PGF5112 - Plasma Physics I

By

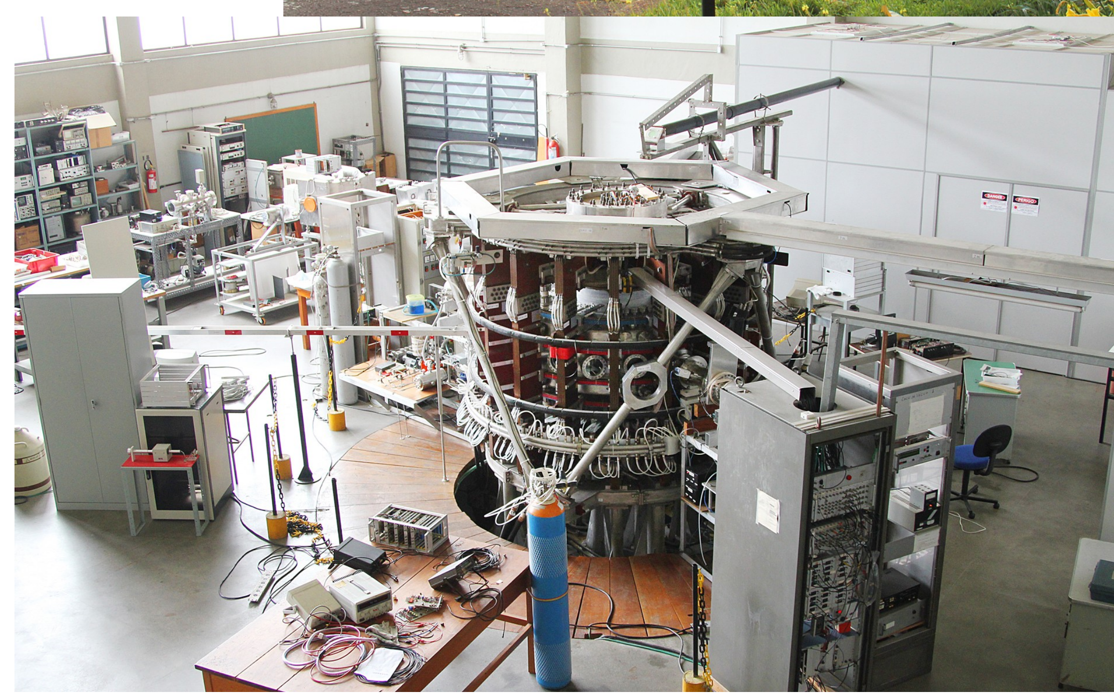
Prof. Gustavo Paganini Canal

Plasma Physics Laboratory

Department of Applied Physics

Institute of Physics

University of São Paulo



Postgraduate course ministered remotely from the

Institute of Physics of the University of São Paulo

IFUSP
Instituto de Física da USP

e-mail: canal@if.usp.br

São Paulo - SP, 30 March 2021

PGF5112 - Plasma Physics I

- **Macroscopic features of plasmas**
  - *Quasi-neutrality*
  - *Debye shielding*
  - *Plasma oscillations*
  - *The plasma definition criteria*

- **Degree of ionization and the Saha equation**

- **Macroscopic features of plasmas**
  - *Quasi-neutrality*
  - *Debye shielding*
  - *Plasma oscillations*
  - *The plasma definition criteria*

- **Degree of ionization and the Saha equation**

Macroscopic features of plasmas: *quasi*-neutrality

- **Plasmas have a natural tendency of remaining electrically neutral**

$$\sum_j Z_j n_j = n_e \quad \text{(Charge neutrality condition)}$$

- **If there exists any charge imbalance within a small spherical region in a plasma:**
  - An electric field is established
  - Electrons are accelerated in the direction of the positive charges
  - In a short time interval, *quasi*-neutrality is restored

Net electric charge due to charge imbalance: $Q = \frac{4}{3}\pi a^3 (n_i - n_e) e$

Electric potential at the surface of the spherical region: $\phi = \frac{1}{4\pi\epsilon_0}\frac{Q}{a} = \frac{e(n_i - n_e)}{3\epsilon_0} a^2$

Macroscopic features of plasmas: *quasi*-neutrality

- **For a plasma with $n_{i0} = n_{e0} = 1\text{x}10^{20}\ m^{-3}$ and a charge imbalance** $\delta n_i / n_{i0} = (n_i - n_{i0})/n_{i0} = 1\ \%$ **within a spherical region of 1 mm of radius:**

$$Q = \frac{4}{3}\pi a^3 (n_i - n_e) e = \frac{4}{3}\pi a^3 (n_{i0} + \delta n_i - n_{e0}) e = \frac{4\pi e n_{i0}}{3}\frac{\delta n_i}{n_{i0}} a^3 = 6.7 \times 10^{-10}\ C$$

$$\phi = \frac{1}{4\pi\epsilon_0}\frac{Q}{a} = \frac{e(n_i - n_e)}{3\epsilon_0} a^2 = 6.0\ kV \qquad E_r = \frac{1}{4\pi\epsilon_0}\frac{Q}{a^2} = \frac{e(n_i - n_e)}{3\epsilon_0} a = 6.0\ MV/m$$

$$|\mathbf{F}| = |e\mathbf{E}| = 1.6 \times 10^{-19} \times 6.0 \times 10^6 \approx 9.6 \times 10^{-13}\ N$$ **(Force doesn't seem so strong)**

$$|\mathbf{a}| = \frac{|\mathbf{F}|}{m_e} = \frac{|e\mathbf{E}|}{m_e} = \frac{1.6 \times 10^{-19} \times 6.0 \times 10^6}{9.11 \times 10^{-31}} \approx 1 \times 10^{18}\ m/s^2$$

(Acceleration is huge!!!)

