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CLASSIFICAÇÃO DE MATRIZES 2x2:

JORDAN SEJA $A \in M_{2\times 2}(\mathbb{R})$. TEMOS EXATAMENTE UMA DAS OPÇÕES ABAIXO
REAL

(1) $\exists$ DOIS AUTOVALORES REAIS $\lambda_1 \neq \lambda_2$.

NESTE CASO $P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $P \in M_{2\times 2}(\mathbb{R})$

(2) $\exists !$ AUTOVALOR REAL λ_0

a) $\dim N(\lambda_0 I - A) = 2 \Rightarrow A = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{pmatrix}$

b) $\dim N(\lambda_0 I - A) = 1 \Rightarrow P^{-1}AP = \begin{pmatrix} \lambda_0 & 0 \\ 1 & \lambda_0 \end{pmatrix}$, $P \in M_{2\times 2}(\mathbb{R})$

(3) $\exists$ DOIS AUTOVALORES $\notin \mathbb{R}$ $\lambda_\pm = a \pm ib$, $b \neq 0$.

NESTE CASO $P^{-1}AP = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, $P \in M_{2\times 2}(\mathbb{R})$

OBSERVAÇÃO: SE USARMOS NÚMEROS COMPLEXOS, ENTÃO EM

(3) $\exists\ \tilde{P} \in M_{2\times 2}(\mathbb{C})$ t.q. $\tilde{P}^{-1}A\tilde{P} = \begin{pmatrix} a+ib & 0 \\ 0 & a-ib \end{pmatrix}$

HOJE: VAMOS DESCREVER QUALITATIVAMENTE AS SOLUÇÕES DE $Y'(t) = AY(t)$, $Y(0) = Y_0$, $A \in M_{2\times 2}(\mathbb{R})$.

CASO 1. $\lambda_1 \neq \lambda_2$, λ_1 E λ_2 SÃO REAIS

i) $0 < \lambda_1 < \lambda_2$.

$\exists P$ T.Q. $P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

SEJAM $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad P = \begin{pmatrix} X_1 & X_2 \end{pmatrix} \Rightarrow Pe_1 = X_1$, $Pe_2 = X_2$

AUTOVETOR AUTOVETOR

SE $Y' = AY$, $Y(0) = Y_0$ ENTÃO $Y(t) = e^{tA}Y_0 = P\exp\begin{pmatrix} t\lambda_1 & 0 \\ 0 & t\lambda_2 \end{pmatrix}P^{-1}Y_0$.

SE $Y_0 = \alpha_1 X_1 + \alpha_2 X_2$, ENTÃO $Y(t) = P\begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix}P^{-1}(\alpha_1 X_1 + \alpha_2 X_2)$

$$= P\begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix}(\alpha_1 e_1 + \alpha_2 e_2)$$

$$= P\begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = P\begin{pmatrix} \alpha_1 e^{t\lambda_1} \\ \alpha_2 e^{t\lambda_2} \end{pmatrix}$$

$$= P(\alpha_1 e^{t\lambda_1} e_1 + \alpha_2 e^{t\lambda_2} e_2)$$

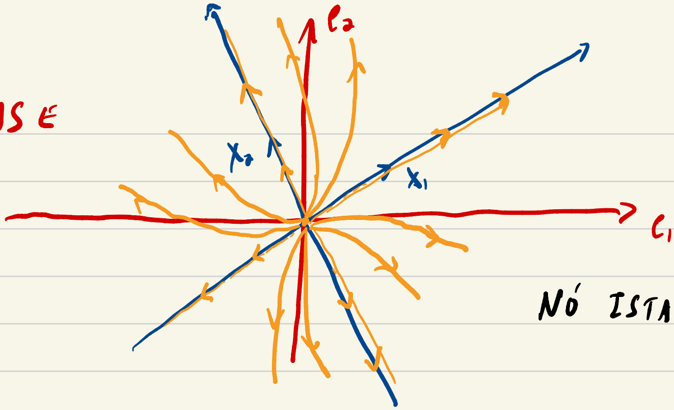
$$= \alpha_1 e^{t\lambda_1} X_1 + \alpha_2 e^{t\lambda_2} X_2.$$

