PGF5112 - Plasma Physics I

By

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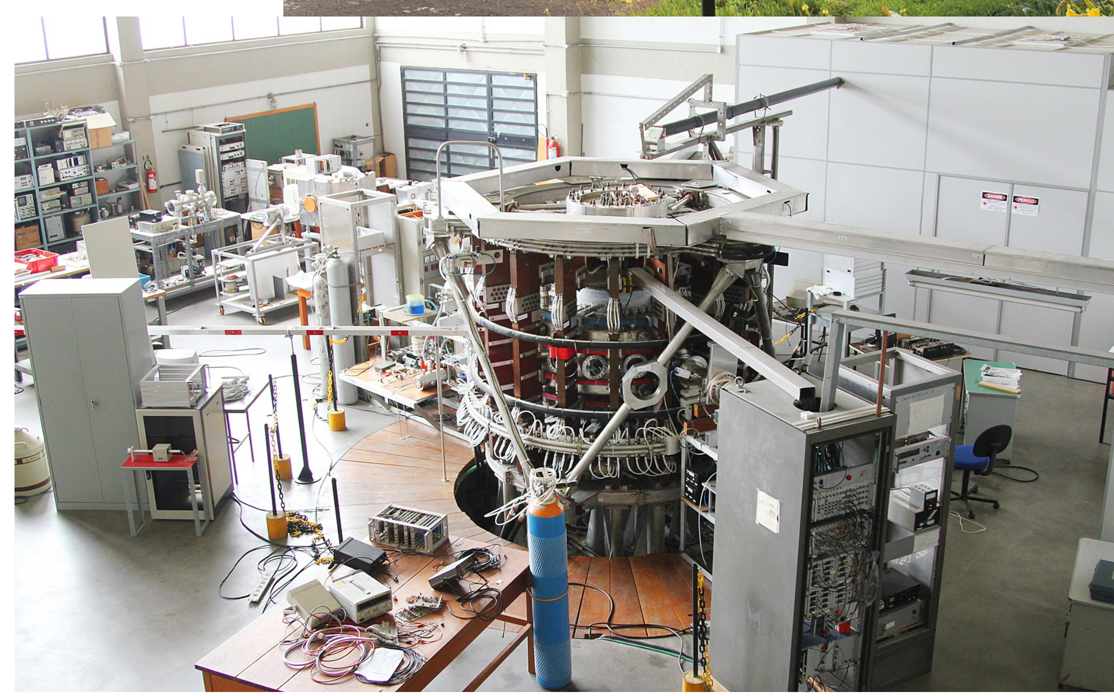

Postgraduate course ministered remotely from the

Institute of Physics of the University of São Paulo

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- **Introduction about the course**
- **Theoretical descriptions of plasma phenomena**
- **Review of basic concepts in kinetic theory of gases**
- **Particle interactions in plasmas**
  - *Collision cross section*
  - *The Rutherford cross section*

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This course aims at providing a broad view about the various phenomena occurring in plasmas

- **What are plasmas?**
  - Plasmas are ionized gases whose atoms have been dissociated (not necessarily all of them) into ions and electrons

- **All ionized gases are considered plasmas?**
  - No, plasmas are ionized gases that exhibit collective effects

- **How plasmas are produced and maintained?**
  - Plasmas are produced by the ionization of atoms, which can happens through a variety of collisional processes
  - To maintain a steady state plasma, particles and/or energy must be supplied constantly

This course aims at providing a broad view about the various phenomena occurring in plasmas

- **How can we describe/model the behavior of plasmas?**
  - From first principles: following the trajectory of each individual particle
  - From a statistical approach: kinetic theory
  - Assuming the plasma is a continuous medium: fluid model

- **Why is it important to study plasma physics?**
  - Plasmas are used in an enormous number of technological applications
  - Important astrophysical phenomena for human life: solar thunderstorms
  - Energy production through thermonuclear fusion: tokamaks

- **How stable are plasmas in tokamaks?**
  - Sometimes, plasmas can find a path towards a lower energy state
  - Plasma instabilities are the result of plasmas accessing a lower energy state

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The particle orbit theory: a first principles plasma model

- **Motion equation of a charged particle of especies α in an electromagnetic field:**

$$\frac{d\mathbf{r}_\alpha}{dt} = \mathbf{v}_\alpha$$

$$m_\alpha \frac{d\mathbf{v}_\alpha}{dt} = \mathbf{F}_\alpha = q_\alpha \left(\mathbf{E} + \mathbf{v}_\alpha \times \mathbf{B}\right)$$

- **Maxwell's equations:**

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

- **Constitutive relations:**

$$\rho = \rho_{ext} + \rho_{plasma} = \rho_{ext} + \sum_\alpha q_\alpha \delta[\mathbf{r} - \mathbf{r}_\alpha(t)] \qquad \mathbf{E} = \mathbf{E}_{\mathbf{ext}} + \mathbf{E}_{\mathbf{plasma}}$$

$$\mathbf{J} = \mathbf{J}_{\mathbf{ext}} + \mathbf{J}_{\mathbf{plasma}} = \mathbf{J}_{\mathbf{ext}} + \sum_\alpha q_\alpha \mathbf{v}_\alpha(t) \delta[\mathbf{r} - \mathbf{r}_\alpha(t)] \qquad \mathbf{B} = \mathbf{B}_{\mathbf{ext}} + \mathbf{B}_{\mathbf{plasma}}$$

The particle orbit theory: a first principles plasma model

- **The particle orbit theory provides a well-defined self-consistent model to describe plasmas, however, this model has limitations in practice**
  - The large number of particles (~10^{20} m^{-3}) makes this model prohibitive
  - The amount of information contained in this model is unnecessarily large: (# of particles) x (3 positions) x (3 velocities) x (# of temporal steps)

