$ is also constant

- **The vector $\mathbf{r_c}$ is called the Larmor radius or gyroradius**

The cyclotron frequency and Larmor radius for some particular cases

- **Larmor radius:** $r_c = v_\perp/\Omega_c = \dfrac{m v_\perp}{|q| B_0}$ One can estimate $v_\perp \approx = v_{th} = \sqrt{\dfrac{k_B T}{m}}$

- **Cyclotron frequency:** $\Omega_c = |q| B_0/m$
  - Electron cyclotron: $f_{\rm ce} = \Omega_{\rm ce}/2\pi = 28.0 \times {\rm B}_0$ (GHz)
  - Ion cyclotron: $f_{\rm ci} = \Omega_{\rm ci}/2\pi = 15.2 \times {\rm B}_0$ (MHz)

- **Cyclotron frequency and Larmor radius in plasmas**
  - Tokamaks ($m_i = 1.67 \times 10^{-27}\ kg;\ B_0 = 1.5\ T;\ T = 1 \times 10^8\ K$)

$$f_{\rm ce} = 42\ GHz,\ f_{\rm ci} = 22.8\ MHz,\ r_{ce} = 0.15\ mm \text{ and } r_{ci} = 6.3\ mm$$

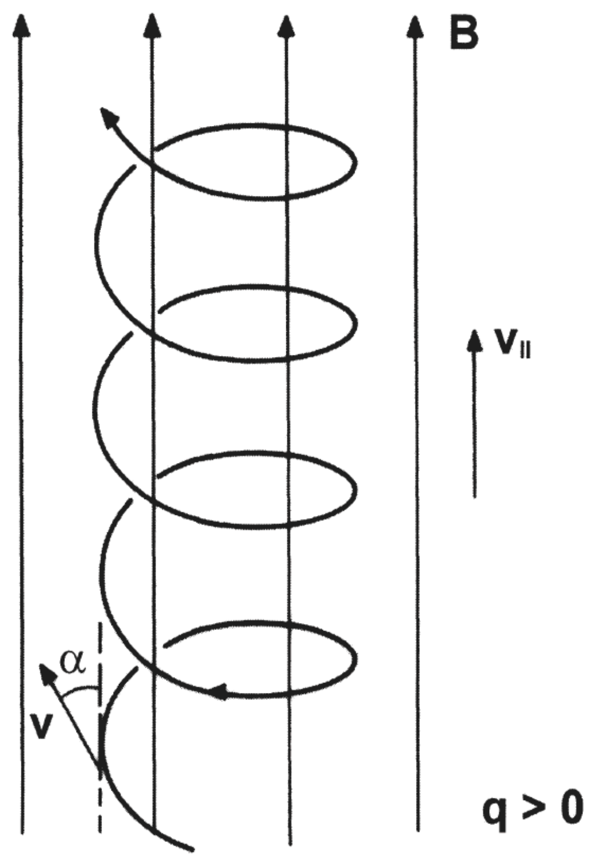
  - Solar corona ($m_i = 1.67 \times 10^{-27}\ kg;\ B_0 = 0.1\ T;\ T = 1 \times 10^6\ K$)

$$f_{\rm ce} = 2.8\ GHz,\ f_{\rm ci} = 1.5\ MHz,\ r_{ce} = 0.22\ mm \text{ and } r_{ci} = 9.5\ mm$$

- **When $r_{ce}, r_{ci} \ll L$ (plasma size), the electrons/ions are said to be magnetized**

Single particle orbits: the trajectories of charged particles in an uniform and static magnetic field

- **The trajectory of the particle is given by the superposition of a uniform motion along $\mathbf{B}_0$ and a cyclotron motion perpendicular to $\mathbf{B}_0$**
  - The particle describes a helix



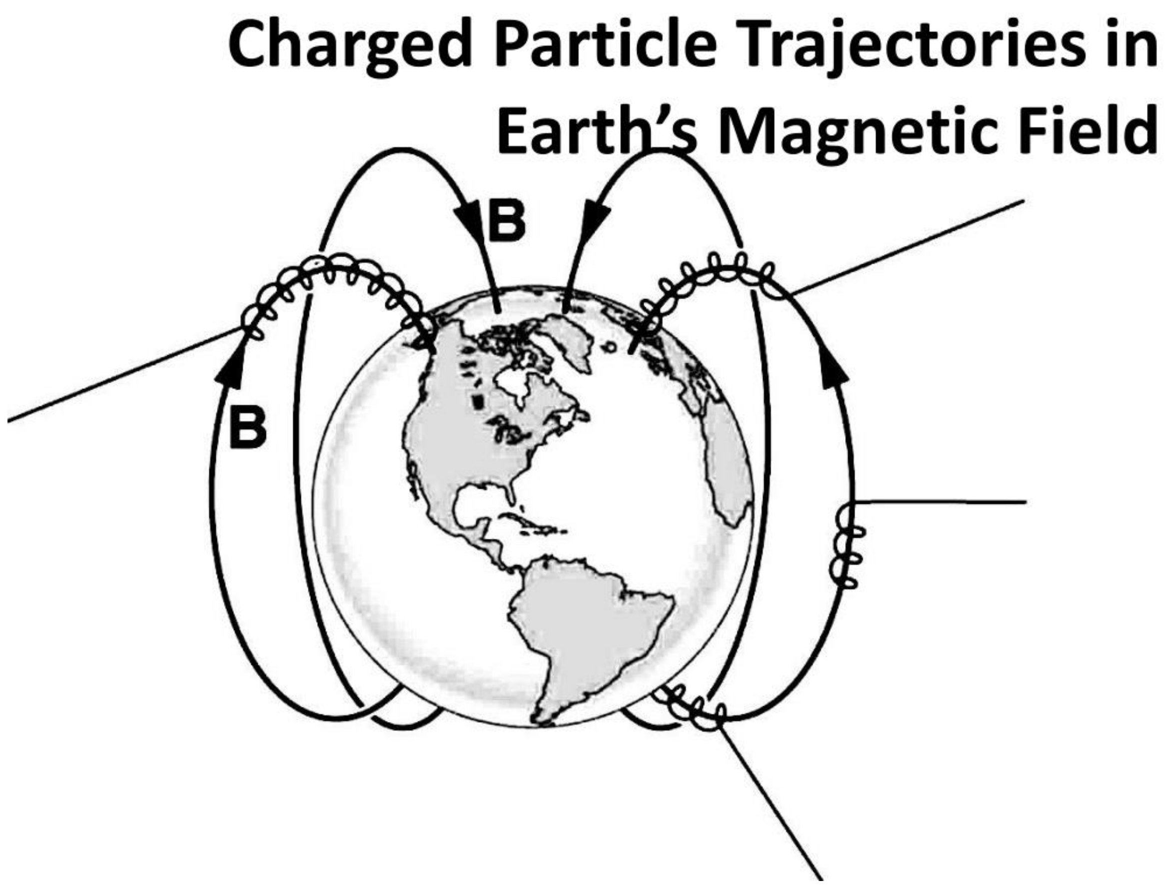
- **The angle between $\mathbf{B}_0$ and the direction of the particle motion is called the pitch angle**