TODA SOLUÇÃO É DA FORMA

$$\alpha_1 e^{t\lambda_1} X_1 + \alpha_2 e^{t\lambda_2} X_2.$$

$\mathbb{R}^2$

RETRATO DE FASE



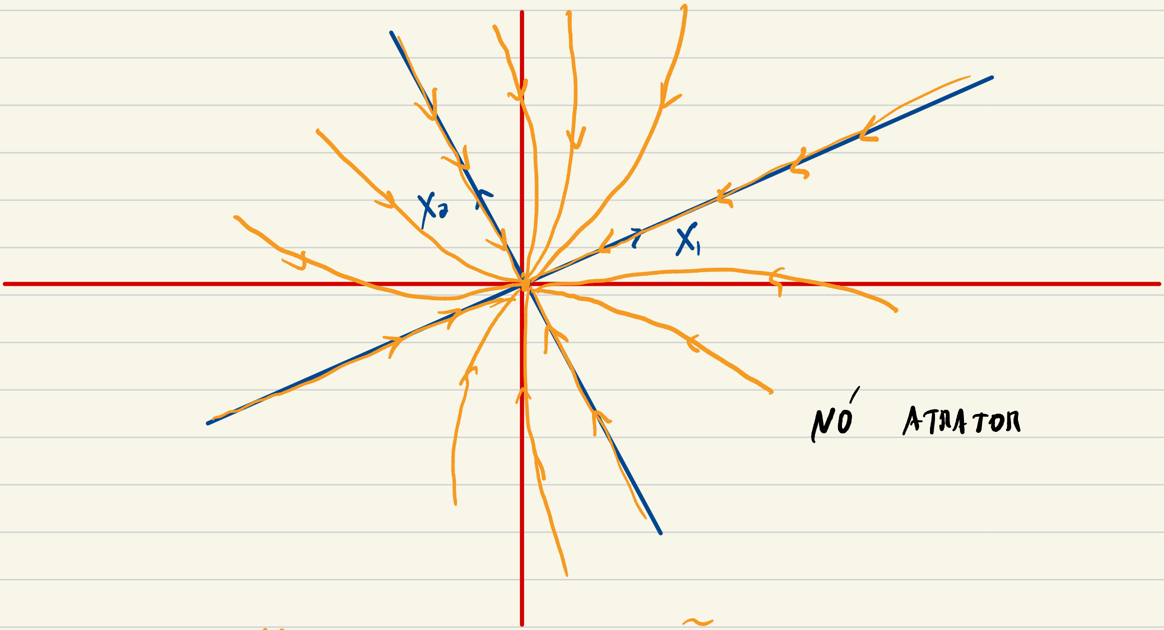
NÓ ISTÁVEL

$Y(t) = \alpha_1 e^{t\lambda_1} X_1 + \alpha_2 e^{t\lambda_2} X_2.$

AS LINHAS EM LARANJA REPRESENTAM AS SOLUÇÕES

ii) $\lambda_2 < \lambda_1 < 0$.

AS SOLUÇÕES SERÃO DA FORMA $Y(t) = \alpha_1 e^{t\lambda_1} X_1 + \alpha_2 e^{t\lambda_2} X_2$