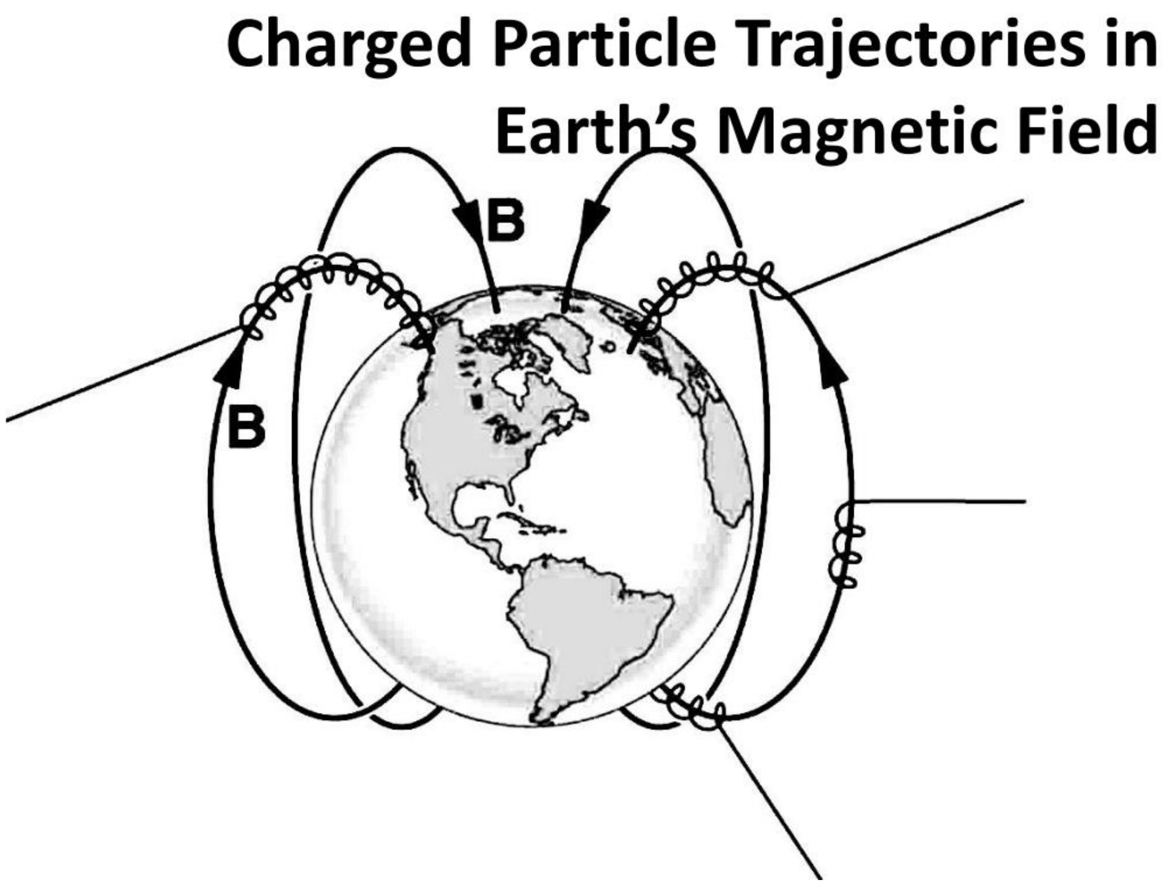
- **To simplify the model, the response/reaction of charged particles to EM fields from other charged particles is neglected ($\mathbf{E}_{\mathbf{ext}} >> \mathbf{E}_{\mathbf{plasma}}$ and $\mathbf{B}_{\mathbf{ext}} >> \mathbf{B}_{\mathbf{plasma}}$)**
  - The charged particle trajectory is, therefore, determined by ONLY the externally applied EM fields
  - This model neglects collective effects (not well suited for plasmas)

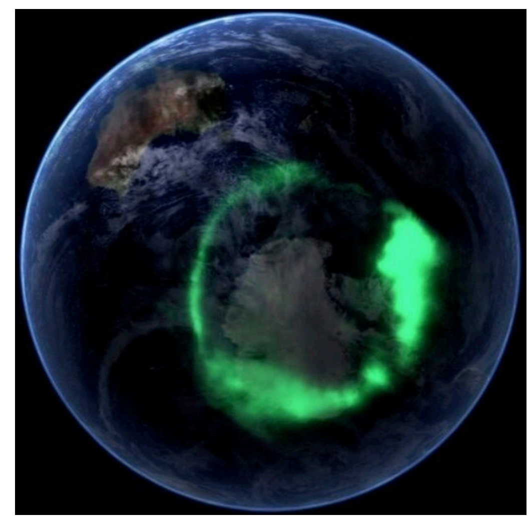
$$\mathbf{E} = \mathbf{E}_{\mathbf{ext}} + \cancel{\mathbf{E}_{\mathbf{plasma}}} \qquad \frac{d\mathbf{r}_\alpha}{dt} = \mathbf{v}_\alpha$$

$$\mathbf{B} = \mathbf{B}_{\mathbf{ext}} + \cancel{\mathbf{B}_{\mathbf{plasma}}} \qquad m_\alpha \frac{d\mathbf{v}_\alpha}{dt} = q_\alpha \left(\mathbf{E}_{\mathbf{ext}} + \mathbf{v}_\alpha \times \mathbf{B}_{\mathbf{ext}}\right)$$

The particle orbit theory is well suited to study the trajectory of charged particles entering Earth's atmosphere

- **Charged particles arriving to Earth's atmosphere are deflected towards the poles by the terrestrial magnetic field**
  - The collision of these charged particles with the air molecules in the atmosphere gives rise to the so-called auroras





Due to its large number of particles, plasmas can also be described by means of a statistical approach: the kinetic theory of plasmas

- **In kinetic theory, all the information of the system is contained in the distribution function, $f_\alpha = f_\alpha(\mathbf{r}, \mathbf{v}, t)$, which is defined for each particle species**

- **The evolution of the system is given by the so-called Boltzmann equation:**

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla f_\alpha + \frac{q_\alpha}{m_\alpha}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_\alpha = \sum_\beta C_{\text{coll}}[f_\alpha, f_\beta]$$

- **Maxwell's equations:**

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

- **Constitutive relations:**

$$\rho = \rho_{ext} + \rho_{plasma} = \rho_{ext} + \sum_\alpha q_\alpha \int_{\mathbf{v}} f_\alpha(\mathbf{r}, \mathbf{v}, t) d\mathbf{v}$$

$$\mathbf{J} = \mathbf{J}_{\mathbf{ext}} + \mathbf{J}_{\mathbf{plasma}} = \mathbf{J}_{\mathbf{ext}} + \sum_\alpha q_\alpha \int_{\mathbf{v}} \mathbf{v} f_\alpha(\mathbf{r}, \mathbf{v}, t) d\mathbf{v}$$

