01/12/20 # Multiplicative ergodic theorem of Oseledets

Def: A flag in $\mathbb{R}^d$ is a finite sequence of sub spaces: $\boxed{1968}$

$$\mathbb{R}^d = V_1 > V_2 > \cdots > V_k > \{0\}.$$

flag is complete $d = k$, $\dim V_2 = d-1$, .. $\dim V_k = 1$

$$\dim V_j = d - j + 1$$

$f: M \to M$ measurable and μ f-invariant; $A: M \to GL(d)$ measurable s.t $\log^+\|A\|, \log^+\|A^{-1}\| \in L^1(\mu)$.

Thm: for μ-a.e $x \in M$, there exist a flag $\boxed{\mathbb{R}^d = V_x^1 > V_x^2 > \cdots > V_x^{k(x)} > \{0\}}$

and numbers $\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_{k(x)}(x)$ s.t

1. $\boxed{A(x) V_x^i = V_{f(x)}^i}$; $\lambda_i(x) = \lambda_i(fx)$ and $\boxed{k(x) = k(fx)}$ for μ-a.e $x \in M$.

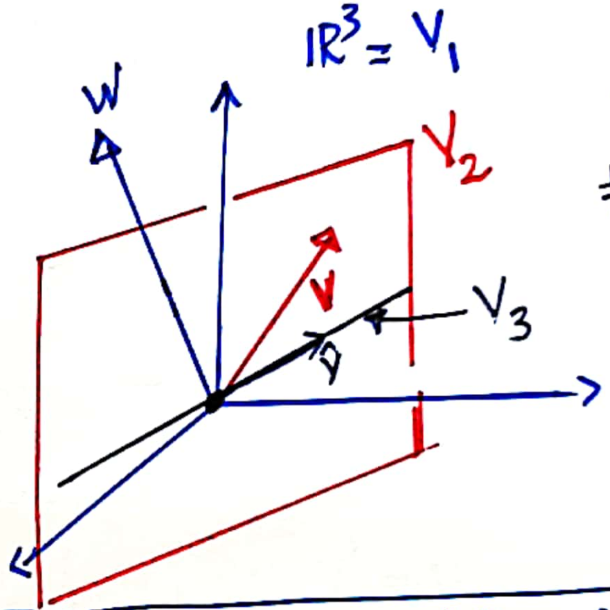
$$V_{f(x)}^i = A(x) V_x^i$$

2/ $\boxed{x \longmapsto V_x^i}$ ← subspace of $\mathbb{R}^d$; $x \longmapsto \lambda_i(x)$ and $x \mapsto k(x)$ are measurable. ②

3/ $\displaystyle \lim_{n\to+\infty} \frac{1}{n} \log \| A^n(x) v^i \| = \lambda_i(x) \quad \forall\, v^i \in V_x^i \setminus V_x^{i+1}$.

<u>d=3</u>: $\mathbb{R}^3 = V_1 > V_2 \overset{V_3 >}{>} \{0\}$.

$\dim V_2 = 2 \qquad \lambda_1 \quad \lambda_2 \quad \lambda_3$



$\dim V_x^i \neq \dim V_y^i$

$$\begin{cases} \displaystyle\lim_{n\to+\infty} \frac{1}{n} \log \| A^n(x) w \| = \lambda_1 \\ \displaystyle\lim_{n\to+\infty} \frac{1}{n} \log \| A^n(x) v \| = \lambda_2 \\ \displaystyle\lim_{n\to+\infty} \frac{1}{n} \log \| A^n(x) \hat{v} \| = \lambda_3 \end{cases}$$

$x \longmapsto V_x^i \qquad \dim V_x^i$

$x \mapsto V_x^i$ is measurable if:

① $\forall\, \ell \in \{1, \dots, d\}$. $E_\ell = \{x : \dim V_x = \ell\}$ is a measurable set

② $\forall\, \ell \in \{1, \dots, d\}$ there exist measurable map $x_j : E_\ell \longrightarrow \mathbb{R}^d$

s.t $\{x_1(x), \dots, x_\ell(x)\}$ is a basis for V_x, $x \in E_\ell$

$V_x^1 > \dots > V_x^{k(x)} \leftarrow$ Oseledets flag of F at x

$\lambda_1(x) > \dots > \lambda_{k(x)}(x) \leftarrow$ Lyapunov exponents of F at x.