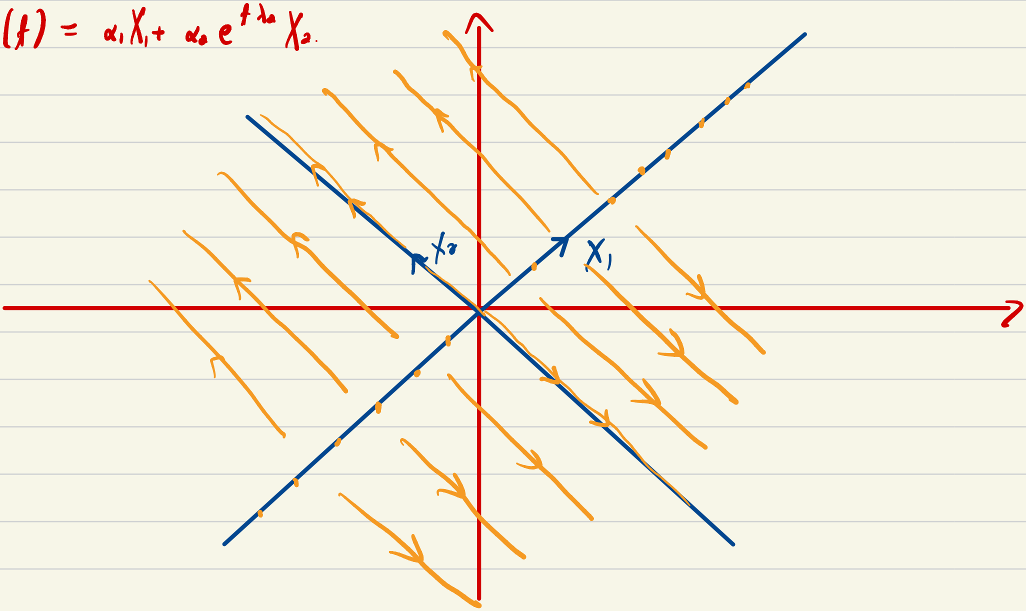


NÓ ATRATOR

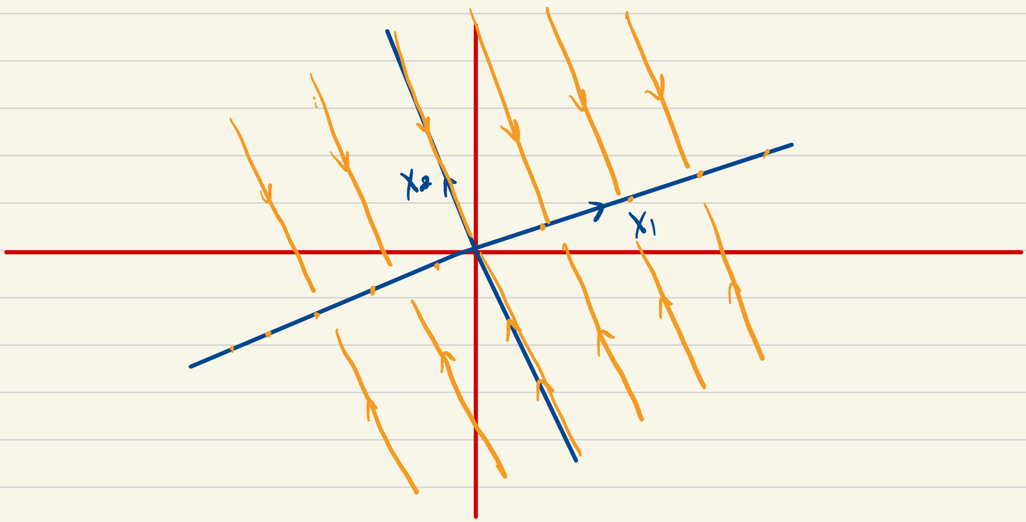
LINHAS LARANJAS SÃO SOLUÇÕES

iii) $\lambda_1 = 0$, $\lambda_2 > 0$.

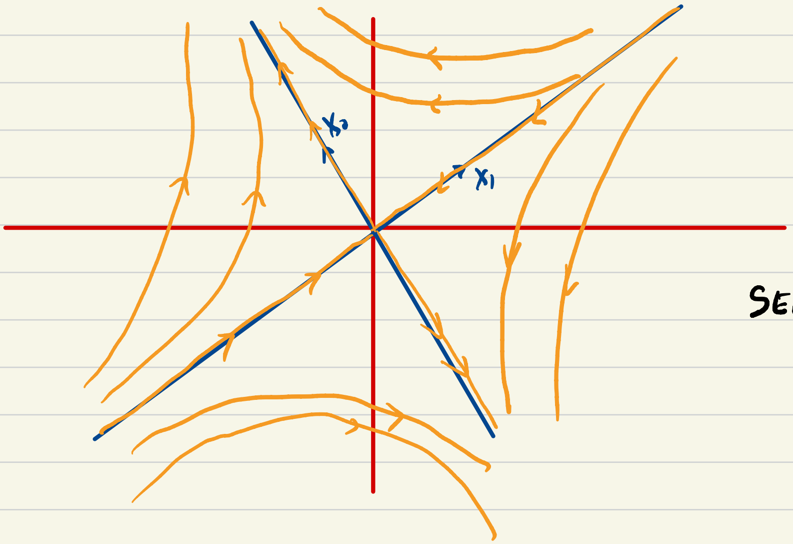
$Y(t) = \alpha_1 X_1 + \alpha_2 e^{t\lambda_2} X_2$.



iv) $\lambda_2 < 0$, $\lambda_1 = 0$ $\quad Y(t) = \alpha_1 X_1 + \alpha_2 e^{t\lambda_2} X_2$



(v) $\lambda_1 < 0 < \lambda_2$. $\quad Y(t) = \alpha_1 e^{t\lambda_1} X_1 + \alpha_2 e^{t\lambda_2} X_2$.



SELA.

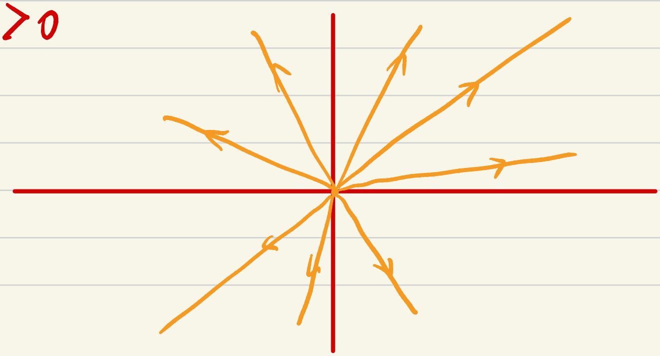
CASO 2 $\quad \lambda_1 = \lambda_2 = \lambda_0$ ($\exists!$ AUTOVALOR REAL).