- **Macroscopic features of plasmas**
  - *Quasi-neutrality*
  - *Debye shielding*
  - *Plasma oscillations*
  - *The plasma definition criteria*

- **Degree of ionization and the Saha equation**

Macroscopic features of plasmas: Debye shielding

- **Charge imbalance occurs naturally in plasmas only in a small region whose typical size is of the order of the so-called Debye length (see Bittencourt ch. 11, section 2)**

$$\lambda_D = \sqrt{\frac{\epsilon_0 k_B T}{n_0 e^2}}$$

- **The Debye length is the typical distance at which the particle electric potencial energy balances its thermal energy**
  - The electric potential energy tend to restore *quasi*-neutrality
  - The thermal energy of the particles tend to break *quasi*-neutrality

Macroscopic features of plasmas: Debye shielding

- **Let's isolate one single charge from a (neutral) plasma and see how this charge interacts with all the other particles**

- **From Poisson's equation:**

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0} = -\frac{\left[Z_i\, e\, n_i(\mathbf{r}) - e\, n_e(\mathbf{r})\right]}{\epsilon_0} - \frac{q_t}{\epsilon_0}\delta(\mathbf{r})$$

- **From Boltzmann's equation (see Bittencourt ch. 7, section 5)**

$$n_j(\mathbf{r}) = n_{j0} \exp\left[-\frac{U(\mathbf{r})}{k_B\, T_j}\right] = n_{j0} \exp\left[-\frac{q_j\, \phi(\mathbf{r})}{k_B\, T_j}\right]$$

- **Suppose that** $n_e(\mathbf{r} \to \infty) = n_i(\mathbf{r} \to \infty) = n_0$

$$\nabla^2 \phi + \frac{e n_0}{\epsilon_0}\left[Z_i \exp\left(-\frac{Z_i e \phi}{k_B T_i}\right) - \exp\left(\frac{e\phi}{k_B T_e}\right)\right] = -\frac{q_t}{\epsilon_0}\delta(\mathbf{r})$$

Macroscopic features of plasmas: Debye shielding

- **Let's suppose that the perturbing electrostatic potential due to the test charge is weak so that the electrostatic potential energy of the particles is much smaller than their mean thermal energy, i.e.** $q_j\phi(r) \ll k_B T_j$ **:**

$$\exp\left(-\frac{q_j\,\phi}{k_B\,T_j}\right) \approx 1 - \frac{q_j\,\phi}{k_B\,T_j}$$

- **Therefore,** $\nabla^2\phi - \frac{\phi}{\lambda_D^2} = -\frac{q_t}{\epsilon_0}\delta(\mathbf{r})$, **with** $\lambda_D = \sqrt{\frac{\epsilon_0 k_B T_{\mathrm{eff}}}{n_0 e^2}}$ **or** $\lambda_D = 7433\sqrt{\frac{T_{\mathrm{eff}}}{n_0}}$

(Debye length)

$T_{\mathrm{eff}} = \frac{T_e\,T_i}{(Z_i T_e + T_i)}$ **in eV**

n_0 **in m**$^{-3}$

- **Debye length for some particular cases**
  - Tokamaks ($n_0 = 1\times10^{20}\ m^{-3};\ T_e = T_i = 1\times10^{8}\ K$): $\lambda_D = 5\times10^{-5}\ m$
  - Solar corona ($n_0 = 1\times10^{12}\ m^{-3};\ T_e = T_i = 1\times10^{6}\ K$): $\lambda_D = 0.05\ m$

Macroscopic features of plasmas: Debye shielding

- **In spherical coordinates, one has that**

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\phi}{dr}\right]-\frac{\phi}{\lambda_D^2}=-\frac{q_t}{\epsilon_0}\delta(\mathbf{r})$$

- **Let's try to solve this equation with the ansatz** $\phi(r)=\phi_c(r)f(r)=\frac{q_t}{4\pi\epsilon_0}\frac{f(r)}{r}$ **, where** $f(r\to 0)=1$ **and** $\phi(r\to\infty)=0$**.**

- **For** $\mathbf{r}\neq 0$**, one has that:** $\frac{d^2f}{dr^2}=\frac{f}{\lambda_D^2}\ \rightarrow\ f(r)=A\exp\left(\frac{r}{\lambda_D}\right)+B\exp\left(-\frac{r}{\lambda_D}\right)$

- **The condition** $\phi(r\to\infty)=0$ **implies that** $A=0$**, while the condition** $f(r\to 0)=1$ **implies that** $B=1$**. Therefore,**

$$\phi(r)=\phi_c(r)\,\exp\left(-\frac{r}{\lambda_D}\right)=\frac{1}{4\pi\epsilon_0}\frac{q_t}{r}\exp\left(-\frac{r}{\lambda_D}\right)$$

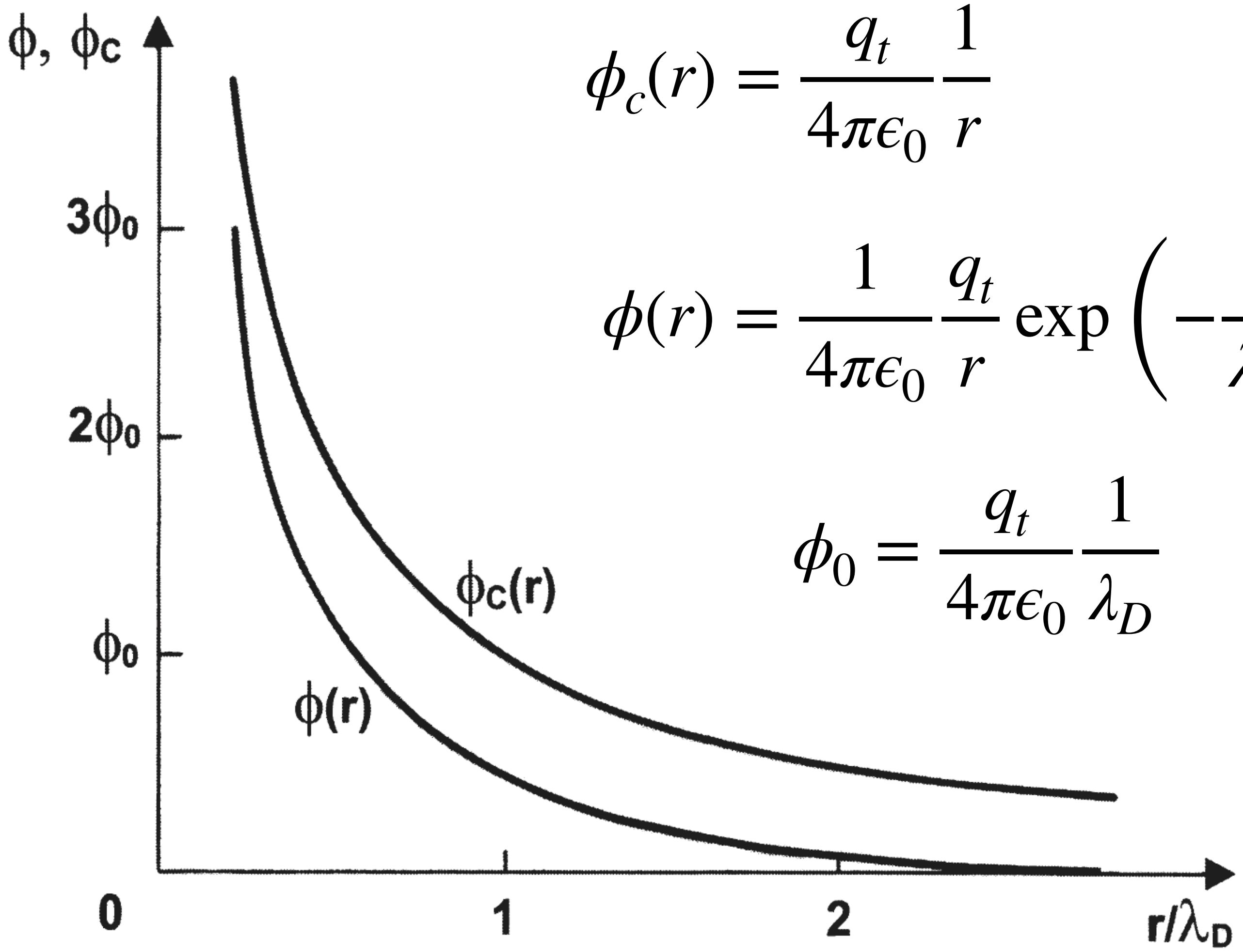
Macroscopic features of plasmas: Debye shielding