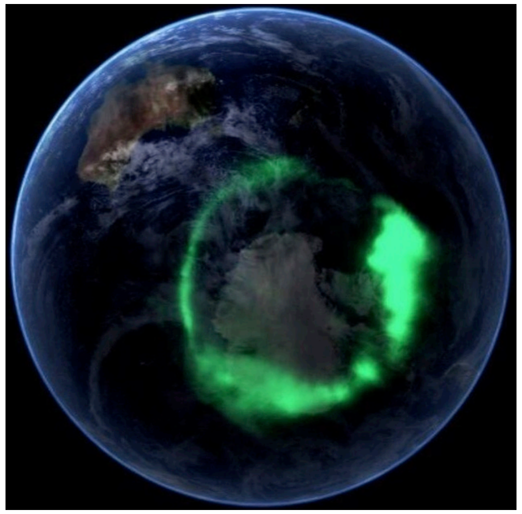
$$\alpha = \sin^{-1}\left(\frac{v_\perp}{v}\right) = \tan^{-1}\left(\frac{v_\perp}{v_\parallel}\right)$$

  - Here, $v = \sqrt{v_\parallel^2 + v_\perp^2}$ is the total speed of the particle
  - When $v_\parallel = 0$ and $v_\perp \neq 0$, then $\alpha = \pi/2$ (Circular/Cyclotron Motion)
  - When $v_\parallel \neq 0$ and $v_\perp = 0$, then $\alpha = 0$ (Uniform Rectilinear Motion)

The particle orbit theory is well suited to study the trajectory of charged particles entering Earth's atmosphere

- **Charged particles arriving to Earth's atmosphere are deflected towards the poles by the terrestrial magnetic field**
  - The collision of these charged particles with the air molecules in the atmosphere gives rise to the so-called auroras





Magnetic moment associated with the cyclotron motion

- **The magnetic moment due to a circulating current I is normal to the area A**

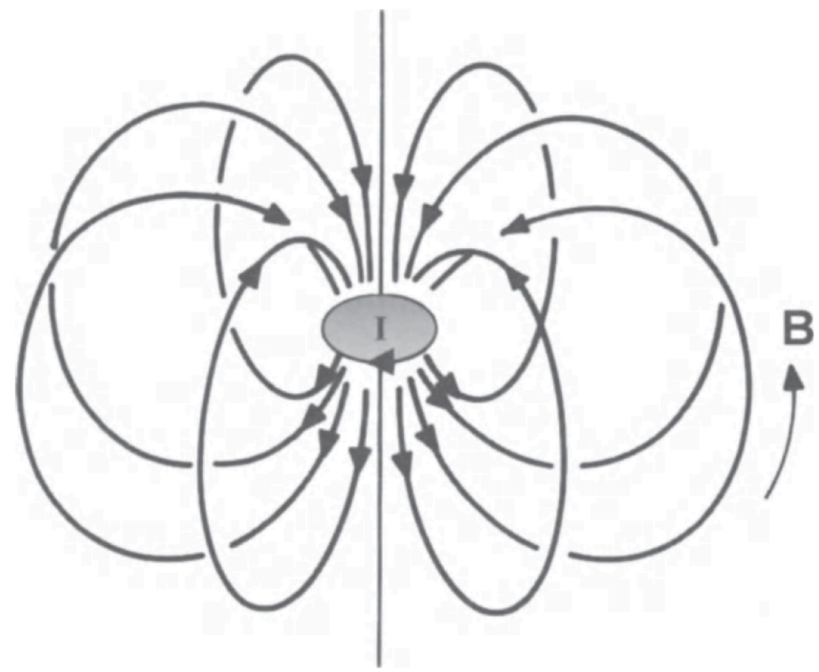
$$|\mathbf{m}| = I\,A$$

  - The current due to cyclotron motion is

$$I = \frac{|q|}{T_c} = \frac{|q|\Omega_c}{2\pi}$$

  - The area of the current loop is

$$A = \pi r_c^2$$



- **Therefore, the magnetic moment becomes**

$$|\mathbf{m}| = \frac{|q|\Omega_c}{2\pi}\pi r_c^2 = \frac{1}{2}|q|\Omega_c r_c^2 = \frac{\frac{1}{2}mv_\perp^2}{B_0} = \frac{W_\perp}{B_0}$$