I) Se $\dim N(\lambda_0 I - A) = 2$, então $\quad A = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{pmatrix}$

$$Y(t) = \exp(tA)\, Y_0 = \begin{pmatrix} e^{t\lambda_0} & 0 \\ 0 & e^{t\lambda_0} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 e^{t\lambda_0} \\ \alpha_2 e^{t\lambda_0} \end{pmatrix} = e^{t\lambda_0} Y_0.$$

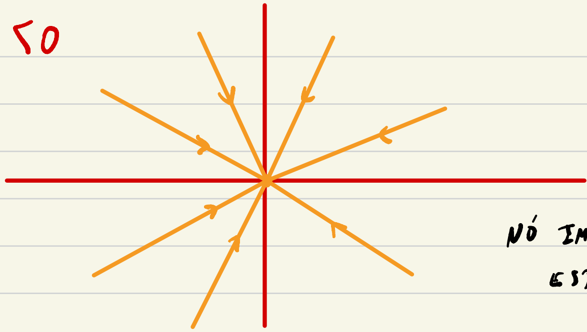
$$Y_0 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \alpha_1 e_1 + \alpha_2 e_2$$

i) $\lambda_0 > 0$



NÓ IMPRÓPRIO ESTRELADO

ii) $\lambda_0 < 0$



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ESTRELADO.

iii) $\lambda_0 = 0$ $\quad Y(t) = \overbrace{e^{t\lambda_0}}^{=1} Y_0 = Y_0$

II) $\dim N(\lambda_0 I - A) = 1.$ $\quad P^{-1}AP = \begin{pmatrix} \lambda_0 & 0 \\ 1 & \lambda_0 \end{pmatrix}$

$P = \begin{pmatrix} X_1 & X_2 \end{pmatrix}$ $\quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad Pe_1 = X_1, \; Pe_2 = X_2$

$(P^{-1}X_1 = e_1, \; P^{-1}X_2 = e_2)$

Se $Y_0 = \alpha_1 X_1 + \alpha_2 X_2$, então

$$e^{tA} Y_0 = P \exp\begin{pmatrix} t\lambda_0 & 0 \\ t & t\lambda_0 \end{pmatrix} P^{-1}(\alpha_1 X_1 + \alpha_2 X_2) = P\begin{pmatrix} e^{t\lambda_0} & 0 \\ te^{t\lambda_0} & e^{t\lambda_0} \end{pmatrix}(\alpha_1 e_1 + \alpha_2 e_2)$$

$$= P\begin{pmatrix} e^{t\lambda_0} & 0 \\ te^{t\lambda_0} & e^{t\lambda_0} \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = P\begin{pmatrix} e^{t\lambda_0}\alpha_1 \\ te^{t\lambda_0}\alpha_1 + e^{t\lambda_0}\alpha_2 \end{pmatrix} = P\left(\alpha_1 e^{t\lambda_0} e_1 + (te^{t\lambda_0}\alpha_1 + e^{t\lambda_0}\alpha_2)e_2\right)$$

$$= \alpha_1 e^{t\lambda_0} X_1 + (\alpha_1 t e^{t\lambda_0} + \alpha_2 e^{t\lambda_0}) X_2.$$

$$\exp\begin{pmatrix} t\lambda_0 & 0 \\ t & t\lambda_0 \end{pmatrix} = \exp\begin{pmatrix} t\lambda_0 & 0 \\ 0 & t\lambda_0 \end{pmatrix} \exp\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}$$

$$\begin{pmatrix} t\lambda_0 & 0 \\ t & t\lambda_0 \end{pmatrix} = \begin{pmatrix} t\lambda_0 & 0 \\ 0 & t\lambda_0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}$$

$$\exp\begin{pmatrix} t\lambda_0 & 0 \\ 0 & t\lambda_0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} e^{t\lambda_0} & 0 \\ 0 & e^{t\lambda_0} \end{pmatrix} \qquad \exp\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

$$\begin{pmatrix} t\lambda_0 & 0 \\ 0 & t\lambda_0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ t^2\lambda_0 & 0 \end{pmatrix}$$

$$=$$

$$\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}\begin{pmatrix} t\lambda_0 & 0 \\ 0 & t\lambda_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ t^2\lambda_0 & 0 \end{pmatrix}$$

$$\exp\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} = I + \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} + \cancelto{0}{\frac{1}{2!}\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}^2} + \cancelto{0}{\frac{1}{3!}\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}^3} + \cancelto{0}{\cdots} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