Particle density

$$n_\alpha(\mathbf{r}, t) = \int_{\mathbf{v}} f_\alpha(\mathbf{r}, \mathbf{v}, t)\ d\mathbf{v}$$

Mean velocity

$$\mathbf{u}_\alpha(\mathbf{r}, t) = \frac{1}{n_\alpha(\mathbf{r}, t)} \int_{\mathbf{v}} \mathbf{v}\ f_\alpha(\mathbf{r}, \mathbf{v}, t)\ d\mathbf{v}$$

Depending of the conditions, each particle species in a plasma can be treated as a separate continuous medium: the multi-fluid model

- **Plasma fluid transport equations can be derived for each species by taking the moments of the Boltzmann equation**

Mass conservation

$$\frac{\partial \rho_{m\alpha}}{\partial t} + \nabla \cdot \left(\rho_{m\alpha} \mathbf{u}_\alpha\right) = S_\alpha$$

Momentum conservation

$$\rho_{m\alpha} \left[\frac{\partial \mathbf{u}_\alpha}{\partial t} + \left(\mathbf{u}_\alpha \cdot \nabla\right) \mathbf{u}_\alpha\right] = n_\alpha q_\alpha \left(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}\right) - \nabla \cdot \mathbf{P}_\alpha + \mathbf{A}_\alpha - \mathbf{u}_\alpha S_\alpha$$

Energy conservation

$$\frac{3}{2}\left[\frac{\partial p_\alpha}{\partial t} + \mathbf{u}_\alpha \cdot \nabla p_\alpha\right] + \frac{3\, p_\alpha}{2}\left(\nabla \cdot \mathbf{u}_\alpha\right) + \left(\mathbf{P}_\alpha \cdot \nabla\right) \cdot \mathbf{u}_\alpha + \nabla \cdot \mathbf{q}_\alpha = M_\alpha - \mathbf{u}_\alpha \cdot \mathbf{A}_\alpha + \frac{1}{2} u_\alpha^2 S_\alpha$$

Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \qquad \mathbf{J} = \sum_\alpha n_\alpha q_\alpha \mathbf{u}_\alpha$$

Depending of the conditions, the whole plasma can be treated as one continuous medium: the single-fluid model

- **The behavior of the plasma as a whole can be determined by adding the contributions of the various particle species in the plasma**

Mass conservation

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{u}) = 0$$

Momentum conservation

$$\rho_m \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} \right] = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B} - \nabla \cdot \mathbf{P}$$

Energy conservation

$$\frac{3}{2} \left[\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p \right] + \frac{3\,p}{2} (\nabla \cdot \mathbf{u}) + (\mathbf{P} \cdot \nabla) \cdot \mathbf{u} + \nabla \cdot \mathbf{q} = \mathbf{J} \cdot \mathbf{E} - \mathbf{u} \cdot (\mathbf{J} \times \mathbf{B}) - \rho \mathbf{u} \cdot \mathbf{E}$$

Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Generalized Ohm's law

$$\frac{\partial \mathbf{J}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{J}' + \mathbf{J}\mathbf{u}) - \frac{e}{m_e} \nabla \cdot \mathbf{P_e} = \frac{ne^2}{m_e} (\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \frac{e}{m_e} \mathbf{J} \times \mathbf{B} - \nu_{ei} \mathbf{J}$$

The magnetohydrodynamic model

- **Within the single-fluid approach, the magnetohydrodynamic (MHD) model focuses on large (plasma size) scale and (relatively) low frequency phenomena**
  - In the MHD model, the information about the plasma is contained in ρ_m, $\mathbf{u}$ e p

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot \left(\rho_m \mathbf{u}\right) = 0$$ (Mass conservation)

$$\rho_m \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u}\right] = \mathbf{J} \times \mathbf{B} - \nabla p - \nabla \cdot \mathbf{\Pi}$$ (Momentum conservation)

$$p = \left(\frac{k_B T}{m_i}\right) \rho_m$$ (Energy conservation)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$ (Faraday's law)

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$ (Ampère's law)

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{J}$$ (Ohm's law)

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Review of basic concepts in the kinetic theory of gases

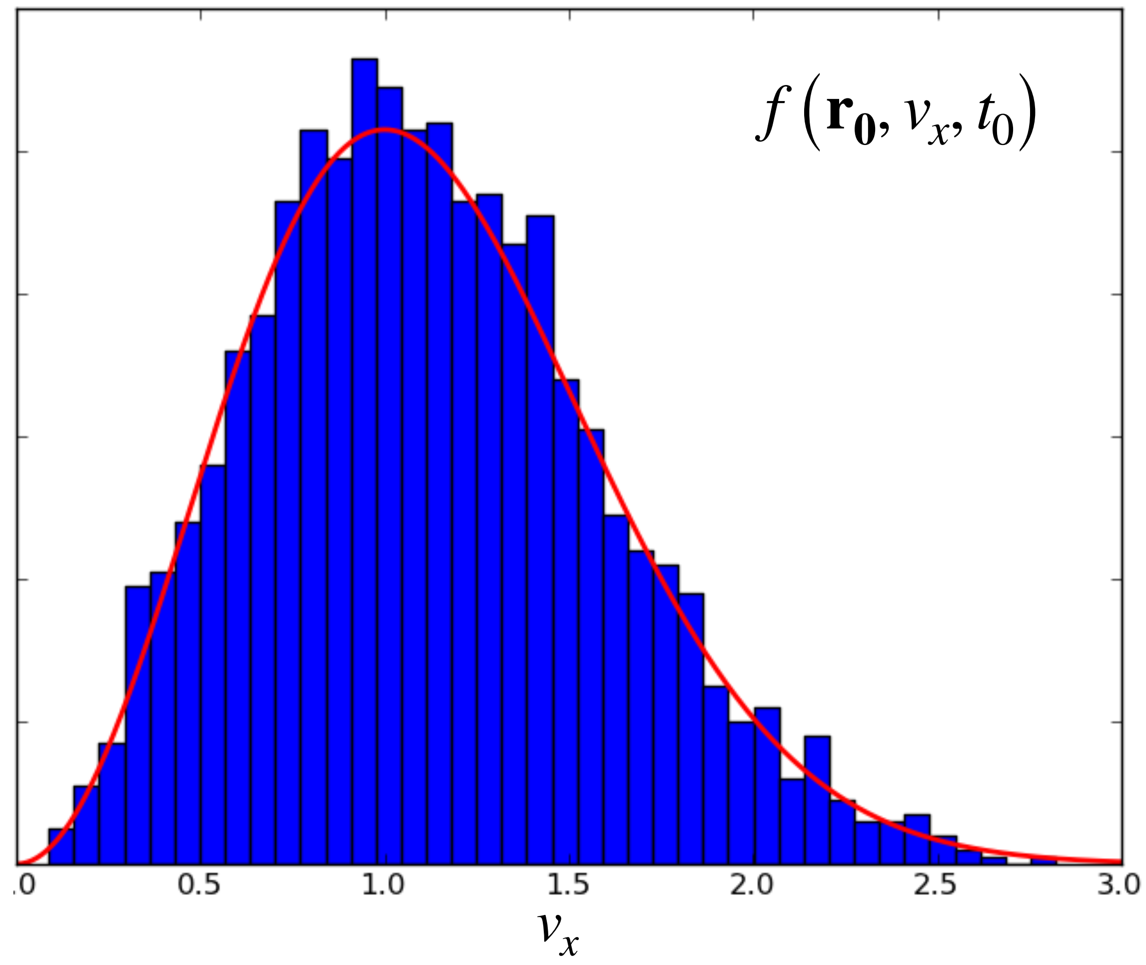
- **Distribution function**
  - Number of particles, per unit of volume, with velocity between $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$
  - $f(\mathbf{r}, \mathbf{v}, t)$ has units of s^3/m^6