- **In vector form, the magnetic moment associated with the cyclotron motion is**

$$\mathbf{m} = -\frac{W_\perp}{B_0^2}\mathbf{B_0}$$

Magnetization current associated with the cyclotron motion

- **The magnetization $\mathbf{M}$ due to the cyclotron motion of several various particles is**

$$\mathbf{M} = \frac{1}{V}\sum_{j=1}^{N}\mathbf{m_j} = \frac{N\,\mathbf{m}}{V} = n\mathbf{m} \qquad \rightarrow \qquad \mathbf{M} = -\frac{nW_\perp}{B_0^2}\mathbf{B_0}$$

- **From classical electrodynamics, the magnetization current is $\mathbf{J_M} = \nabla\times\mathbf{M}$. Writing the total current density as $\mathbf{J_{total}} = \mathbf{J} + \mathbf{J_M}$, where $\mathbf{J}$ is the current due to free charges, the Ampère-Maxwell equation becomes**

$$\nabla\times\mathbf{B} = \mu_0\left(\mathbf{J} + \mathbf{J_M} + \epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right) = \mu_0\left(\mathbf{J} + \nabla\times\mathbf{M} + \epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)$$

$$\nabla\times\left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = \mathbf{J} + \epsilon_0\frac{\partial\mathbf{E}}{\partial t} \qquad \rightarrow \qquad \nabla\times\mathbf{H} = \mathbf{J} + \epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$, where $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$

- **A simple linear relation between $\mathbf{B}$ and $\mathbf{H}$ exists when $\mathbf{M}$ is proportional to $\mathbf{B}$ or $\mathbf{H}$**
  - E.g. $\mathbf{M} = \chi_m\mathbf{H}$, where χ_m is the magnetic susceptibility of the medium

- **In a plasma, however, $M \propto 1/B$ (non-linear). Therefore, it is NOT convenient to treat a plasma as a magnetic medium**

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Single particle orbits: the trajectories of charged particles in uniform and static electric and magnetic fields

- **For the case in which $\mathbf{E}_{\mathrm{ext}} = \mathbf{E_0}$ and $\mathbf{B}_{\mathrm{ext}} = \mathbf{B_0}$ are uniform and static, one has**

$$m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{E_0} + \mathbf{v} \times \mathbf{B_0}\right)$$

- **Decompose $\mathbf{v}$ and $\mathbf{E_0}$ in their parallel and perpendicular (to $\mathbf{B_0}$) components**

$$m\frac{d\mathbf{v}_{||}}{dt} + m\frac{d\mathbf{v}_{\perp}}{dt} = q\left(\mathbf{E}_{\mathbf{0},||} + \mathbf{E}_{\mathbf{0},\perp} + \mathbf{v}_{\perp} \times \mathbf{B_0}\right)$$

  - Parallel direction

$$m\frac{d\mathbf{v}_{||}}{dt} = q\mathbf{E}_{\mathbf{0},||} \quad \rightarrow \quad \mathbf{v}(t) = \mathbf{v}_{||}(0) + \frac{q}{m}\mathbf{E}_{\mathbf{0},||}\, t \quad \rightarrow \quad \mathbf{r}_{||}(t) = \mathbf{r}_{||}(0) + \mathbf{v}_{||}(0)t + \frac{q}{2m}\mathbf{E}_{\mathbf{0},||}\, t^2$$

  - Perpendicular direction

$$m\frac{d\mathbf{v}_{\perp}}{dt} = q\left(\mathbf{E}_{\mathbf{0},\perp} + \mathbf{v}_{\perp} \times \mathbf{B_0}\right)$$

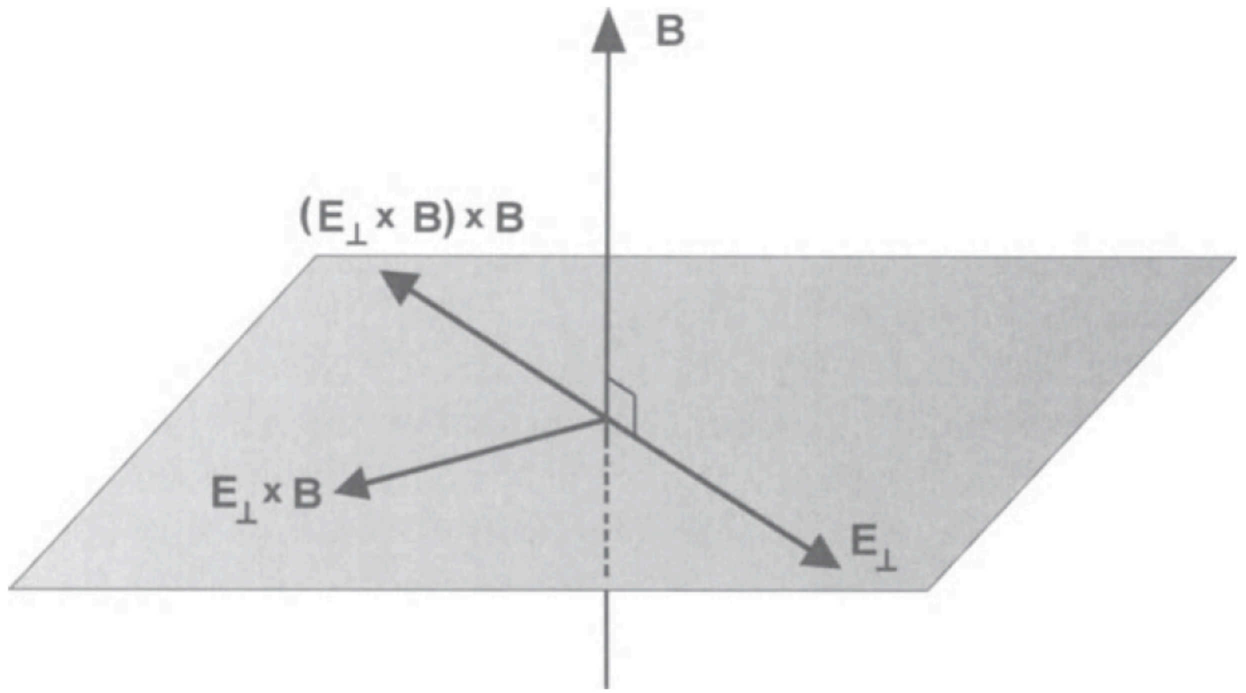
Single particle orbits: the trajectories of charged particles in uniform and static electric and magnetic fields