- Near the test particle ($r \ll \lambda_D$), the electric potential created by the test charge is given by the Coulomb potential

- Far from the test particle ($r \gg \lambda_D$), the electric potential is significantly smaller that the Coulomb potential of the test charge

- The number of particles that interact collectively with the test charge is about the number of charges within the Debye sphere

$$N_D = \frac{4\pi}{3}\lambda_D^3 n_0$$



$$\phi_c(r) = \frac{q_t}{4\pi\epsilon_0}\frac{1}{r}$$

$$\phi(r) = \frac{1}{4\pi\epsilon_0}\frac{q_t}{r}\exp\left(-\frac{r}{\lambda_D}\right)$$

$$\phi_0 = \frac{q_t}{4\pi\epsilon_0}\frac{1}{\lambda_D}$$

Macroscopic features of plasmas: Debye shielding

- **Close to the test charge, the assumption $q_j\phi(r) \ll k_B T_j$ is unlikely to be fulfilled**

- **Taking $q_t = e$, let's calculate the parameter**

$$\frac{e\phi}{k_B T} = \frac{e^2}{4\pi\epsilon_0 k_B T}\frac{\exp(-r/\lambda_D)}{r}$$

- **If one sets $r = n_0^{-1/3}$ (average distance between charged particles) one has that**

$$\left.\frac{e\phi}{k_B T}\right|_{r=n_0^{-1/3}} = \frac{e^2}{4\pi\epsilon_0 k_B T}\frac{\exp(-n_0^{-1/3}/\lambda_D)}{n_0^{-1/3}}$$

- **Defining the so-called Plasma Parameter $g = \dfrac{1}{n_0\lambda_D^3}$**

$$\left.\frac{e\phi}{k_B T}\right|_{r=n_0^{-1/3}} = \frac{g^{3/2}\exp(-g^3)}{4\pi}$$

- **Therefore, a necessary condition for Debye shielding to occur is $g \ll 1$**

Exercises

- **(1) What is the total charge of the plasma? Is the plasma globally neutral?**

$$Q = \int \rho \, dV \qquad \textbf{(Total charge)}$$

- **(2) If $T_e \gg T_i$, which particle species determines the Debye length? Why?**