Rmk: $\lambda_+(x) = \lambda_1(x)$ and $\lambda_-(x) = \lambda_{k(x)}(x)$ Extremum Lyapunov exponents.

Construction of Oseledets flag

For each $v \in \mathbb{R}^d \setminus \{0\}$ and $x \in M$: $\boxed{\lambda(x,v) = \limsup_{n \to +\infty} \frac{1}{n} \log \|A^n(x) v\|}$

Lemma:

✓ a) $\lambda(x, cv) = \lambda(x, v)$ $\forall v \in \mathbb{R}^d \setminus \{0\}$ $c \neq 0$ ✓

✓ b) $\lambda(x, v+v') = \max\{\lambda(x,v); \lambda(x,v')\}$ if $v+v' \neq 0$ ✓

$a_n, b_n > 0$ $\limsup_n \frac{1}{n} \log(a_n + b_n) = \max\{\limsup_n \frac{1}{n} \log a_n; \limsup_n \frac{1}{n} \log b_n\}$

✓ c) $\lambda(x,v) = \lambda(f(x), A(x)v)$ $\quad \log \|A^n(f(x)) A(x) v\| = \log \|A^{n+1}(x) v\|$

For every $x \in M$, the function : $\mathbb{R}^d \setminus \{0\} \longrightarrow \mathbb{R}$ can only take ④

$v \longmapsto \lambda(x,v)$ finitely many values

$v_1, \dots, v_d$ st $\cancel{v_i \neq v_j}$ $i \neq j$ $1 \leq i,j \leq d$

$\lambda(x, v_i) \neq \lambda(x, v_j)$ $\Rightarrow$ v_i are linearly indpt

$w \in \mathbb{R}^d \setminus \{0\}$: $w = \sum_{i=1}^{d} \alpha_i v_i$ $\lambda(x, w) = \lambda(x, v_{i_0})$

let $\lambda_1(x) > \lambda_2(x) > \dots > \lambda_{k(x)}(x)$ be those values in decreasing order

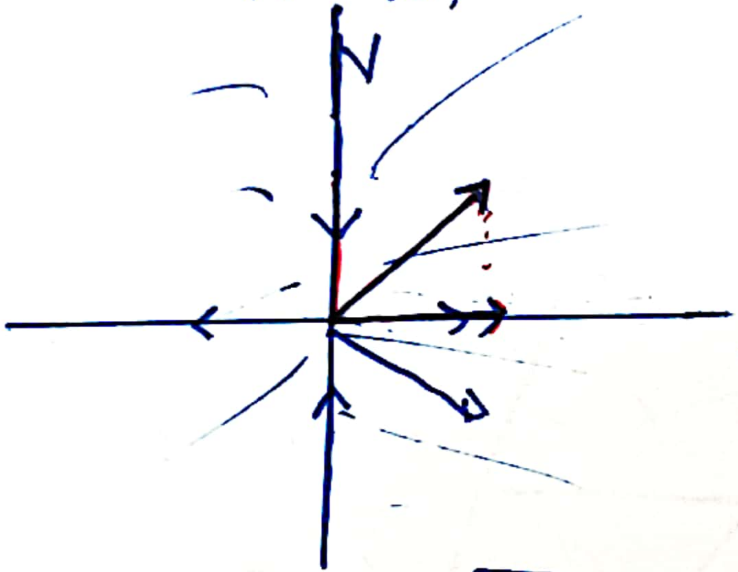
$$\boxed{V_x^i = \{ v \in \mathbb{R}^d \setminus \{0\} : \lambda(x,v) \leq \lambda_i(x) \} \cup \{0\}} \quad \forall i = 1, \dots, k(x).$$

V_x^i are subspaces of $\mathbb{R}^d$. $\underline{\forall v \in V_x^i \setminus V_x^{i+1} : \lambda(x,v) = \lambda_i(x)}$.