- **To solve the perpendicular equation, let's change referencial:** $\mathbf{v}_\perp(t) = \mathbf{v_c}(t) + \mathbf{v_{ExB}}$

$$m\frac{d\mathbf{v_c}}{dt} = q\left(\mathbf{E_{0,\perp}} + \mathbf{v_c}\times\mathbf{B_0} + \mathbf{v_{ExB}}\times\mathbf{B_0}\right)$$

- **Choose the constant velocity $\mathbf{v_{ExB}}$ as**

$$\mathbf{v_{ExB}} = \frac{\mathbf{E_{0,\perp}}\times\mathbf{B_0}}{B_0^2}$$



- **The equation of motion becomes**

$$\frac{d\mathbf{v_c}}{dt} = \frac{q}{m}\mathbf{v_c}\times\mathbf{B_0}$$

  - The solution of this equation is the cyclotron motion

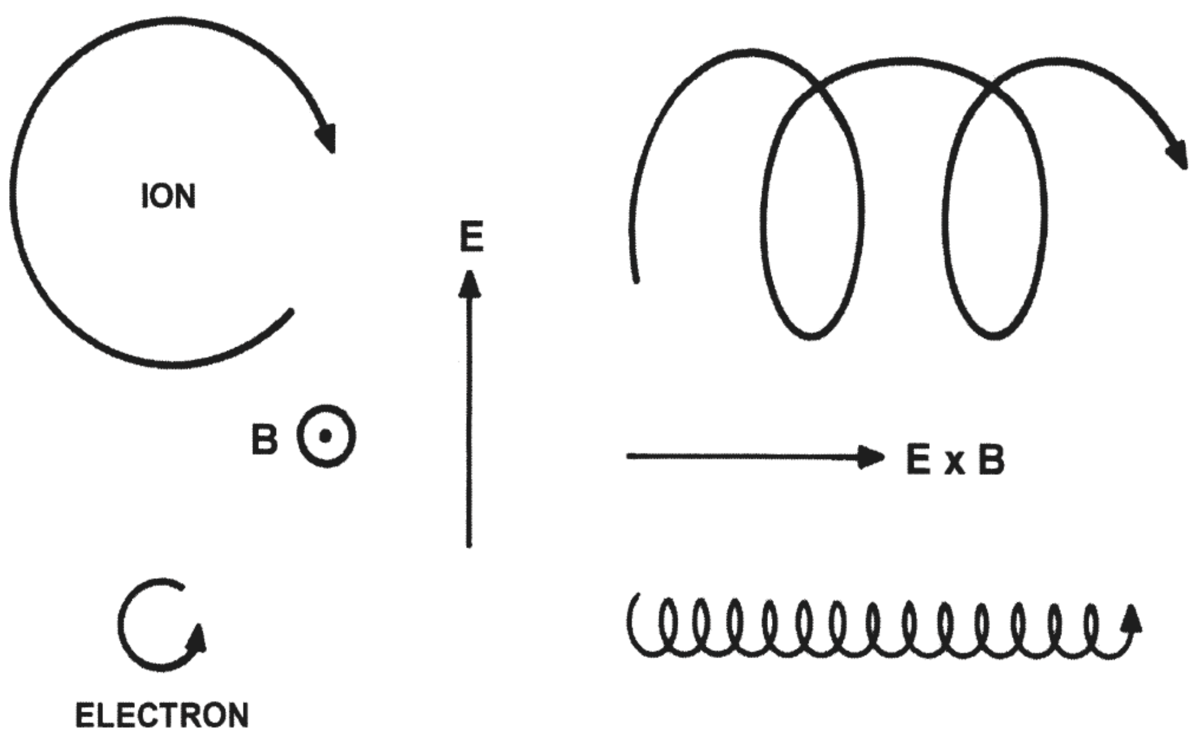
$$\mathbf{v_c}(t) = \mathbf{\Omega_c}\times\mathbf{r_c}(t)$$

Single particle orbits: the trajectories of charged particles in uniform and static electric and magnetic fields

- **Therefore, the solution of this problem is**

$$\mathbf{v}(t) = \mathbf{\Omega_c} \times \mathbf{r_c}(t) + \mathbf{v_{ExB}} + \mathbf{v}_{\|}(0) + \frac{q\,\mathbf{E}_{\mathbf{0},\|}}{m}t$$

- **The constant velocity $\mathbf{v_{ExB}} = \mathbf{E}_{\mathbf{0},\perp} \times \mathbf{B_0}/B_0^2$ is termed the ExB drift velocity**
  - Note that $\mathbf{v_{ExB}}$ is independent of the particle mass and charge
  - Since $\mathbf{E}_{\mathbf{0},\|} \times \mathbf{B_0} = 0$, one can also write $\mathbf{v_{ExB}} = \mathbf{E_0} \times \mathbf{B_0}/B_0^2$



Drift due to an external force

- **For the case in which, in addition to the EM fields, there is a force $\mathbf{F}$ acting on the particle, the equation of motion becomes**

$$m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{E_0} + \mathbf{v}\times\mathbf{B_0}\right) + \mathbf{F}$$