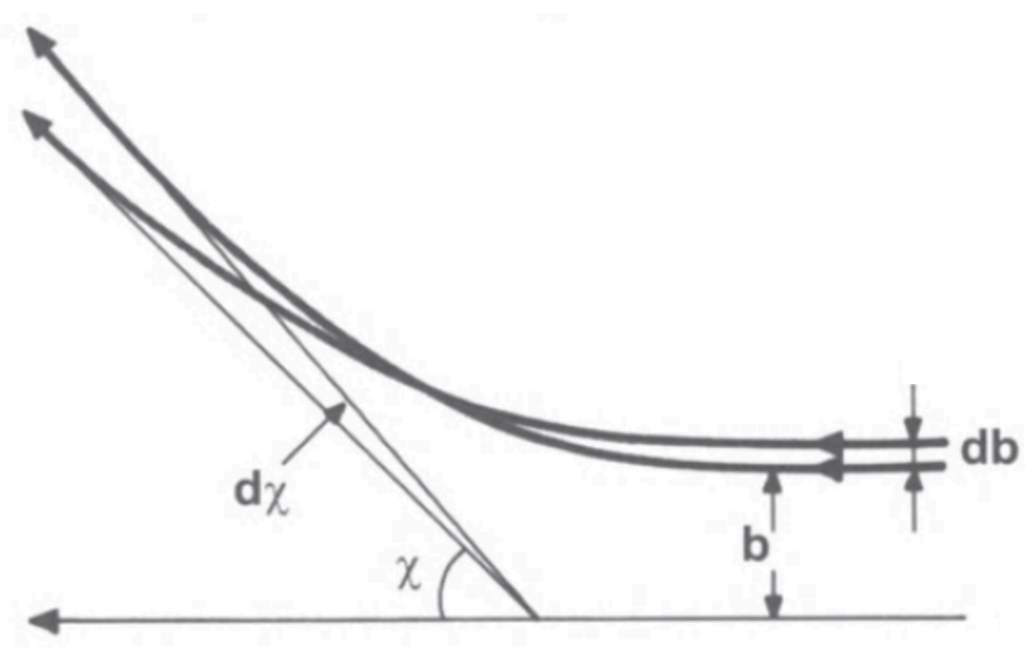
Screening of the Coulomb potential

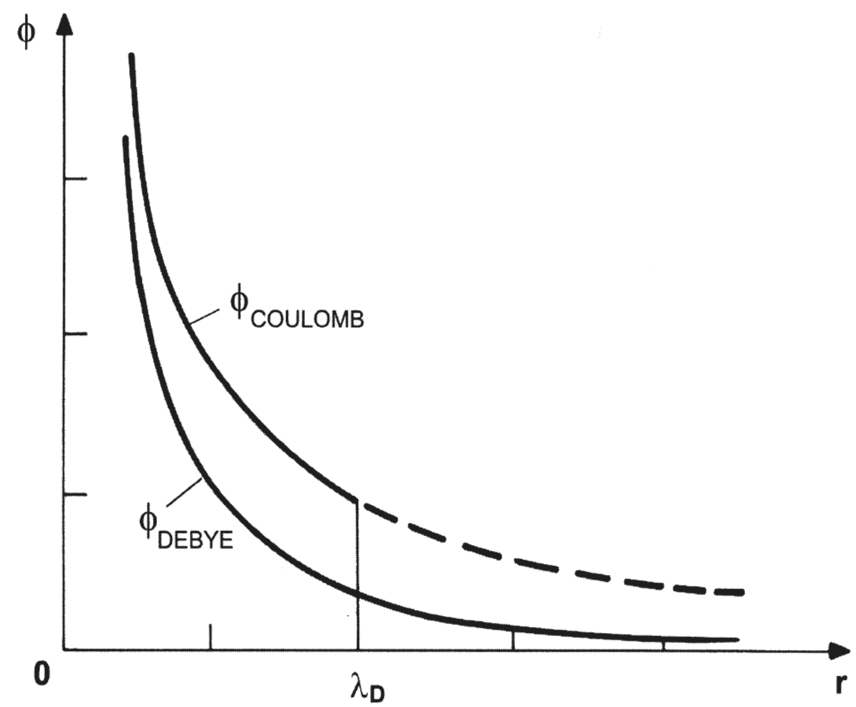
- **Total cross section (considering $\sigma(\chi,\epsilon)d\Omega = b\,db\,d\epsilon$ independent of ϵ)**

$$\sigma_t = \int \sigma(\chi,\epsilon)d\Omega = \int_{b_{\min}}^{b_c}\int_0^{2\pi} b\,db\,d\epsilon = 2\pi\int_{b_{\min}}^{b_c} b\,db = \pi(b_c^2 - b_{\min}^2)$$

- **Deflections with $0 < b < b_0$, which yield scattering angles $\pi/2 < \chi < \pi$ are usually called** *large-angle deflections* **or** *close encounters*

$$\sigma_{t,large} = \pi b_0^2$$





Screening of the Coulomb potential

- **Total cross section (considering $\sigma(\chi,\epsilon)d\Omega = b\,db\,d\epsilon$ independent of ϵ)**

$$\sigma_t = \int \sigma(\chi,\epsilon)d\Omega = \int_{b_{\min}}^{b_c}\int_0^{2\pi} b\,db\,d\epsilon = 2\pi\int_{b_{\min}}^{b_c} b\,db = \pi(b_c^2 - b_{\min}^2)$$

- **Deflections with $0 < b < b_0$, which yield scattering angles $\pi/2 < \chi < \pi$ are usually called** *large-angle deflections* **or** *close encounters*

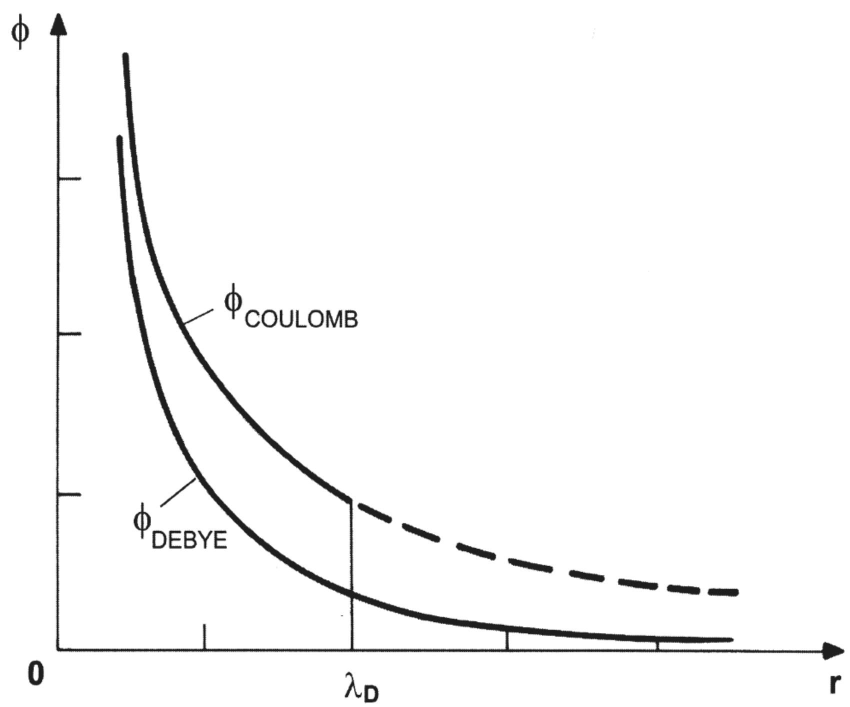
$$\sigma_{t,large} = \pi b_0^2$$

- **Deflections with $b_0 < b < \lambda_D$, which yield scattering angles $\chi < \pi/2$, are usually called** *small-angle deflections*

$$\sigma_{t,small} = \pi(\lambda_D^2 - b_0^2) \approx \pi\lambda_D^2 \qquad (\lambda_D \gg b_0)$$

- **Small-angle deflections are much more frequent than large-angle deflections**

$$\frac{\sigma_{t,small}}{\sigma_{t,large}} = \frac{\lambda_D^2}{b_0^2} - 1 \approx \frac{\lambda_D^2}{b_0^2} \gg 1$$



Momentum transfer cross section due to the Coulomb potential

- **Momentum transfer cross section (see Bittencourt ch. 20, section 8)**

$$\sigma_m = \int (1 - \cos\chi)\sigma(\chi, \epsilon)d\Omega$$

- **For the special case of an isotropic interaction potential**

$$\sigma_m = 2\pi \int_0^{\pi} (1 - \cos\chi)\sigma(\chi)\sin\chi \, d\chi$$

- **Using the result,** $\sigma(\chi) = b_0^2/(1 - \cos\chi)^2$ **one has that**

$$\sigma_m = 4\pi b_0^2 \ln \left[\frac{1}{\sin(\chi_{\min}/2)} \right]$$

- **Using that** $\sin(\chi_c/2) = \left(1 + b_c^2/b_0^2\right)^{-1/2}$ **one obtains** $\sigma_m = 4\pi b_0^2 \ln\Lambda$ **, where** $\Lambda = \dfrac{\lambda_D}{b_0}$

- **It is interesting to note that the parameter** $\Lambda = \dfrac{12\pi\epsilon_0 k_B T}{e^2}\lambda_D = 12\pi n_0 \lambda_D^3 = \dfrac{12\pi}{g} = 9N_D$