$$d^6N = f(\mathbf{r}, \mathbf{v}, t)\, d^3v\, d^3x$$

$$d^6N = f(\mathbf{r}, \mathbf{v}, t)\, d\mathbf{r}\, d\mathbf{v}$$

$$N(t) = \int_{\mathbf{r}}\int_{\mathbf{v}} d^6N\ d\mathbf{r}\, d\mathbf{v}$$

$$N(t) = \int_{\mathbf{r}}\int_{\mathbf{v}} f(\mathbf{r}, \mathbf{v}, t)\, d\mathbf{r}\, d\mathbf{v}$$



Review of basic concepts in the kinetic theory of gases

- **Particle density**
  - Number of particles per unit of volume (independent of their velocity) within a volume d^3x around $\mathbf{r}$

$$n(\mathbf{r},t) = \iiint f(\mathbf{r},\mathbf{v},t)\, d\mathbf{v}$$

- **Mean velocity**
  - Average velocity of the particles within a volume d^3x around $\mathbf{r}$

$$\mathbf{u}(\mathbf{r},t) = \frac{\iiint \mathbf{v}\, f(\mathbf{r},\mathbf{v},t)\, d\mathbf{v}}{n(\mathbf{r},t)}$$

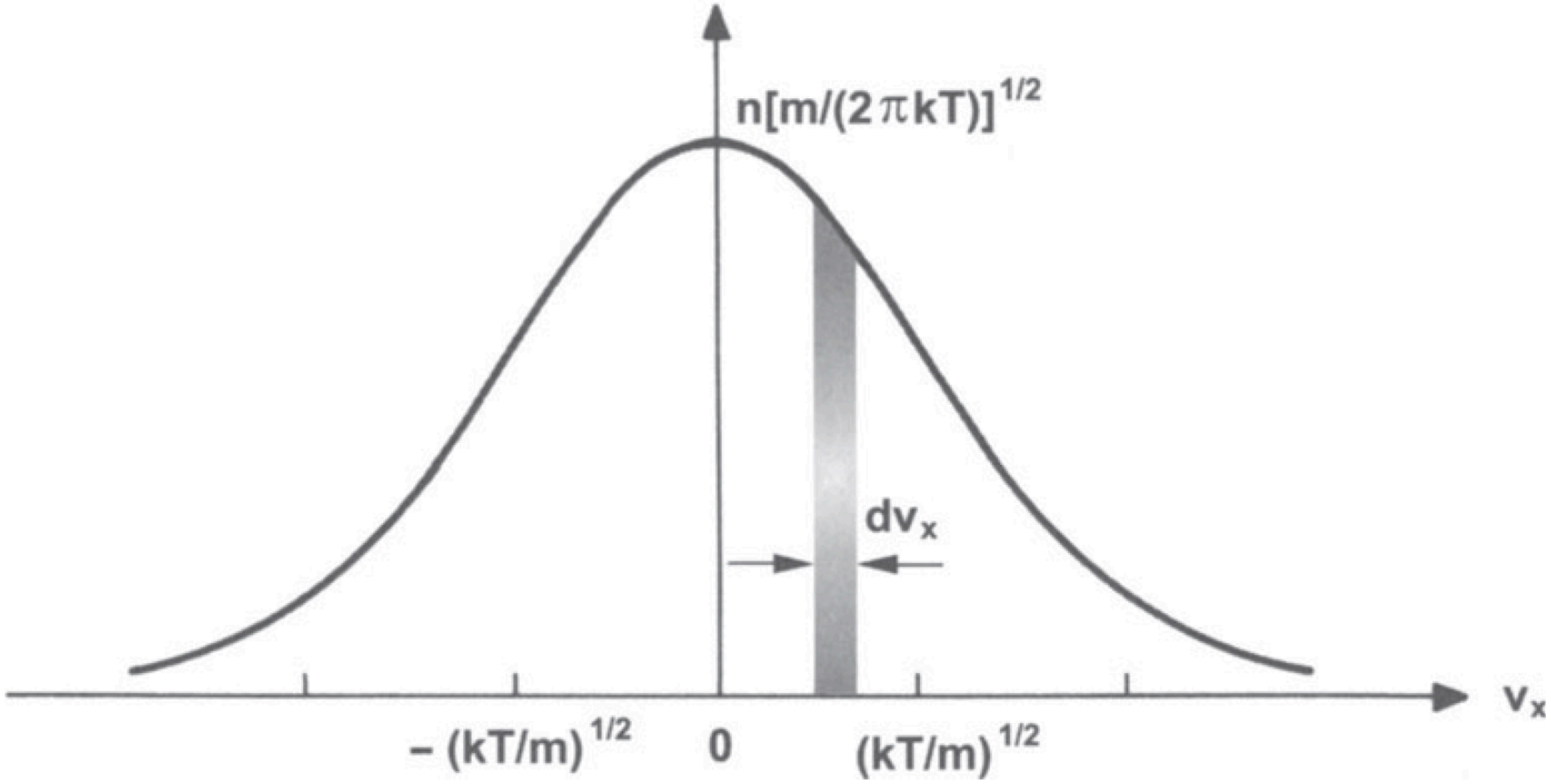
- **Mean energy**
  - Average energy of the particles within a volume d^3x around $\mathbf{r}$

