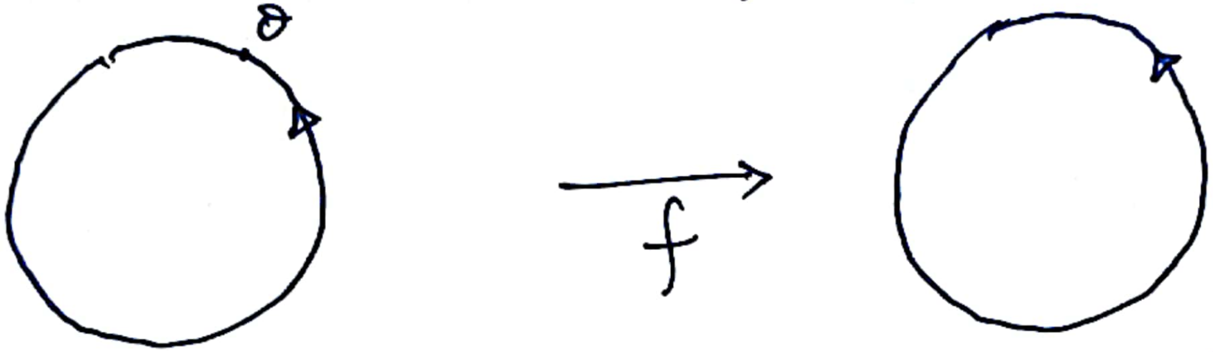
25/11/20 : Example (Michael Herman)

$f: S^1 \to S^1$ be a covering map with degree k

Ex: $z \mapsto z^k$

$\{z \in \mathbb{C} : |z| = 1\}$

$k=2$ doubling ...



$A: S^1 \to SL(2)$

$\theta \mapsto \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}}_{A_0} \boxed{R_\theta}$ (product of a rotation with a hyp matrix)

Claim: The corresponding cocycle is not uniformly hyperbolic unless $\boxed{k \in \{-1, 0, 2; 3\}}$.

Assume that the cocycle is UH.

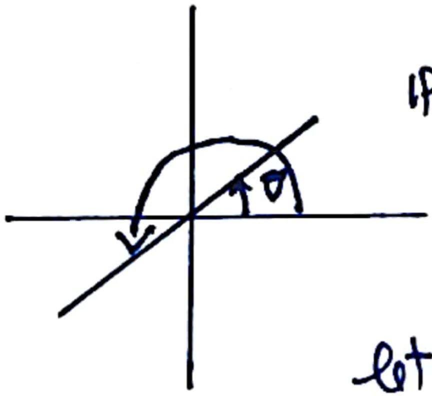
$\forall \theta \in S^1 : \mathbb{R}^2 = E^u_\theta \oplus E^s_\theta$

$\theta \mapsto E^u_\theta$; $\theta \mapsto E^s_\theta$ are continuous.

$$\Psi : S^1 \longrightarrow \mathbb{P}\mathbb{R}^2$$
$$\theta \longmapsto E^u_\theta$$

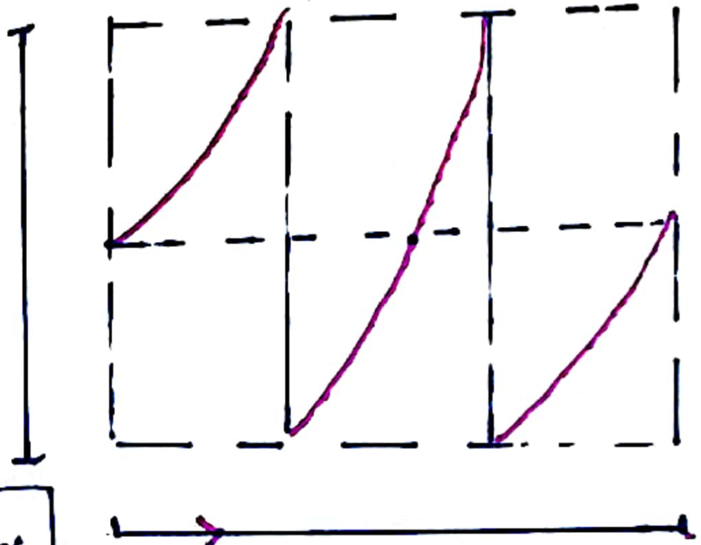
$\dim E^u_\theta = 1$

Ψ is continuous.