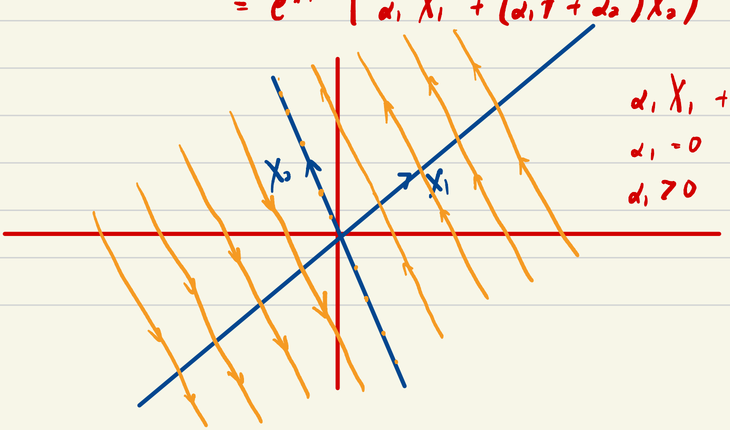
$$\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

CONCLUSÃO $e^{tA} = \begin{pmatrix} e^{t\lambda_0} & 0 \\ 0 & e^{t\lambda_0} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} e^{t\lambda_0} & 0 \\ te^{t\lambda_0} & e^{t\lambda_0} \end{pmatrix}$

A SOLUÇÃO É $Y(t) = \alpha_1 e^{t\lambda_0} X_1 + (\alpha_1 t e^{t\lambda_0} + \alpha_0 e^{t\lambda_0}) X_0$

$= e^{t\lambda_0} (\alpha_1 X_1 + (\alpha_1 t + \alpha_2) X_0)$

iv) $\lambda_0 = 0$

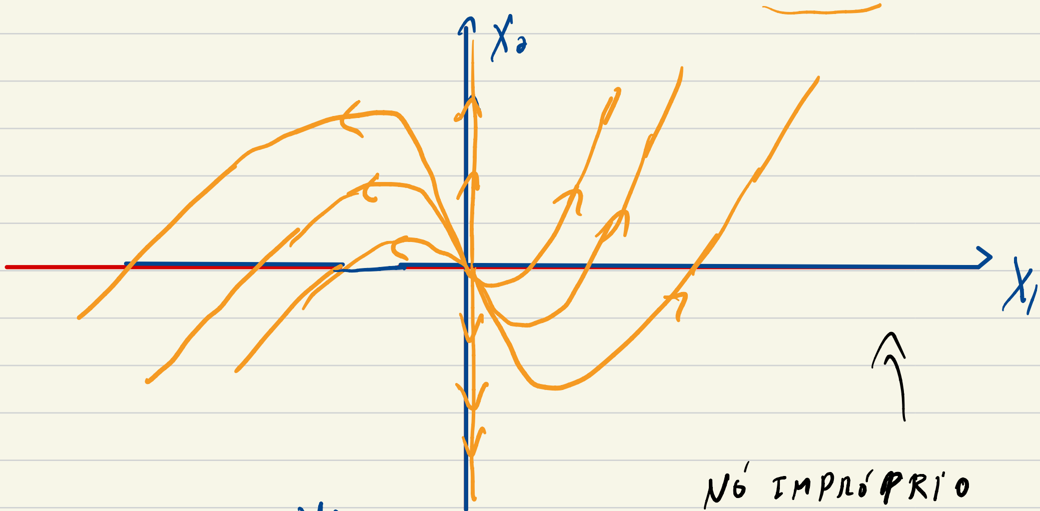


$\alpha_1 X_1 + (\alpha_1 t + \alpha_0) X_0$

$\alpha_1 = 0 \quad Y(t) = \alpha_0 X_0.$

$\alpha_1 > 0 \quad Y(t) = \alpha_1 X_1 + \underbrace{(\alpha_0 + \alpha_1 t)}_{>0} X_0$

v) $\lambda_0 > 0$ $\quad Y(t) = e^{\lambda t}\left(\alpha_1 X_1 + \underline{(\alpha_1 t + \alpha_2)} X_2\right)$



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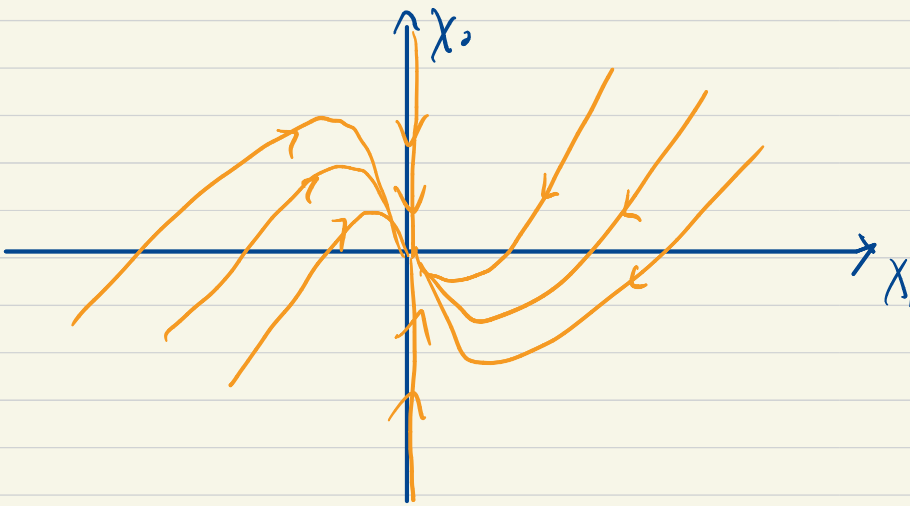
$\alpha_1 = 0 \quad Y(t) = \alpha_2 e^{\lambda t} X_2$