$$K(\mathbf{r},t) = \frac{\iiint \frac{1}{2} m v^2 f(\mathbf{r},\mathbf{v},t)\, d\mathbf{v}}{n(\mathbf{r},t)}$$

Review of basic concepts in the kinetic theory of gases

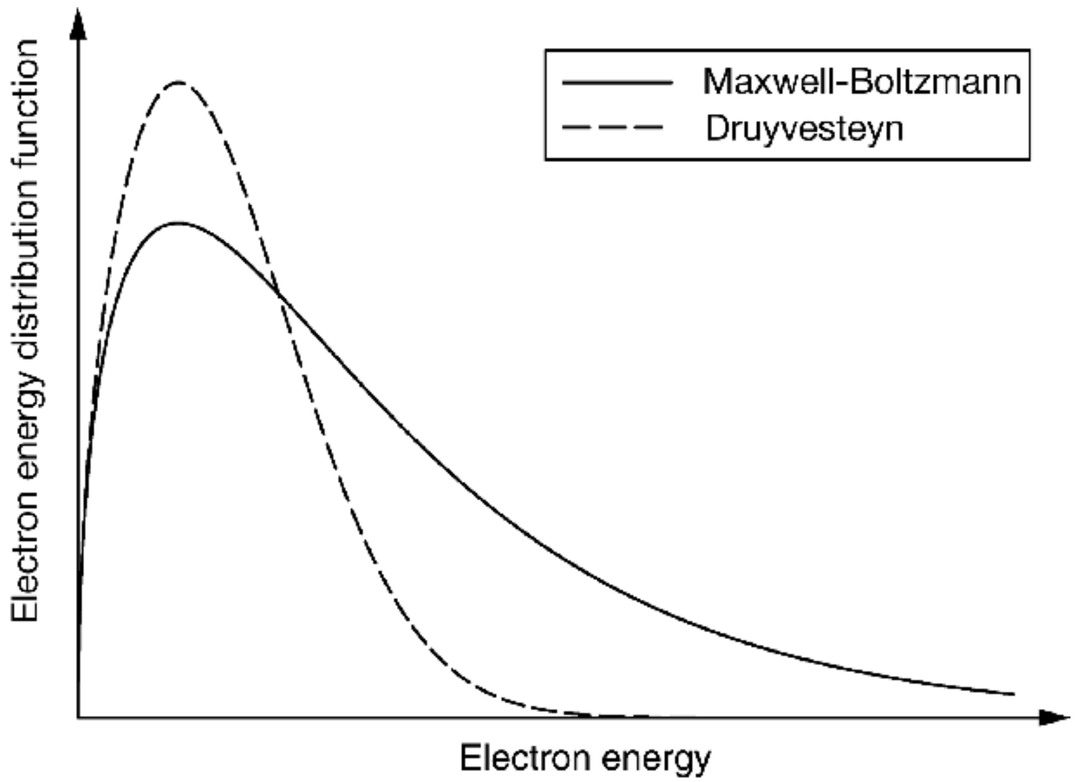
- **In the kinetic theory of gases, one can show (H-theorem) that the velocity distribution function of the particles tends to the Maxwell-Boltzmann distribution when the gas reaches thermodynamic equilibrium through collisions**

$$f_M(v) = n\left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{mv^2}{2k_B T}\right) \qquad v^2 = v_x^2 + v_y^2 + v_z^2$$



Review of basic concepts in the kinetic theory of gases

- **In plasmas, however, the general form of the H-theorem does not hold**
  - There are frequent situations in what the electrons distribution function evolves away from the Maxwell-Boltzmann distribution (for example, a Druyvesteyn distribution)



  - The shape of the distribution function has an impact on the reaction rates in what electrons play a significant role

Review of basic concepts in the kinetic theory of gases

- **Mean velocity in a Maxwell-Boltzmann distribution**

$$\langle v_x \rangle = \frac{1}{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v_x \, n \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp \left[- \frac{m \left(v_x^2 + v_y^2 + v_z^2 \right)}{2 k_B T} \right] dv_x \, dv_y \, dv_z$$

$$\langle v_x \rangle = \left(\frac{m}{2\pi k_B T} \right)^{3/2} \underbrace{\left[\int_{-\infty}^{\infty} v_x \exp \left(- \frac{m v_x^2}{2 k_B T} \right) dv_x \right]}_{0} \underbrace{\left[\int_{-\infty}^{\infty} \exp \left(- \frac{m v_y^2}{2 k_B T} \right) dv_y \right]}_{\sqrt{\frac{2\pi k_B T}{m}}} \underbrace{\left[\int_{-\infty}^{\infty} \exp \left(- \frac{m v_z^2}{2 k_B T} \right) dv_z \right]}_{\sqrt{\frac{2\pi k_B T}{m}}}$$

- **In a Maxwell-Boltzmann distribution, the mean velocities** $\langle v_x \rangle = \langle v_y \rangle = \langle v_z \rangle = 0$