PGF5112 - Plasma Physics I

- **Macroscopic features of plasmas**
  - *Quasi-neutrality*
  - *Debye shielding*
  - *Plasma oscillations*
  - *The plasma definition criteria*

- **Degree of ionization and the Saha equation**

Macroscopic features of plasmas: plasma oscillations

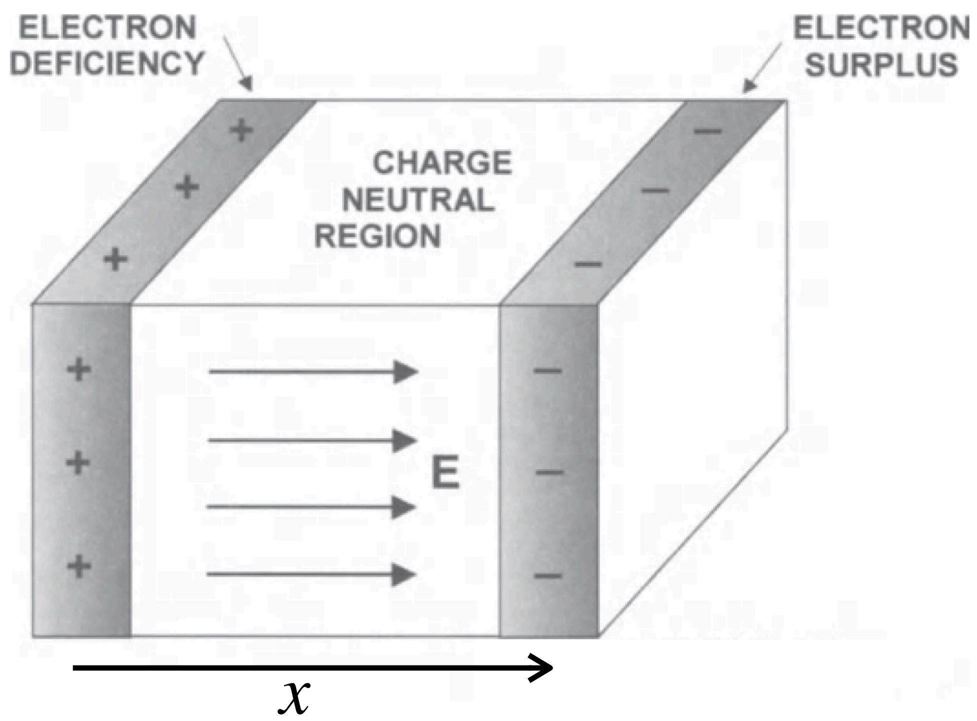
- **The Debye shielding is a dynamic effect**

- **When charge imbalance is created in the plasma, e.g. by thermal fluctuations**
  - A strong electric field arises
  - Electrons accelerate towards positive charges
  - Most of the electrons have small-angle deflections
  - Electrons oscillate around the test particle

From Gauss law: $E_x = \left(\frac{n_0 e}{\epsilon_0} \right) x$

From the motion equation of an electron:

$$\frac{d^2x}{dt^2} = -\frac{eE_x}{m_e} = -\omega_{pe}^2 x \qquad \omega_{pe} = \sqrt{\frac{n_0 e^2}{\epsilon_0 m_e}}$$



Electron plasma frequency (ω_{pe})

Macroscopic features of plasmas: plasma oscillations

- **The average time an electron takes to complete an oscillation depends on its average thermal speed** $v_{th,e} = \sqrt{k_B T_e / m_e}$

- **Note that there is a relation between** ω_{pe}, λ_{De} **and** $v_{th,e}$

$$\lambda_{De} = \sqrt{\frac{\epsilon_0 k_B T_e}{n_0 e^2}} = \sqrt{\frac{\epsilon_0 m_e}{n_0 e^2}\frac{k_B T_e}{m_e}} = \frac{v_{th,e}}{\omega_{pe}} \quad \rightarrow \quad v_{th,e} = \lambda_{De}\omega_{pe}$$

- **Electron plasma frequency**

$$\omega_{pe} = \sqrt{\frac{n_0 e^2}{\epsilon_0 m_e}} \qquad f_{pe} = \frac{\omega_{pe}}{2\pi} = 9.0\sqrt{n_0} \qquad \textbf{(}n_0 \textbf{ in m}^{-3}\textbf{)}$$

- **Electron plasma frequency for some particular cases**
  - Tokamaks ($n_0 = 1 \times 10^{20}\ m^{-3}$): $f_{pe} = 90\ GHz$
  - Solar corona ($n_0 = 1 \times 10^{12}\ m^{-3}$): $f_{pe} = 9\ MHz$

Macroscopic features of plasmas: plasma oscillations

- **The collision between electrons and neutral gas particles might prevent electron plasma oscillations to be established thus preventing the Debye shielding**
  - Typical time between collision between electrons and neutral gas

$$\tau_{en} = \frac{1}{\nu_{en}} = \frac{1}{n_n \sigma_{en} v_{th,e}}$$

  - Taking $\sigma_{en} \approx \pi a_0^2$ (with a_0 being the Bohr radius - Hydrogen) and $v_{th,e} = \sqrt{\frac{k_B T}{m_e}}$

$$\tau_{en} \approx \frac{1 \times 10^{17}}{n_n \sqrt{T}}$$

(given n_i, n_n can be calculated from the Saha equation)

  - Tokamaks ($n_i = 1 \times 10^{20}\ m^{-3};\ T = 1 \times 10^{8}\ K$): $\tau_{en} = 2.4 \times 10^{6}\ s$
  - Solar corona ($n_i = 1 \times 10^{12}\ m^{-3};\ T = 1 \times 10^{6}\ K$): $\tau_{en} = 2 \times 10^{20}\ s$

- **For the cases above, the criterion $\tau_{en} \gg 2\pi/\omega_{pe}$ is well satisfied**

- **Macroscopic features of plasmas**
  - *Quasi-neutrality*
  - *Debye shielding*
  - *Plasma oscillations*
  - *The plasma definition criteria*

- **Degree of ionization and the Saha equation**

Macroscopic features of plasmas: the plasma definition criteria

- **The main difference between ionized gases and plasmas is the presence of collective effects, which are present if**
  - (1) The plasma dimensions be much larger than the Debye length

$$L \gg \lambda_D$$

  - (2) The # of electrons within a Debye sphere be much larger than unity

$$N_D = \frac{4}{3}\pi\lambda_D^3 n_0 \gg 1 \qquad\qquad g = \frac{1}{n_0\lambda_D^3} \ll 1 \quad \textbf{(Plasma Parameter)}$$