$\lambda_{i_0}(x) \Rightarrow v \in V_x^{i_0} < V_x^{i-1}$

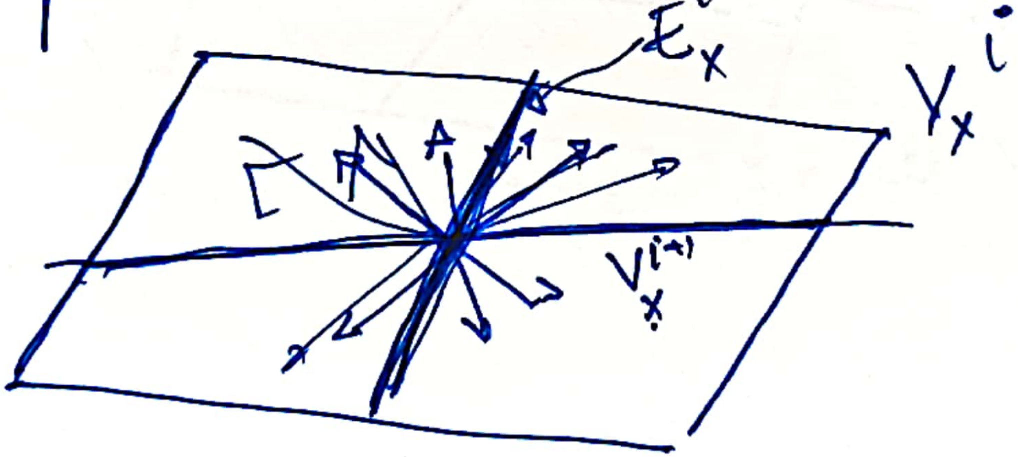
Example:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \in SL(2) \qquad \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$



$$\lim_{n\to+\infty} \frac{1}{n} \log \| A^n v \| = \begin{cases} \log 2 & \text{if } v \text{ is not vertical} \\ -\log 2 & \text{if } v \text{ is vertical} \end{cases}$$

$$\lim_{n\to-\infty}$$

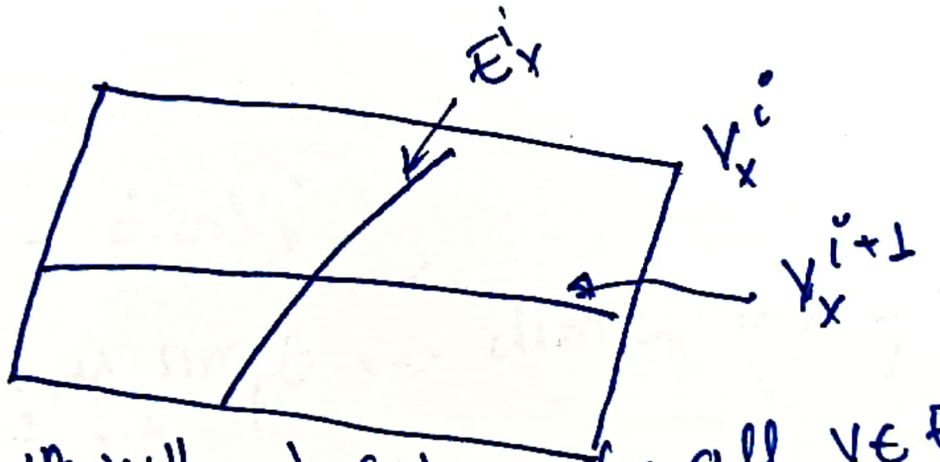


$$\lambda(x,v) = \lambda_i(x)$$

<u>Thm</u> (Oseledets) $f: M \longrightarrow M$ is invertible Then for μ-a.e $x \in M$

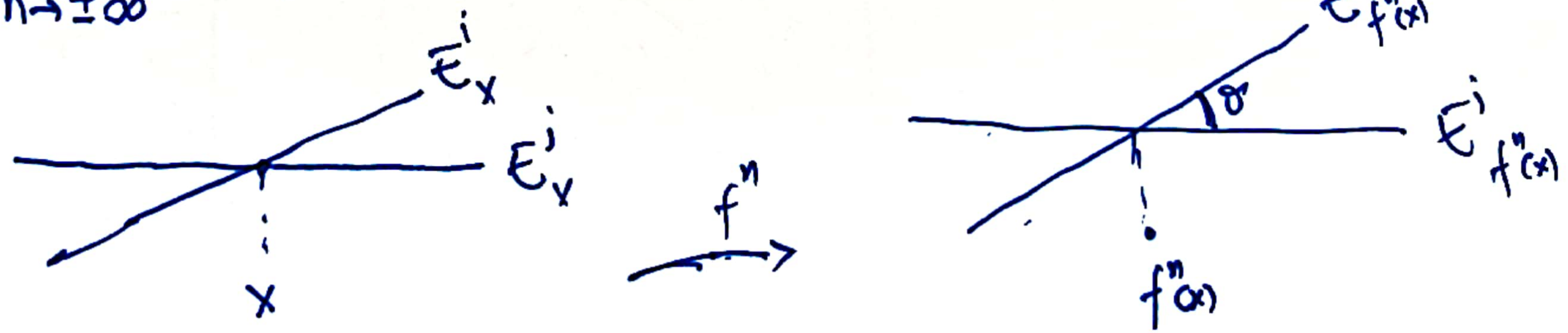
$$\mathbb{R}^d = E^1_x \oplus E^2_x \oplus \cdots \oplus E^{k(x)}_x \quad \text{s.t } \forall\, i = 1, \ldots, k.$$