Review of basic concepts in the kinetic theory of gases

- **Mean energy in a Maxwell-Boltzmann distribution:** $\langle E\rangle = \frac{1}{2}m\langle v^2\rangle = \frac{1}{2}m\left(\langle v_x^2\rangle + \langle v_y^2\rangle + \langle v_z^2\rangle\right)$

$$\langle v_x^2\rangle = \frac{1}{n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_x^2\, n\left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left[-\frac{m\left(v_x^2+v_y^2+v_z^2\right)}{2k_B T}\right] dv_x\, dv_y\, dv_z$$

$$\langle v_x^2\rangle = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \underbrace{\left[\int_{-\infty}^{\infty} v_x^2 \exp\left(-\frac{m v_x^2}{2k_B T}\right) dv_x\right]}_{\frac{\sqrt{\pi}}{2}\left(\frac{2k_B T}{m}\right)^{3/2}} \underbrace{\left[\int_{-\infty}^{\infty} \exp\left(-\frac{m v_y^2}{2k_B T}\right) dv_y\right]}_{\sqrt{\frac{2\pi k_B T}{m}}} \underbrace{\left[\int_{-\infty}^{\infty} \exp\left(-\frac{m v_z^2}{2k_B T}\right) dv_z\right]}_{\sqrt{\frac{2\pi k_B T}{m}}}$$

- **In a Maxwell-Boltzmann distribution, the mean velocities** $\langle v_x^2\rangle = \langle v_y^2\rangle = \langle v_z^2\rangle = \frac{k_B T}{m}$

- **This result is consistent with the theorem of equipartition of energy:**

$$\frac{1}{2}m\langle v_j^2\rangle = \frac{1}{2}k_B T \qquad\qquad \langle E\rangle = \frac{3}{2}k_B T$$

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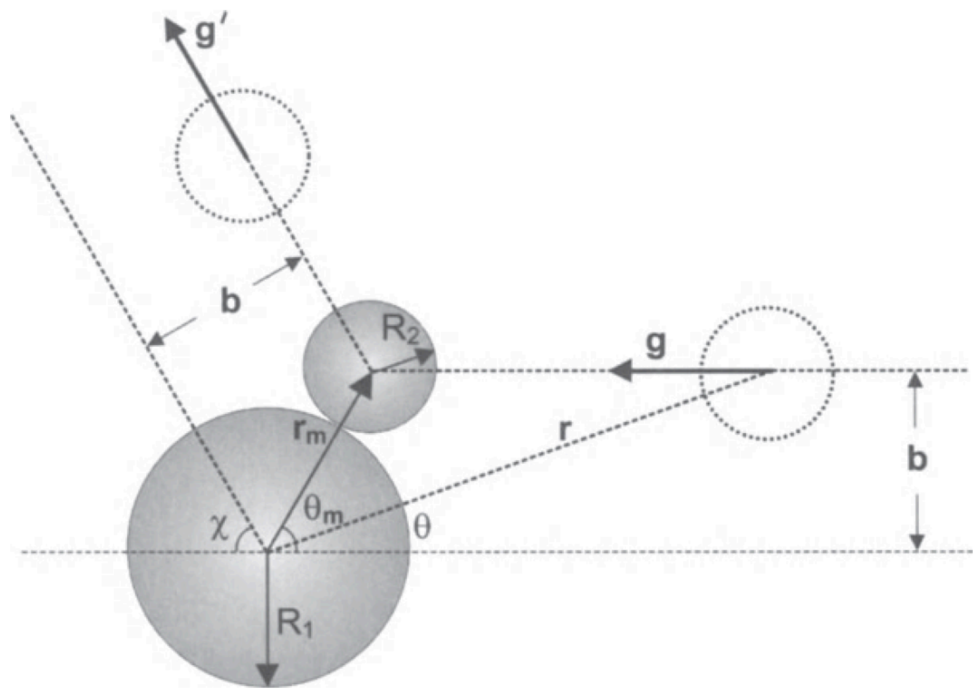
The most basic concept of cross section

- **The cross section of a certain collision process corresponds to the effective area of the target particle assuming that the projectile is a point particle**

$$\sigma_t = \pi b_0^2$$

- **As an example, consider the collision between two neutrals (or between a neutral and a charged particle)**

$$\sigma_t = \pi \left(R_1 + R_2\right)^2$$



- **Considering** $R_1 = R_2 = a_0 = 0.5 \times 10^{-10}$ m

$$\sigma_t = 0.8 \times 10^{-20}\ \mathrm{m}^2$$

The most basic concept of cross section

- **The cross section of a certain collision process corresponds to the effective area of the target particle assuming that the projectile is a point particle**

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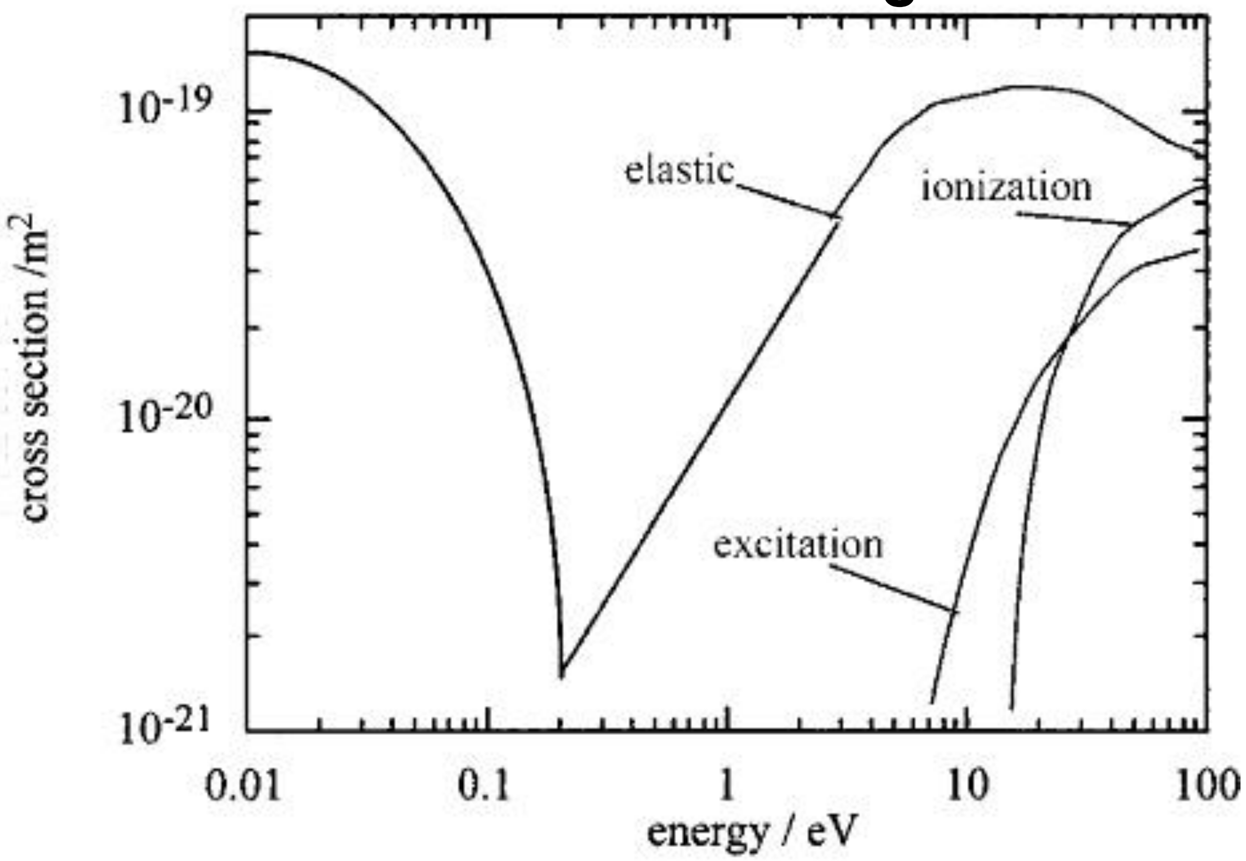
$$\sigma_t = 0.8 \times 10^{-20}\ \text{m}^2$$