- **The effect of the force is, in a formal sense, analogous to the effect of $\mathbf{E_0}$**

$$\mathbf{v_F} = \frac{\mathbf{F}\times\mathbf{B_0}}{qB_0^2}$$

- **In the case of a uniform gravitational field ($\mathbf{F} = m\mathbf{g}$), the drift velocity is**

$$\mathbf{v_F} = \frac{m}{q}\frac{\mathbf{g}\times\mathbf{B_0}}{B_0^2}$$

- **Associated to the gravitational drift, there is an electric current density**

$$\mathbf{J_g} = \frac{1}{\delta V}\sum_j q_j\,\mathbf{v_j} = \frac{1}{\delta V}\left(\sum_j m_j\right)\frac{\mathbf{g}\times\mathbf{B_0}}{B_0^2} = \rho_m\frac{\mathbf{g}\times\mathbf{B_0}}{B_0^2}$$

  - This current contributes to the so-called equatorial electrojet

Exercise

- **What happens with the ExB drift velocity when the magnetic field tends to zero while the electric field remains finite? What is the validity of the ExB drift expression?**

PGF5112 - Plasma Physics I

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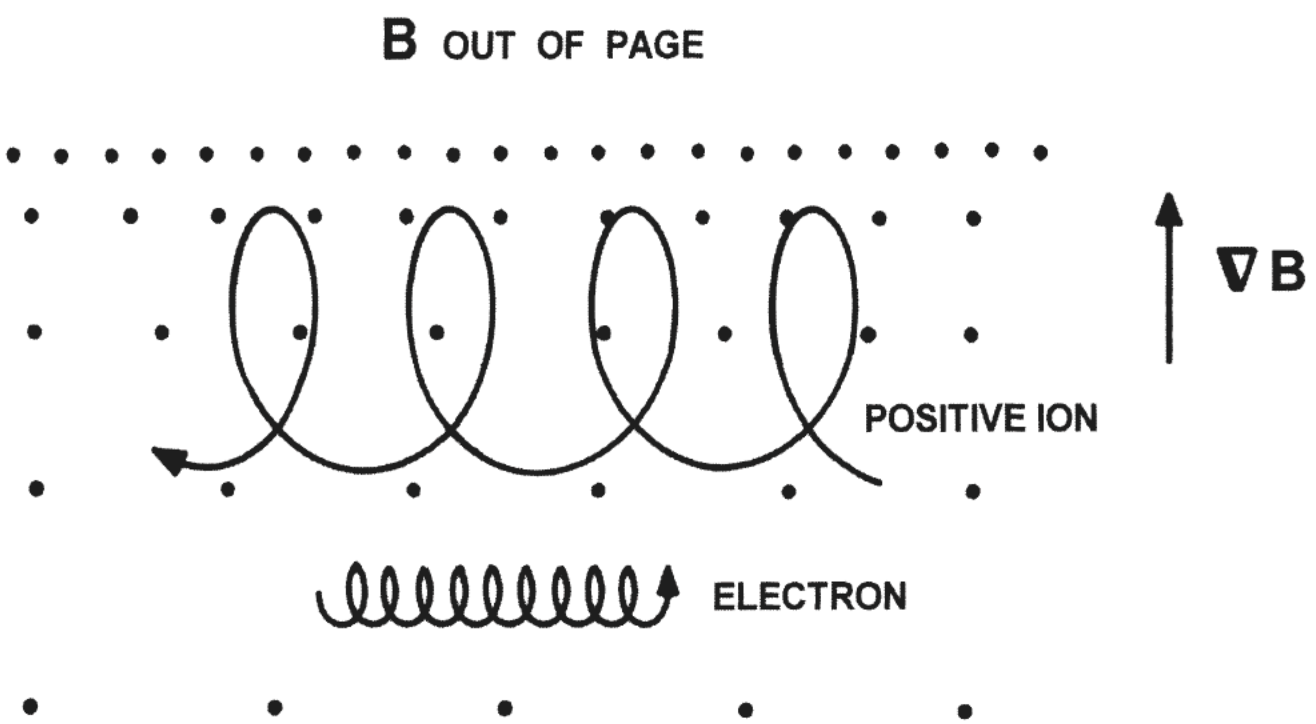
Drift due to magnetic field gradient (physical insight)

- **One can expect that if the magnetic field varies over the Larmor radius, a drift velocity might arise**

Larmor radius

$$r_c = \frac{m v_\perp}{|q| B_0}$$



Magnetic force due to ∇B_0

$$\langle \mathbf{F} \rangle_L = -|\mathbf{m}| \nabla B_0$$

- **The magnetic drift associated to the gradient of the magnetic field (∇B_0) is**

$$\mathbf{v}_{\nabla \mathbf{B}} = \frac{\langle \mathbf{F} \rangle_L \times \mathbf{B_0}}{q B_0^2} = -\frac{|\mathbf{m}|}{q} \frac{\nabla B_0 \times \mathbf{B_0}}{B_0^2}$$

Drift due to magnetic field curvature (physical insight)