(a) $A(x) \cdot E^i_x = E^i_{f(x)}$ and $V^i_x = \bigoplus_{j=i}^{k(x)} E^j_x = V^{i+1}_x \oplus E^i_x$

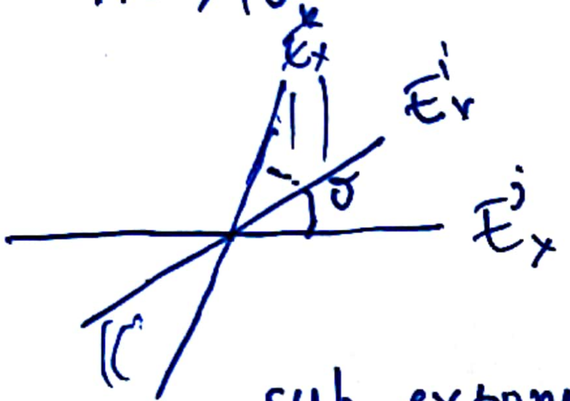


(b) $\lim_{n \to \pm\infty} \frac{1}{n} \log \|A^n(x) v\| = \lambda_i(x)$ for all $v \in E^i_x \setminus \{0\}$.

(c)



ⓒ $\lim\limits_{n\to+\infty} \frac{1}{n}\log\left|\sin\measuredangle\left(\bigoplus_{i\in I} E^i_{f^n_x},\ \bigoplus_{j\in J} E^j_{f^n_x}\right)\right| = 0$ ← whenever $\underbrace{I\cap J=\phi}$