Argon Plasma



For energies lower than 1 eV, a quantic, resonant effect causes a decrease in the cross section (The Ramsauer effect)

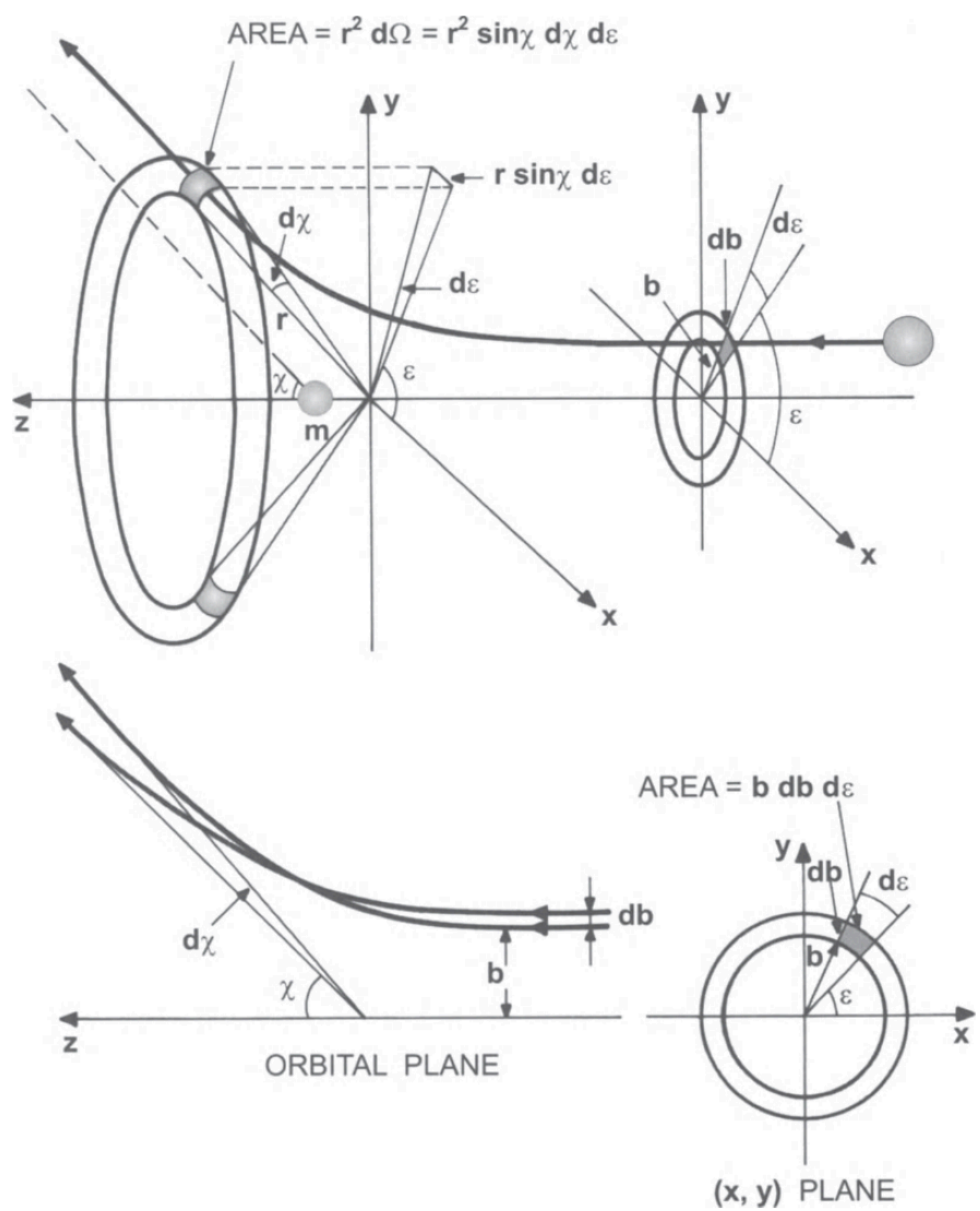
The precise concept of cross section

- **The precise definition of a cross section accounts for the scattering of an incoming particle as follows**
  - Γ is the particle flux (#/m²/s)
  - $\sigma(\chi, \epsilon)$ is the differential cross section
  - b is the impact parameter
  - $\dot{N}$ is the # of particles scattered, per unit time, into $d\Omega$

$$\frac{dN}{dt} = \Gamma\, d\sigma_t = \Gamma\, \sigma(\chi, \epsilon)\, d\Omega = \Gamma\, b\, d\epsilon\, db$$

- **Since $d\Omega = \sin(\chi)\, d\chi\, d\epsilon$, one finds that**

$$\sigma(\chi, \epsilon) = \frac{b}{\sin(\chi)} \left| \frac{db}{d\chi} \right|$$