  - (3) The average time between collision of electrons and neutral particles be much larger than the time for one electron plasma oscillation to take place

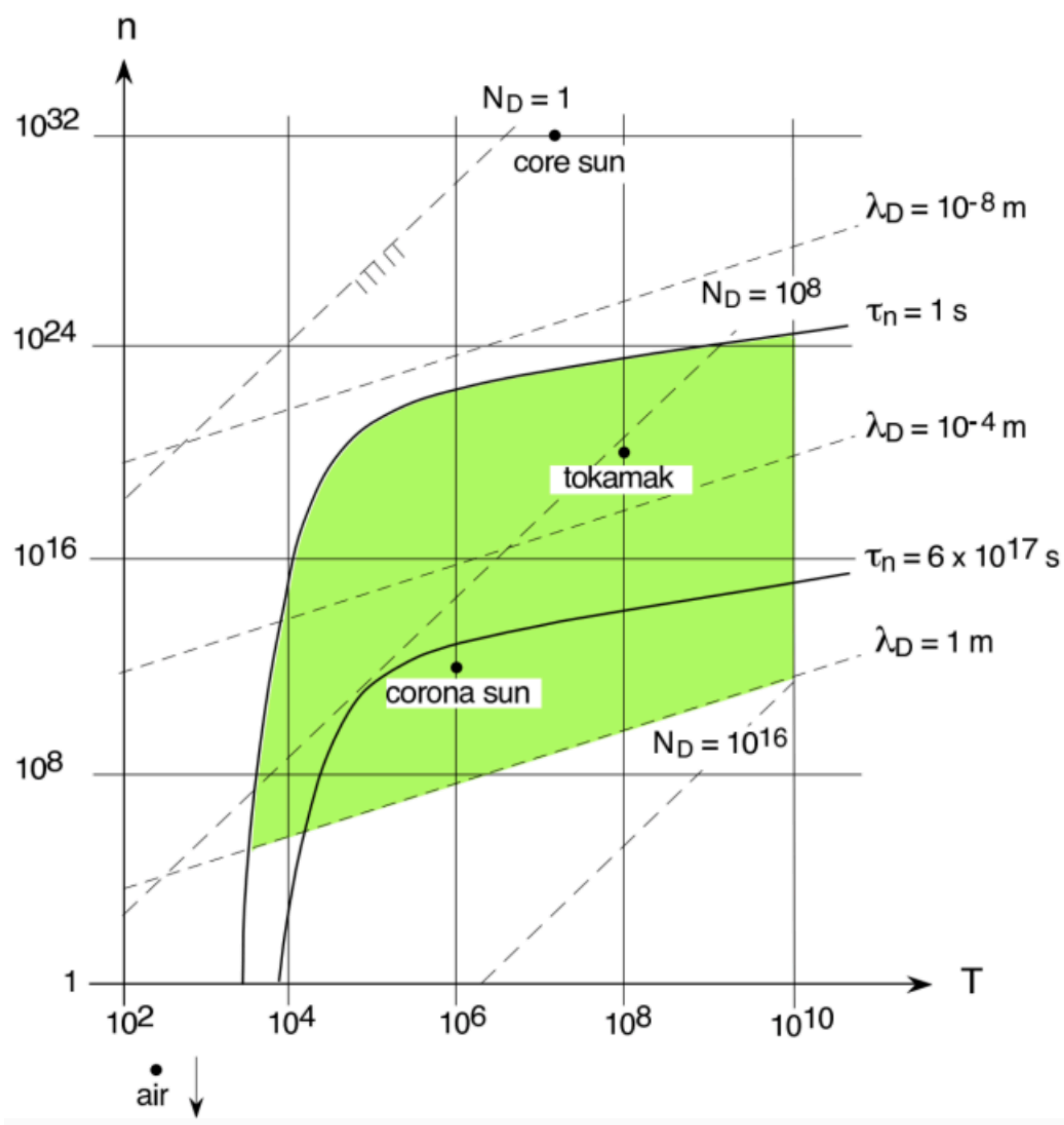
$$\tau_{en} \gg \tau_{pe} = \frac{1}{f_{pe}} = \frac{2\pi}{\omega_{pe}}$$

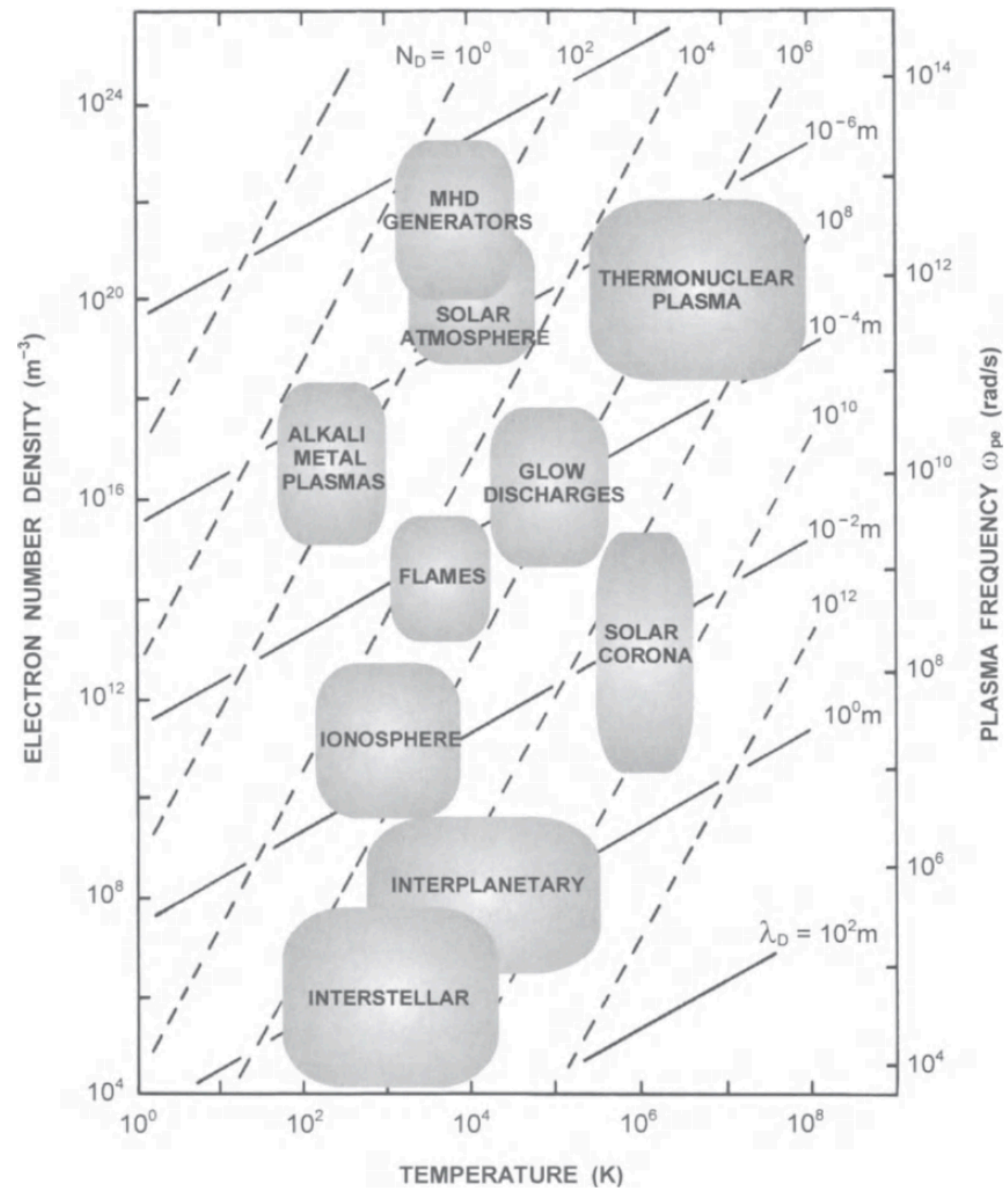
- **Note that the first criterion already implies macroscopic charge neutrality**
  - Sometimes, charge neutrality is considered a 4th criterion, even though it is not an independent one