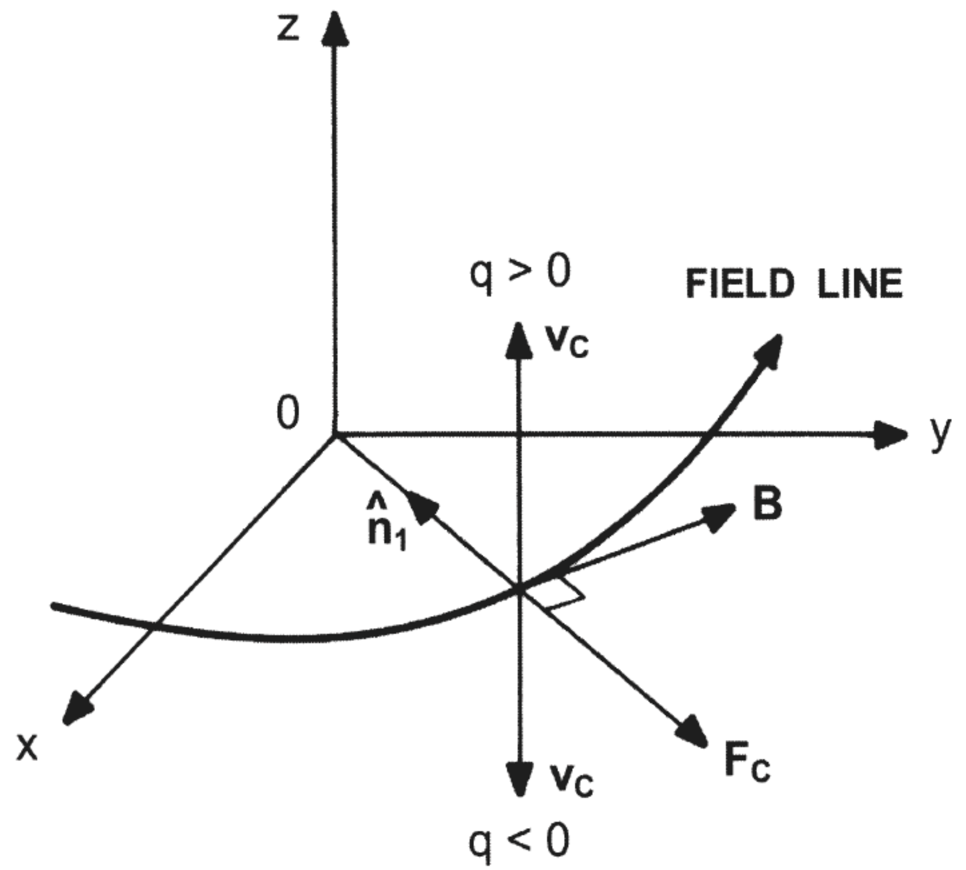
- **One can also expect that if the magnetic field direction varies over the Larmor radius, a drift velocity might arise**

Magnetic force due to the $\mathbf{B_0}$-curvature

$$\langle \mathbf{F} \rangle_{\mathrm{L}} = -\frac{m v_{\parallel}^2}{R} \hat{\mathbf{n}}_1$$

- **Magnetic drift due to the $\mathbf{B}_0$-curvature**

$$\mathbf{v}_{\mathbf{curv}} = \frac{\langle \mathbf{F} \rangle_{\mathrm{L}} \times \mathbf{B_0}}{q B_0^2} = -\frac{m v_{\parallel}^2}{Rq} \frac{\hat{\mathbf{n}}_1 \times \mathbf{B_0}}{B_0^2}$$



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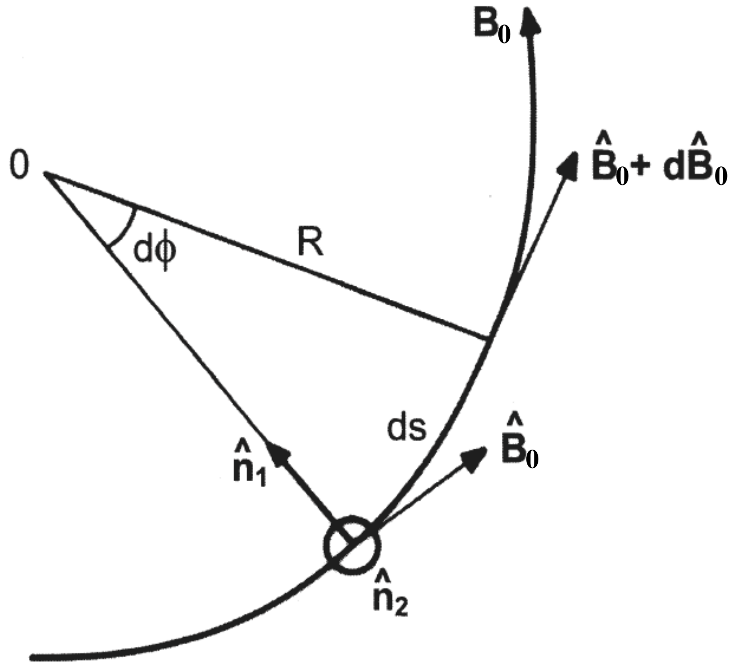
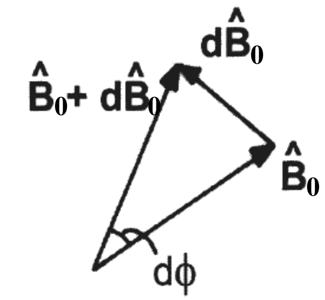




Magnetic force due to the $\mathbf{B_0}$-curvature

$$\langle \mathbf{F} \rangle_{\mathrm{L}} = -\frac{mv_{\parallel}^2}{R}\hat{\mathbf{n}}_1$$