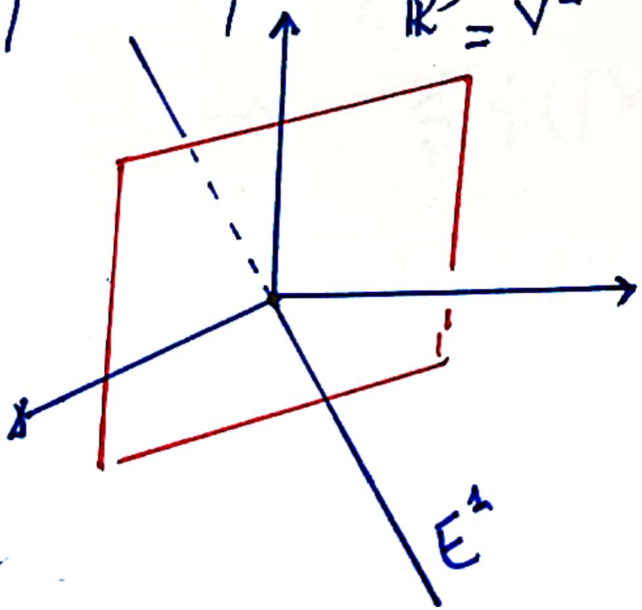
$i\neq j$

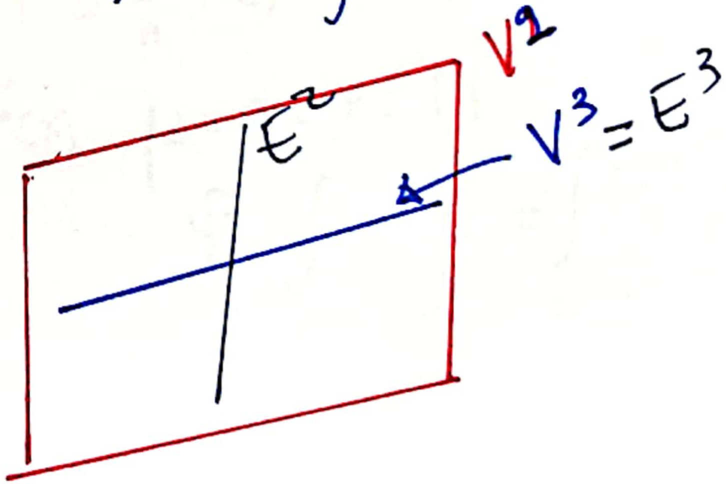
$$\underbrace{\left|\sin\measuredangle\left(\bigoplus_{i\in I} E^i_{f^n_x},\ \bigoplus_{j\in J} E^j_{f^n_x}\right)\right| \geqslant e^{-n\varepsilon}}$$

sub-exponential decay of angles.

$\dim E^i_x = \dim V^i_x - \dim V^{i+1}_x$; E^i_x : Oseledets subspace.

Lyapunov spectrum is simple $\Leftrightarrow$ $\dim E^i_x = 1$ for every i.





Lemma 1 : $f: M \to M$ measurable, μ an f-invariant ergodic μ $\phi: M \to \mathbb{R}$ be a measurable funct$^{\underline{o}}$. Then if $\boxed{\phi \circ f - \phi}$ is integrable w.r.t μ then: $\boxed{\lim_{n \to +\infty} \frac{1}{n} \phi(f^n x) = 0}$ μ-a.e $x \in M$.

Proof : $\Psi = \phi \circ f - \phi$

$$\Psi \in L^1(\mu) \underset{\text{Birkhoff}}{\Longrightarrow} \frac{1}{n} \sum_{J=0}^{n-1} \Psi(f^n x) \longrightarrow \tilde{\Psi}(x) \quad \mu\text{-a.e } x \in M.$$

$$\frac{1}{n} \sum_{J=0}^{n-1} \Psi(f^n x) = \frac{1}{n} \sum_{J=0}^{n-1} \phi(f^{j+1} x) - \phi(f^j x) = \frac{1}{n} \phi(f^n x) - \frac{1}{n} \phi(x)$$

$$\Rightarrow \quad \frac{1}{n} \phi(f^n x) \longrightarrow \tilde{\Psi}(x) \quad \mu\text{-a.e } x \in M$$

$$\tilde{\Psi} = \int \Psi \, d\mu = \underbrace{\int \phi \circ f \, d\mu - \int \phi \, d\mu}_{= 0}$$

$\forall \varepsilon > 0$

$$E_n = \left\{ x \in M : \left| \frac{1}{n} \phi(f^n x) - 0 \right| \geq \varepsilon \right\}$$

$$E_n = \left\{ x \in M : \quad |\phi(f^n x)| \geq n\varepsilon \right\} = f^{-n}\Big(\underbrace{\{y \in M : |\phi(y)| \geq n\varepsilon\}}_{F_n}\Big)$$

$\mu(E_n) = \mu(F_n)$; $F_{n+1} \subset F_n$ $\qquad F = \bigcap_{n \geq 1} F_n = \phi$

$$0 = \mu(F) = \lim_{n \to +\infty} \mu(F_n) = \lim_{n \to +\infty} \mu(E_n)$$

$\frac{1}{n} \phi \circ f^n \longrightarrow 0$ in measure / cvg a.e $\Rightarrow$ cvg in measure

$\Rightarrow \frac{1}{n} \phi \circ f^n \longrightarrow \tilde{\varphi}$ in measure) uniqueness of limit $\tilde{\varphi} = 0$ μ-a.e x.

Lemma 2 $L : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ invertible linear transformat° v, w non zero vector

then : $$\frac{1}{\|L\| \|L^{-1}\|} \leq \frac{|\sin \measuredangle(Lv, Lw)|}{|\sin \measuredangle(v, w)|} \leq \|L\| . \|L^{-1}\|$$