$$\sum_j Z_j n_j = n_e$$

Macroscopic features of plasmas: the plasma definition criteria

- **Conditions for collective plasma behavior are satisfied within the green region**





- **Macroscopic features of plasmas**
  - *Quasi-neutrality*
  - *Debye shielding*
  - *Plasma oscillations*
  - *The plasma definition criteria*

- **Degree of ionization and the Saha equation**

Degree of ionization of a gas or a plasma

- **From statistical mechanics, one can determine the degree of ionization of a gas (or plasma) in thermal equilibrium at some temperature T**

- **A considerable degree of ionization can be achieved even when the average thermal energy of the particles is far below the ionization potential**

- **From statistical mechanics, one has that**

$$\frac{n_a}{n_b} = \frac{g_a}{g_b} \exp\left[-\frac{(U_a - U_b)}{k_B T}\right]$$

  - Here, g_a and g_b are the statistical weights (degeneracy factor) associated to the energies U_a and U_b

Degree of ionization of a gas or a plasma

- **For the particular case of a system having only 2 energy levels, the fraction of particles that are in the higher energy state (U_a) is**

$$\alpha = \frac{n_a}{n_a + n_b} = \frac{n_a}{n_b}\left(\frac{n_a}{n_b} + 1\right)^{-1}$$

Element	U(eV)
Helium (He)	24.59
Argon (A)	15.76
Nitrogen (N)	14.53
Oxygen (O)	13.62
Hydrogen (H)	13.60
Mercury (Hg)	10.44
Iron (Fe)	7.87
Sodium (Na)	5.14
Potassium (K)	4.34
Cesium (Cs)	3.89

- **Which can be written as**

$$\alpha = \frac{\left(g_a/g_b\right)\exp\left(-U/k_B T\right)}{\left(g_a/g_b\right)\exp\left(-U/k_B T\right) + 1}$$

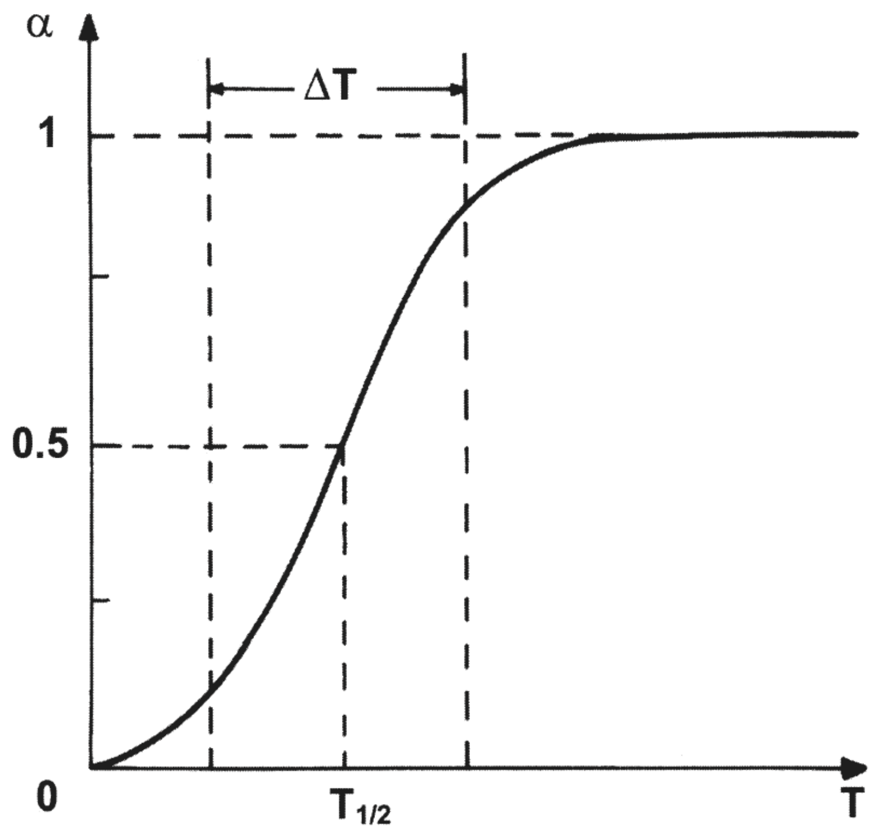
- **For the ionization problem, state a is taken as that of the electron-ion pair, state b is taken as the neutral atom, and $U = U_a - U_b$ is the ionization energy**
  - The temperature for which $\alpha = 0.5$ is equal to

$$\left(g_a/g_b\right)\exp\left(-U/k_B T_{1/2}\right) = 1 \quad \rightarrow \quad T_{1/2} = \frac{U}{k_B \ln\left(g_a/g_b\right)}$$