$\alpha_1 \neq 0$

vi) $\lambda_0 < 0$ $\quad Y(t) = e^{\lambda t}\left(\alpha_1 X_1 + (\alpha_1 t + \alpha_2) X_2\right)$



CASO 3) $\lambda_{\pm} = a \pm ib$, $b \neq 0$. Logo

$$P^{-1}AP = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

$P = \begin{pmatrix} X_1 & X_2 \end{pmatrix}$ $\quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad Pe_1 = X_1 \quad Pe_2 = X_2.$

$$e^{tA}(\alpha_1 X_1 + \alpha_2 X_2) = P \exp\begin{pmatrix} ta & bt \\ -bt & at \end{pmatrix} P^{-1}(\alpha_1 X_1 + \alpha_2 X_2)$$

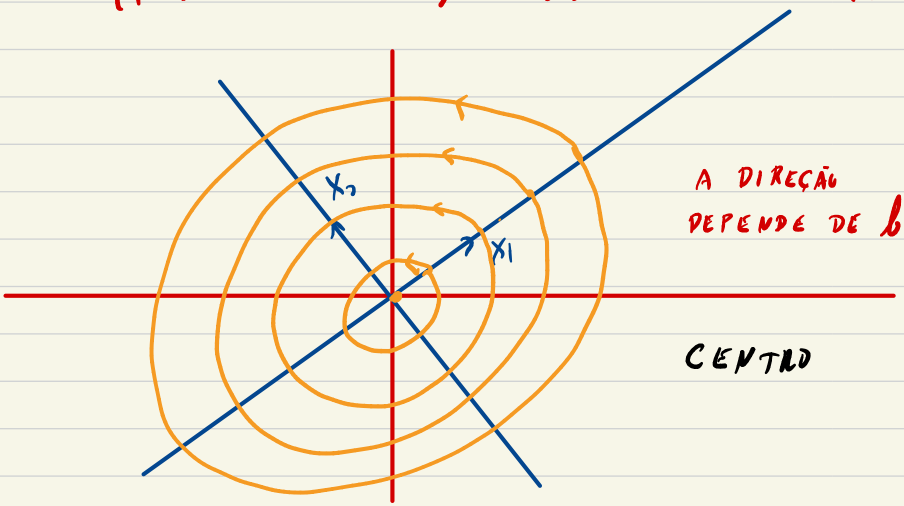
$\alpha_1 e_1 + \alpha_2 e_2 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$

$$= P \begin{pmatrix} e^{ta}\cos(bt) & e^{ta}\operatorname{sen}(bt) \\ -e^{ta}\operatorname{sen}(bt) & e^{ta}\cos(bt) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

$$= P \begin{pmatrix} e^{ta}\cos(bt)\alpha_1 + e^{ta}\operatorname{sen}(bt)\alpha_2 \\ -e^{ta}\operatorname{sen}(bt)\alpha_1 + e^{ta}\cos(bt)\alpha_2 \end{pmatrix}$$

$$\Rightarrow Y(t) = e^{ta}\Big(\big(\cos(bt)\alpha_1 + \alpha_2 \operatorname{sen}(bt)\big) X_1 + \big(-\operatorname{sen}(bt)\alpha_1 + \cos(bt)\alpha_2\big) X_2 \Big)$$

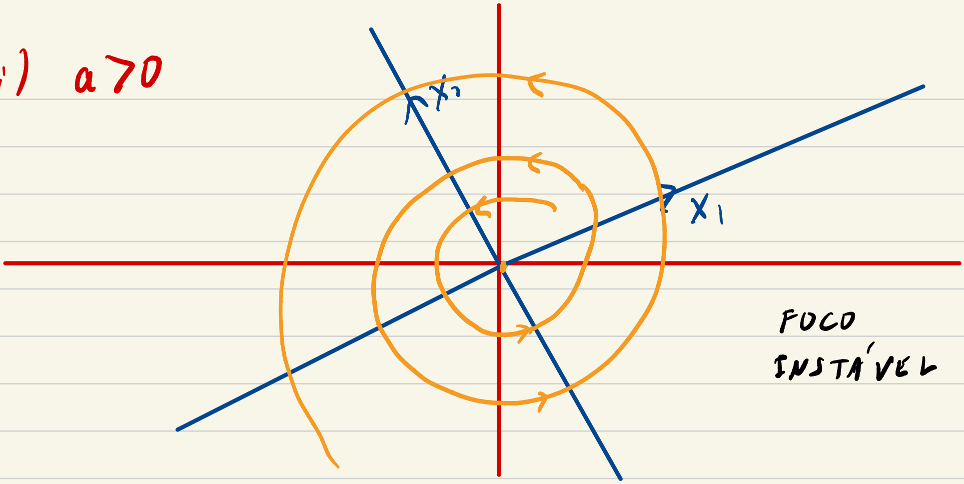
i) $a = 0$



A DIREÇÃO DEPENDE DE b.