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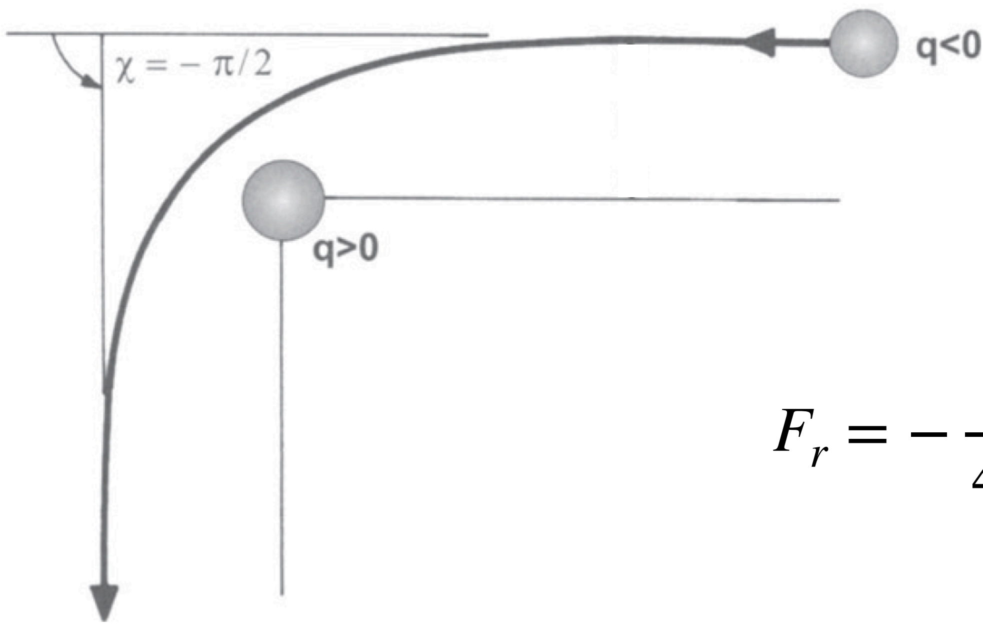
The Rutherford cross section corresponds to the elastic scattering of charged particles due to the Coulomb interaction

- **When an electron (q=-e) approaches a positive ion (q=Ze) by a distance b, it experiences an attractive force (the Coulomb force)**



$$F_r = -\frac{Ze^2}{4\pi\epsilon_0 b^2}$$

- **To have a substantial change in the trajectory of the electron, the energy of the interaction must be of the same order of the electron's kinetic energy**

$$\frac{Ze^2}{4\pi\epsilon_0 b} \approx \frac{1}{2}mv^2 \qquad b \approx \frac{2Ze^2}{4\pi\epsilon_0 v^2} \qquad \sigma \approx \pi b^2 \approx \frac{4\pi Z^2 e^4}{(4\pi\epsilon_0)^2 v^4} \propto \frac{1}{v^4}$$

This dependence has important consequences on plasma resistivity and diffusion:

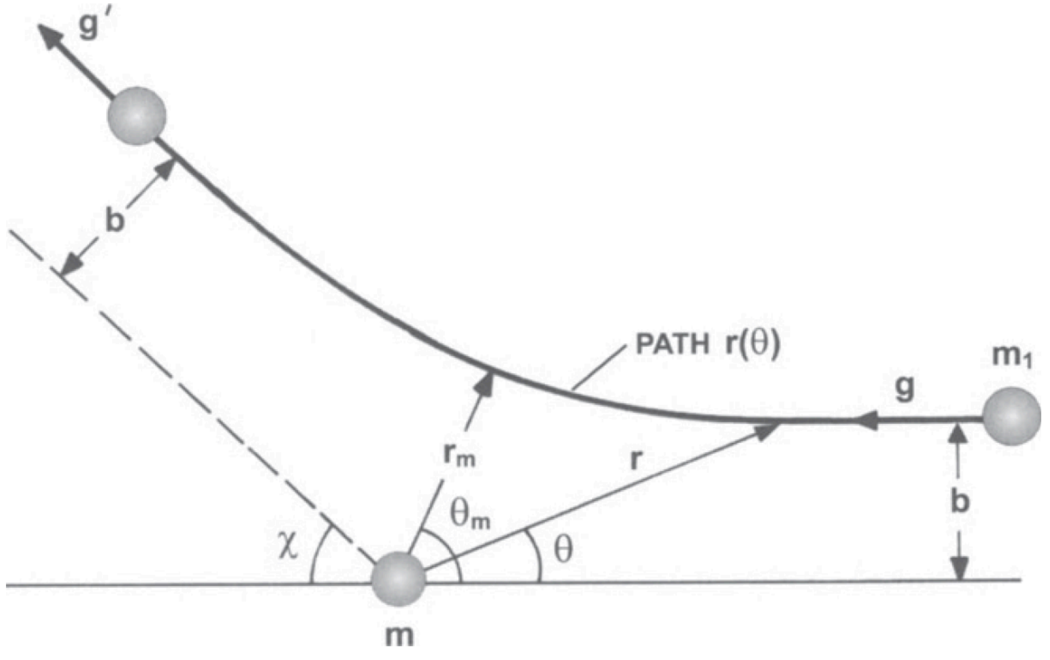
Collisions in plasmas become less frequent at higher velocities/temperatures

Exercise

- **Using polar coordinates, centered on an ion scattering center, and using energy and momentum conservation, show that**

$$\tan\left(\frac{\chi}{2}\right) = \frac{Ze^2}{4\pi\epsilon_0\mu g^2 b}$$

(Eq. 20.4.13 from Bittencourt's book)



  - $\mu = \dfrac{m\, m_1}{m + m_1}$ is the reduced mass
  - $\mathbf{g} = \mathbf{v_1} - \mathbf{v}$ is the relative velocity

References

- **Theoretical descriptions of plasma phenomena**
  - *Bittencourt: Chap. 1 - Section 5*
- **Review of basic concepts in kinetic theory of gases**
  - *Bittencourt: Chap. 7*
- **Boltzmann's H theorem**
  - *Bittencourt: Chap. 21 - Section 3*
- **Particle interactions in plasmas**
  - *Bittencourt: Chap. 20*