- **Magnetic drift due to the $\mathbf{B_0}$-curvature**

$$\mathbf{v_{curv}} = \frac{\langle \mathbf{F} \rangle_{\mathrm{L}} \times \mathbf{B_0}}{qB_0^2} = -\frac{mv_{\parallel}^2}{Rq}\frac{\hat{\mathbf{n}}_1 \times \mathbf{B_0}}{B_0^2}$$

$$\mathbf{v_{curv}} = -\frac{mv_{\parallel}^2}{q}\frac{(\mathbf{B_0} \cdot \nabla)\mathbf{B_0} \times \mathbf{B_0}}{B_0^4}$$

$$ds = Rd\phi \qquad \hat{\mathbf{B}}_0 = \frac{\mathbf{B_0}}{B_0}$$

$$|d\hat{\mathbf{B}}_0| = |\hat{\mathbf{B}}_0| d\phi \qquad d\hat{\mathbf{B}}_0 = \hat{\mathbf{n}}_1 d\phi$$

$$\frac{\hat{\mathbf{n}}_1}{R} = \frac{d\hat{\mathbf{B}}_0}{ds}$$

$$\frac{\hat{\mathbf{n}}_1}{R} = (\hat{\mathbf{B}}_0 \cdot \nabla)\hat{\mathbf{B}}_0 = \frac{(\mathbf{B_0} \cdot \nabla)\mathbf{B_0}}{B_0^2}$$

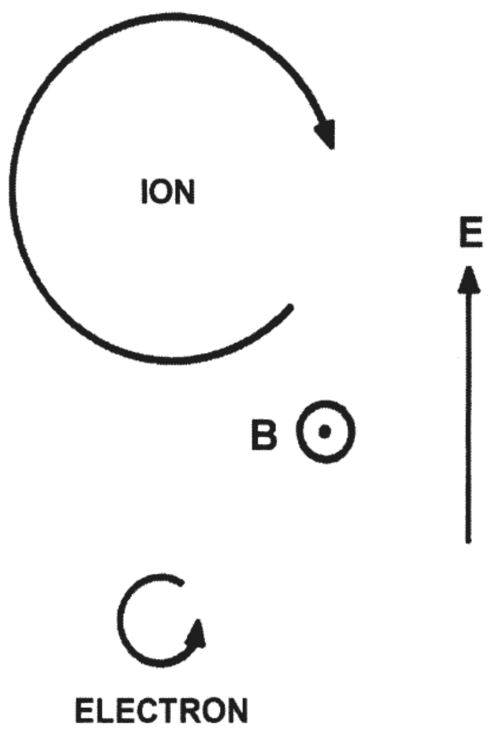
PGF5112 - Plasma Physics I

- **Single particle orbits: the motion of charged particles in electromagnetic fields**
  - *Introduction*
  - *Uniform and static electric field*
  - *Uniform and static magnetic field*
  - *Uniform and static electric and magnetic fields*
  - *Non-uniform and static magnetic field (physical insight)*
  - *Non-uniform and static electric field (physical insight)*
  - *Non-uniform and time-dependent electric and magnetic fields*

Drift due to electric field non-uniformities (physical insight)

- **One can also expect that if the electric field varies over the Larmor radius, a drift velocity might arise**

- **At first order, in which the electric field varies linearly, a charged particle executing a cyclotron motion pass by a region with stronger $\mathbf{E_0}$-field and pass by a region with weaker $\mathbf{E_0}$-field**
  - On average, the first order correction cancels out
  - Therefore, $\mathbf{E_0}$-field non-uniformities are important only as 2nd order corrections



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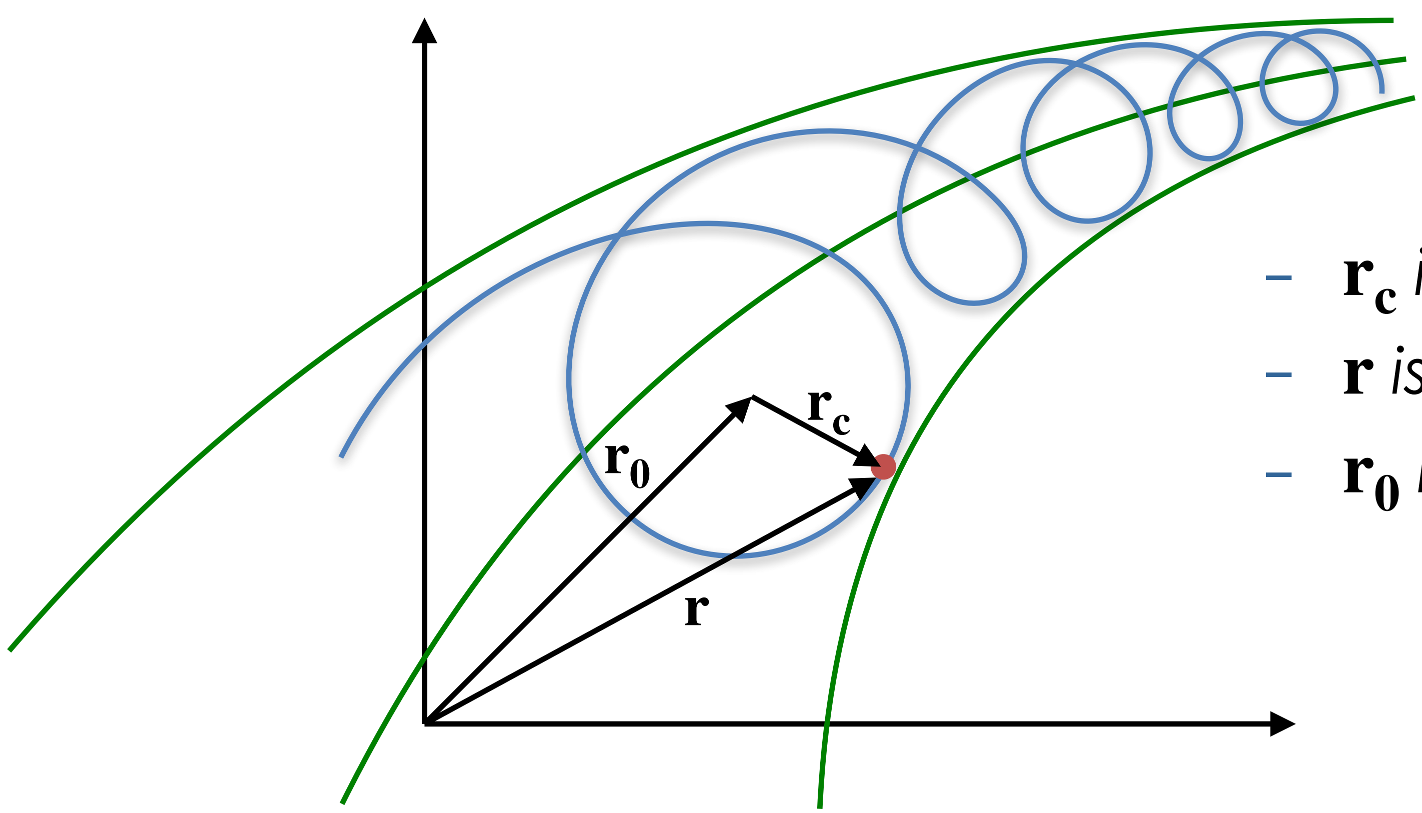
The trajectories of charged particles in non-uniform and time-dependent electric and magnetic fields