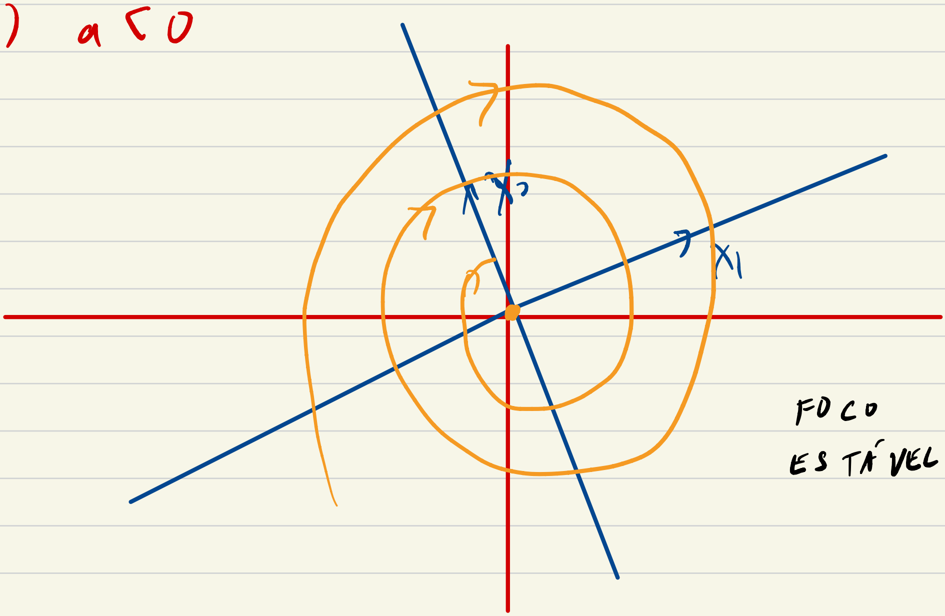
CENTRO

iii) $a > 0$



FOCO INSTÁVEL

iii) $a < 0$



FOCO ESTÁVEL.

FIM!

APLICAÇÃO: MOLA COM AMORTECIMENTO

$F = ma.$ (2º LEI DE NEWTON)

$$F = -kx - bx' \quad , \; b \geq 0, \; k > 0.$$

$$\begin{cases} m\ddot{x} = -kx - b\dot{x} \\ x(0) = x_0 \\ \dot{x}(0) = v_0 \end{cases}$$

DESCREVA QUALITATIVAMENTE AS SOLUÇÕES.

$$q(t) = x(t) \qquad \dot{q}(t) = \dot{x}(t) = \frac{p(t)}{m}$$

$$p(t) = m\dot{x}(t) \qquad \dot{p}(t) = m\ddot{x}(t) = -kx - b\dot{x} = -kq - \frac{b}{m}p.$$

OBTEMOS O SISTEMA:

$$\underbrace{\boxed{\begin{pmatrix} q'(t) \\ p'(t) \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{m} \\ -k & -\frac{b}{m} \end{pmatrix} \begin{pmatrix} q(t) \\ p(t) \end{pmatrix}}}_{:= A.}$$

VAMOS ACHAR AUTOVALORES.

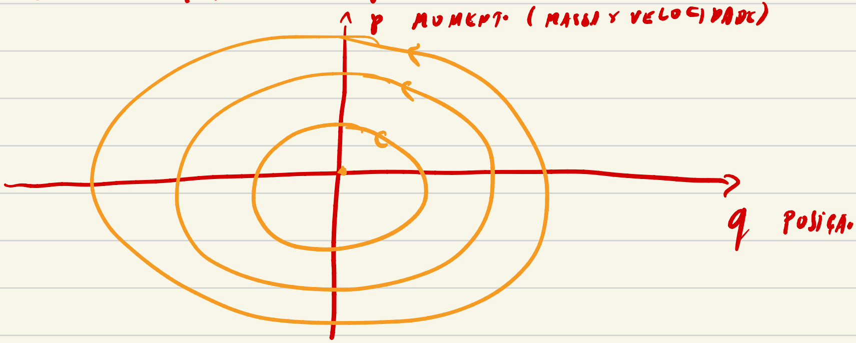
$$\underbrace{\det\begin{pmatrix} \lambda & -\frac{1}{m} \\ k & \lambda + \frac{b}{m} \end{pmatrix}}_{p_A(\lambda)} = \lambda\left(\lambda + \frac{b}{m}\right) + \frac{k}{m} //$$