Exercise

$$\varphi(x) = \log \left| \sin \measuredangle(\hat{E}^1_x, \hat{E}^2_x) \right|$$

$$\begin{cases} \hat{E}^1_x = \bigoplus_{i \in I} E^i_x \\ \hat{E}^2_x = \bigoplus_{j \in J} E^j_x \end{cases}$$

$$|\varphi(fx) - \varphi(x)| = \left| \log \frac{\left| \sin(\measuredangle \hat{E}^1_{f(x)}, \hat{E}^2_{f(x)}) \right|}{\left| \sin \measuredangle(\hat{E}^1_x, \hat{E}^2_x) \right|} \right|$$

$$\leq \underbrace{\log \|A(x)\|}_{\in L^1(\mu)} + \underbrace{\log \|A(x)^{-1}\|}_{\in L^1(\mu)}$$

$$\Longrightarrow \quad \varphi \circ f - \varphi \in L^1(\mu)$$

lemma 1 $\Longrightarrow \dfrac{1}{n} \log \left| \sin \measuredangle(\hat{E}^1_{f^n x}, \hat{E}^2_{f^n x}) \right| \xrightarrow[n \to +\infty]{} 0$

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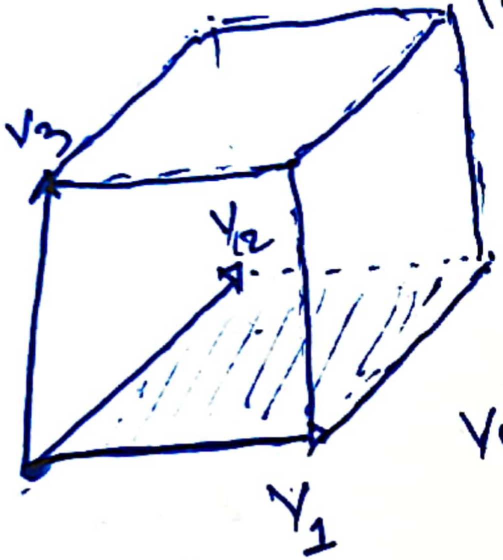
Application : $F: M\times\mathbb{R}^d \longrightarrow M\times\mathbb{R}^d$

$(x, v) \longmapsto (f(x), A(x)v)$

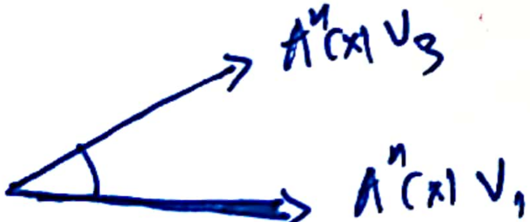


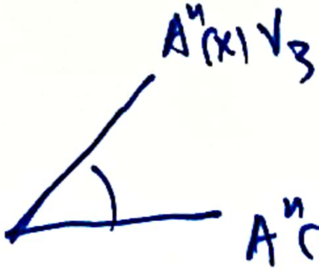
P What can we say about $\mathrm{Vol}(A^n(x)P)$ $n\to+\infty$?

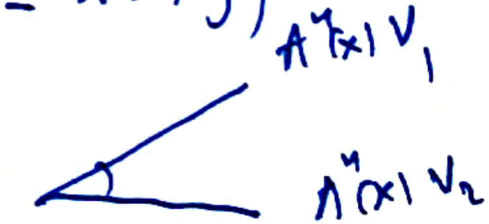
$$\mathrm{Vol}(A^n(x)P) = \|A^n(x)v_1\|\cdot\|A^n(x)v_2\|\cdot\|A^n(x)v_3\|\cdot \Xi(\text{angles})$$

$$\|A^n(x)v_j\| \approx e^{n\theta_j} \qquad \theta_j = \lambda(x, v_j)^*$$

$j = 1, 2, 3$







$$\lim_{n\to+\infty}\frac{1}{n}\log \alpha_n = 0$$

$$\frac{1}{n}\log \mathrm{Vol}(A^n(x)P) \longrightarrow \underbrace{\vartheta_1+\vartheta_2+\vartheta_3}.$$

$\lambda(x, v_1)$

possible values of ϑ_j.

some of 3 Lyapune.

$\underline{d=5}$: $E^1_x \oplus E^2_x \oplus E^3_x$

$\dim E^1_x = 3$ dim 1   dim 1

$$\vartheta_1+\vartheta_2+\vartheta_3 \in \{\lambda_1+\lambda_1+\lambda_1;\ \lambda_1+\lambda_2+\lambda_3;\ \lambda_1+\lambda_1+\lambda_2;\ \lambda_1+\lambda_1+\lambda_3\}$$

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