Degree of ionization of a gas or a plasma

- The range in which α changes from nearly zero to nearly one is defined as

$$\left.\frac{d\alpha}{dT}\right|_{T_{1/2}} = \frac{1}{\Delta T}$$



- Therefore,

$$\Delta T = \frac{4T_{1/2}}{k_B \ln\left(g_a/g_b\right)} = \frac{4U}{\left[k_B \ln\left(g_a/g_b\right)\right]^2}$$

- Since $g_a \gg g_b$, the curve usually looks like a step function near $T_{1/2}$

- From quantum mechanics, one can estimate (h is Planck's constant)

$$\frac{g_a}{g_b} = \frac{1}{\lambda_{th}^3 n_i} = \left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2} \frac{1}{n_i} = 2.405 \times 10^{21} \, \frac{T^{3/2}}{n_i}$$

(λ_{th} is the electron thermal de Broglie wavelength)

The Saha equation

- **From quantum mechanics, one can estimate (T in Kelvin and n_i in m^{-3})**

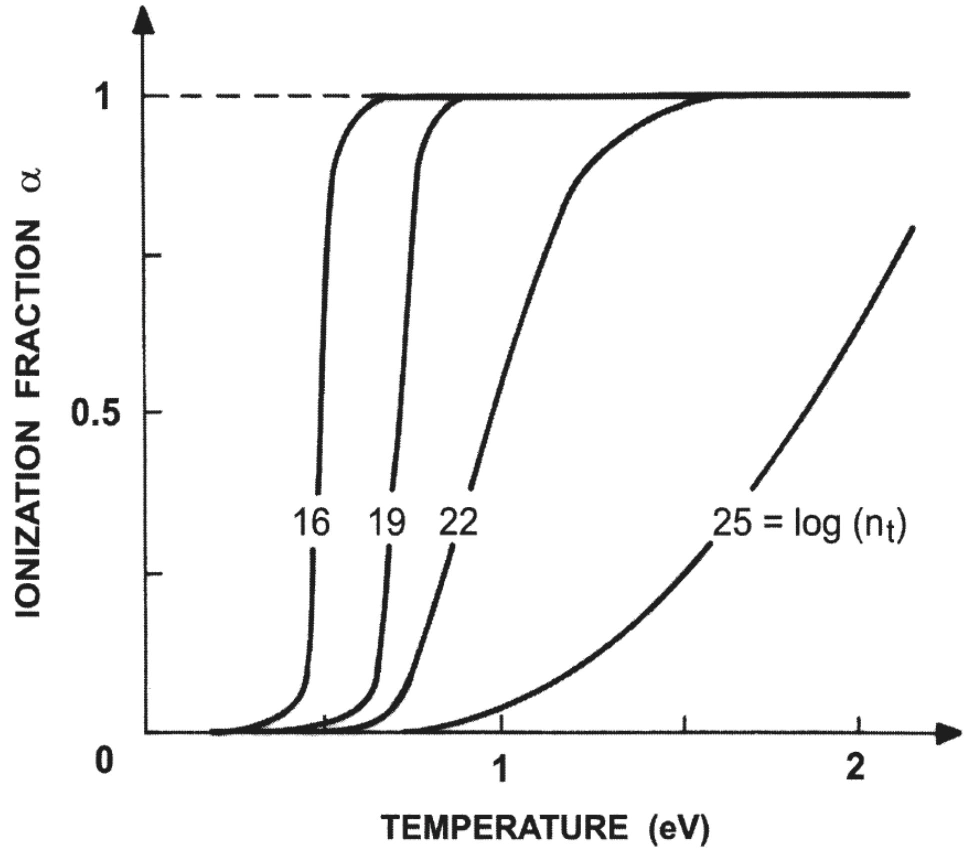
$$\frac{g_a}{g_b} = \left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2} \frac{1}{n_i} = 2.405 \times 10^{21} \frac{T^{3/2}}{n_i}$$

- **Using this results, one can write**
  - This equation is known as the Saha equation

$$\frac{n_i}{n_n} = 2.405 \times 10^{21} \frac{T^{3/2}}{n_i} \exp\left(-\frac{U}{k_B T}\right)$$

- **Since 1 eV = 11,600 K, the Saha equation can be written as (T in eV and n_i in m^{-3})**

$$\frac{n_i}{n_n} = 3.00 \times 10^{7} \frac{T^{3/2}}{n_i} \exp\left(-\frac{U}{T}\right)$$



Degree of ionization for some particular cases

- **Air at room temperature**

$$n_n = 3\times10^{25}\ m^{-3} \qquad T = 300\ K \qquad U = 14.5\ eV\ \text{(Nitrogen)}$$

$$\frac{n_i}{n_n} = 2\times10^{-122} \ll 1$$

- **Tokamak**

$$n = n_e = n_i = 1\times10^{20}\ m^{-3} \qquad T = 1\times10^{8}\ K \qquad U = 13.6\ eV\ \text{(Hydrogen)}$$

$$\frac{n_i}{n_n} = 2.4\times10^{13} \gg 1$$

- **Plasma torch**

$$n_n = 3\times10^{25}\ m^{-3}\ (1\ \text{Atm}) \qquad T = 1\times10^{4}\ K \qquad U = 13.6\ eV\ \text{(Hydrogen)}$$

$$\frac{n_i}{n_n} = 3\times10^{-4} \ll 1$$

Degree of ionization for some particular cases

- **Solar corona**

$$n = 1\times 10^{12}\ m^{-3} \qquad T = 1\times 10^{6}\ K \qquad U = 13.6\ eV\ \text{(Hydrogen)}$$

$$\frac{n_i}{n_n} = 2.4\times 10^{18} \gg 1$$ **(Excellent Plasma!!!!!)**

- **Sun's core**

$$n = n_e = n_i = 1\times 10^{32}\ m^{-3} \qquad T = 1.6\times 10^{7}\ K \qquad U = 13.6\ eV\ \text{(Hydrogen)}$$

$$\frac{n_i}{n_n} = 1.5$$ **(Surprisingly, the sun's core is not fully ionized, but nuclear fusion reactions can occur)**

Note that, from our previous definitions, the gas does not need to be almost fully ionized to be considered a plasma, as far as collective effects are present

References

- **Quasi-neutrality**
  - *Bittencourt: Ch. 1, section 2*
- **Debye shielding**
  - *Bittencourt: Ch. 11, section 2*
  - *Bittencourt: Ch. 7, section 5*
  - *Bittencourt: Ch. 20, section 8*
- **Plasma oscillations**
  - *Bittencourt: Ch. 1, section 2*
- **Degree of ionization and the Saha equation**
  - *Bittencourt: Ch. 7, section 6*