$\mathbb{P}\mathbb{R}^2 = [0,\pi]/_{0\sim\pi}$

$\mathbb{P}\mathbb{R}^2$



θ

S^1

$\deg \Psi.$

let $\boxed{n = \deg \Psi \in \mathbb{Z}}$

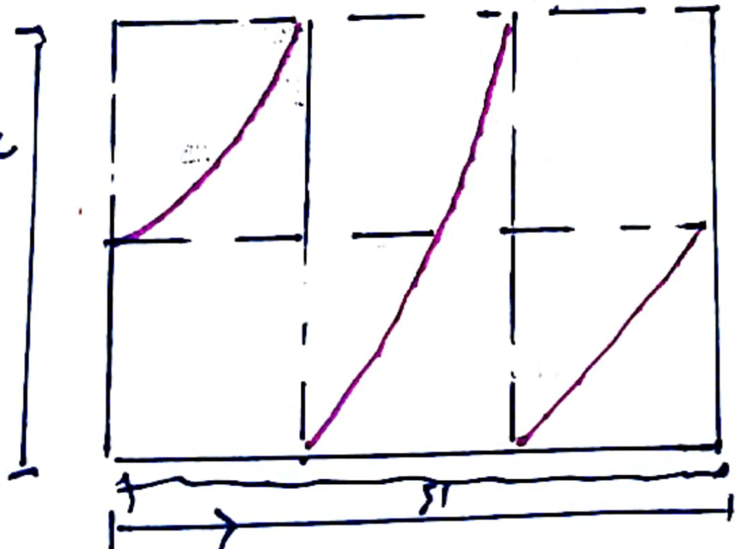
$\{(\theta, E^u_\theta) : \theta \in S^1\}$

(f, θ) cocycle

$\{(f(\theta), E^u_{f(\theta)}) : \theta \in S^1\}$

$E^u_{f(\theta)} = \underbrace{A(\theta)}\, E^u_\theta$

$\xrightarrow{\text{cocycle}}$ f, A $\mathbb{P}\mathbb{R}^2$



S^1

θ

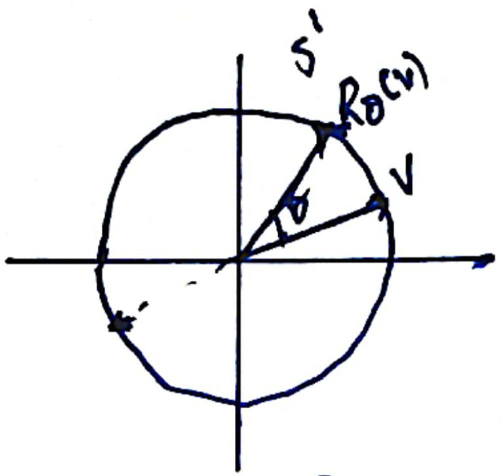
S^1

After iterat$^{\underline{o}}$ the graph goes $\boxed{K.n}$ around $\mathbb{P}\mathbb{R}^2$ when θ goes once around S^1

$A(\theta) = A_0 \cdot R(\theta)$ ← rotation

$\boxed{A(\theta)\, E^u_\theta}$

$\boxed{2+n}$ around $\mathbb{P}\mathbb{R}^2$



θ goes once around S^1

$R_\theta v$ " once around S^1

$R_\theta v$ 2- around $\mathbb{P}\mathbb{R}^2$

$K \cdot n = 2 + n$

$\Longleftrightarrow \; n = \dfrac{2}{K-1}$ has integer solution

$\Longleftrightarrow \; \boxed{K \in \{-1; 0; 2; 3\}}$

Def: A diffeomorphism $f: M \to M$ is called Anosov if the derivative cocycle $Df: TM \circlearrowleft$ is uniformly hyp.

$\chi(M) = 0$

on surface: only on torus.