- To study the trajectory of charged particles in non-uniform and time-dependent electric and magnetic fields, let's expand the fields around a position $\mathbf{r_0}$, which is the guiding center position of the particle

$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}(\mathbf{r_0},t) + \left[(\mathbf{r}-\mathbf{r_0})\cdot\nabla\right]\mathbf{B}(\mathbf{r},t)\Big|_{\mathbf{r}=\mathbf{r_0}} + O^2$$

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r_0},t) + \left[(\mathbf{r}-\mathbf{r_0})\cdot\nabla\right]\mathbf{E}(\mathbf{r},t)\Big|_{\mathbf{r}=\mathbf{r_0}} + \frac{1}{2}\left[(\mathbf{r}-\mathbf{r_0})\cdot\nabla\right]^2\mathbf{E}(\mathbf{r},t)\Big|_{\mathbf{r}=\mathbf{r_0}} + O^3$$



- $\mathbf{r_c}$ *is the Larmor/cyclotron radius*
- $\mathbf{r}$ *is the instantaneous particle position*
- $\mathbf{r_0}$ *is the guiding center position*

$$\mathbf{r}(t) = \mathbf{r_0}(t) + \mathbf{r_c}(t)$$

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$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}(\mathbf{r_0},t) + \left[(\mathbf{r}-\mathbf{r_0})\cdot\nabla\right]\mathbf{B}(\mathbf{r},t)\Big|_{\mathbf{r}=\mathbf{r_0}} + O^2$$

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r_0},t) + \left[(\mathbf{r}-\mathbf{r_0})\cdot\nabla\right]\mathbf{E}(\mathbf{r},t)\Big|_{\mathbf{r}=\mathbf{r_0}} + \frac{1}{2}\left[(\mathbf{r}-\mathbf{r_0})\cdot\nabla\right]^2\mathbf{E}(\mathbf{r},t)\Big|_{\mathbf{r}=\mathbf{r_0}} + O^3$$

- **Using the definition of the instantaneous particle position:** $\mathbf{r}(t) = \epsilon^0\mathbf{r_0}(t) + \epsilon^1\mathbf{r_c}(t)$
  - Here, ϵ is a parameter introduced to explicit the order of the expansion
  - Therefore, the fields become (in a simplified notation)

$$\mathbf{B}(\mathbf{r},t) = \epsilon^0\mathbf{B_0} + \epsilon^1\left(\mathbf{r_c}\cdot\nabla\right)\mathbf{B_0}$$

$$\mathbf{E}(\mathbf{r},t) = \epsilon^0\mathbf{E_0} + \epsilon^1\left(\mathbf{r_c}\cdot\nabla\right)\mathbf{E_0} + \frac{\epsilon^2}{2}\left(\mathbf{r_c}\cdot\nabla\right)^2\mathbf{E_0}$$

  - Note that $\mathbf{E_0} = \mathbf{E}(\mathbf{r_0},t)$ and $\mathbf{B_0} = \mathbf{B}(\mathbf{r_0},t)$ still depend on time

The trajectories of charged particles in non-uniform and time-dependent electric and magnetic fields

- **Let's now write the particle velocity v making the zeroth, first and second order contributions explicit**

$$\mathbf{v}(t) = \epsilon^0 \mathbf{v_0}(t) + \epsilon^1 \mathbf{w_1} + \epsilon^2 \mathbf{w_2}$$

  - Note that the first and second corrections are assumed to be constant

- **Combining all these assumptions with the charged particle equation of motion:**

$$\epsilon^0 \frac{d\mathbf{v_0}}{dt} = \frac{q}{m}\left[\mathbf{E} + \left(\epsilon^0 \mathbf{v_0}(t) + \epsilon^1 \mathbf{w_1} + \epsilon^2 \mathbf{w_2}\right) \times \mathbf{B}\right]$$

$$\epsilon^0 \frac{d\mathbf{v_0}}{dt} = \frac{q}{m}\left[\epsilon^0 \mathbf{E_0} + \epsilon^1 \left(\mathbf{r_c} \cdot \nabla\right) \mathbf{E_0} + \frac{\epsilon^2}{2}\left(\mathbf{r_c} \cdot \nabla\right)^2 \mathbf{E_0}\right] +$$

$$+ \frac{q}{m}\left[\left(\epsilon^0 \mathbf{v_0}(t) + \epsilon^1 \mathbf{w_1} + \epsilon^2 \mathbf{w_2}\right) \times \left(\epsilon^0 \mathbf{B_0} + \epsilon^1 \left(\mathbf{r_c} \cdot \nabla\right) \mathbf{B_0}\right)\right]$$

References

- **The single particle orbit theory**
  - *Bittencourt: Ch. 2, 3 and 4*