RAÍZES $\lambda^2 + \frac{b}{m}\lambda + \frac{k}{m} = 0 \Rightarrow \lambda_{\pm} = -\frac{b}{2m} \pm \frac{\sqrt{\left(\frac{b}{m}\right)^2 - 4\frac{k}{m}}}{2}$

$$\lambda_{\pm} = -\frac{b}{2m} \pm \frac{\sqrt{\left(\frac{b}{m}\right)^2 - 4\frac{k}{m}}}{2}$$

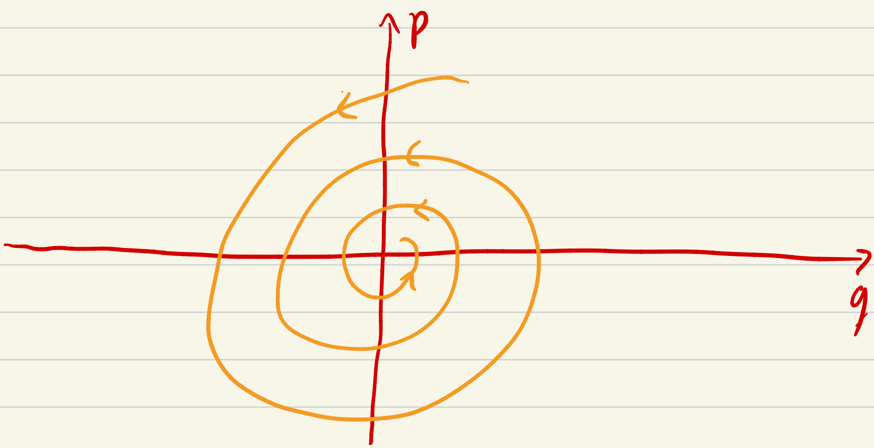
$$\lambda_\pm = -\frac{b}{2m} \pm \frac{1}{2}\sqrt{\left(\frac{b}{m}\right)^2 - 4\frac{k}{m}} \qquad b \geq 0$$

CASO 1 $\quad b = 0 \qquad \lambda_\pm = \pm i\sqrt{\frac{k}{m}}$



CASO 2 $\quad b > 0 \qquad \left(\frac{b}{m}\right)^2 < 4\frac{k}{m}$

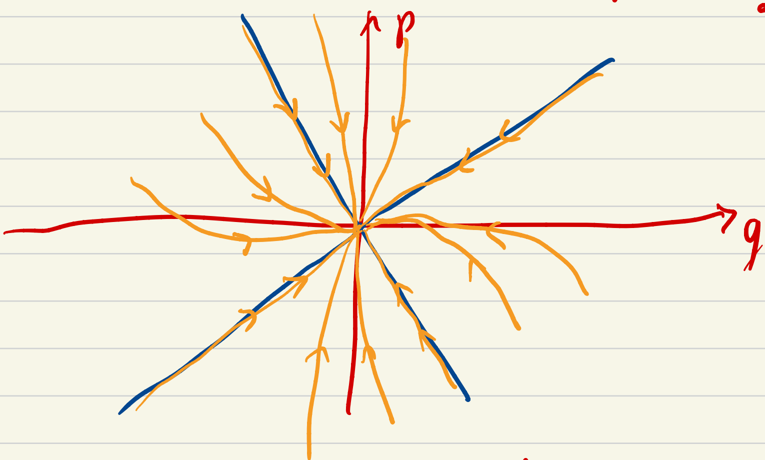
$$\lambda_\pm = -\frac{b}{2m} \pm \frac{1}{2}\, i\sqrt{4\frac{k}{m} - \left(\frac{b}{m}\right)^2}$$



CASO 3 $\quad b > 0 \quad \left(\frac{b}{m}\right)^2 > 4\frac{k}{m}$

$$\lambda_\pm = -\frac{b}{2m} \pm \frac{\sqrt{\left(\frac{b}{m}\right)^2 - 4\frac{k}{m}}}{2}$$

TEMOS 2 RAIZES NEGATIVAS $\left(\frac{\sqrt{\left(\frac{b}{m}\right)^2 - 4\frac{k}{m}}}{2} < \frac{b}{2m} \right)$



CASO 4 $\quad b > 0 \quad \left(\frac{b}{m}\right)^2 = 4\frac{k}{m}$.

$$\lambda_\pm = -\frac{b}{2m}. \qquad A = \begin{pmatrix} 0 & \frac{1}{m} \\ -k & -\frac{b}{m} \end{pmatrix} \neq \lambda \cdot I.$$