New topic

Sub-additive ergodic theorem

$\mathcal{B}$

M measurable space

$f: M \to M$ measurable map

μ probability measure on M. (μ f-invariant)

$A \in \mathcal{B}$ measurable subset : $\mu(f^{-1}(A)) = \mu(A)$

$f_* \mu = \mu$.

$\Leftrightarrow$ " $\forall \psi: M \to \mathbb{R}$ μ-int

$\int \psi \circ f \, d\mu = \int \psi \, d\mu$ "

Def: A sequence $\varphi_n : M \to \mathbb{R}$ (measurable) is sub-additive if

$$\boxed{\varphi_{n+m}(x) \leq \varphi_m(f^n(x)) + \varphi_n(x)} \quad \forall n, m \geq 1 \quad \mu\text{.a.e } x \in M$$

Example 1: Given $\varphi: M \to \mathbb{R}$; $\varphi_n(x) := \varphi(x) + \varphi(f(x)) + \dots + \varphi(f^{n-1}x)$

$$\varphi_{n+m}(x) = \underbrace{\varphi(x) + \varphi(fx) + \dots + \varphi(f^{n-1}_x)}_{} + \underbrace{\varphi(f^n_x) + \dots + \varphi(f^{n+m-1}_x)}_{m\text{-terms}}$$

$$\varphi_{n+m}(x) = \varphi_n(x) + \varphi_m(f^n_x) \quad \varphi_n(x)$$

additive.

Example 2: $A : M \to GL(d)$

$$\varphi_n(x) = \log \| A^n(x) \|$$

$$\|B\| = \sup_{v \neq 0} \frac{\|Bv\|}{\|v\|}$$

$$\|BC\| \leq \|B\| \cdot \|C\|$$

sub-multiplicat

$$\varphi_{n+m}(x) = \log \| A^{n+m}(x) \|$$
$$= \log \| A^m(f^n x) \cdot A^n(x) \|$$
$$\leq \log \| A^m(f^n x) \| + \log \| A^n(x) \| = \varphi_m(f^n x) + \varphi_n(x)$$

$\Rightarrow (\varphi_n)_n$ sub additive.

Example 3: $\varphi_n(x) = \log m(A^n(x))$

$$m(\underset{GL(d)}{A}) = \inf_{v \neq 0} \frac{\|Av\|}{\|v\|}$$

$$\varphi_{n+m}(x) \geq \varphi_m(f^n x) + \varphi_n(x)$$ "super-additive"

$m(BC) \geq m(B) \cdot m(C)$

super-multiplicativ

$(-\varphi_n)_n$ is sub-additive.

$\varphi : M \to \mathbb{R}$

positive part $\varphi^+(x) = \max\{\varphi(x), 0\}$

Thm : (sub-additive ergodic thm) Kingman. $f : M \to M$, μ

let $\varphi_n : M \to [-\infty, +\infty)$, $n \geq 1$ be a sub-additive sequence of measurable funct°; s.t φ^+_1 is integrable w.r.t μ. L^1

Then $(\varphi_n / n)_n$ converges μ-a.everywhere to some function

$$\tilde{\varphi} : M \to [-\infty, +\infty) \text{ Moreover}$$

$\frac{\varphi_n(x)}{n} = e^{n}$ e

$e^{n+m} \leq e^n e^m$

ⓐ $\tilde{\varphi} \circ f(x) = \tilde{\varphi}(x)$ μ-a.e $x \in M$.

ⓑ $\tilde{\varphi}^+$ is integrable w.r.t μ and

$$\int \tilde{\varphi} \, d\mu = \lim_n \frac{1}{n} \int \varphi_n \, d\mu = \inf \left\{ \frac{1}{n} \int \varphi_n \, d\mu \right\}$$

Rmk If φ_1 is integrable $\Rightarrow$ $\forall n$ φ_n is also integrable

$n = (n-1) + 1$ $\quad \varphi_n(x) \leq \varphi_{(n-1)+1}(x) \leq \varphi_{n-1}(fx) + \varphi_1(x)$

$$\leq \varphi_1(f^{n-1}x) + \dots + \varphi_1(fx) + \varphi_1(x)$$

$$\forall n: \quad \varphi_n(x) \le \varphi_1(x) + \varphi_1(f(x)) + \cdots + \varphi_1(f^{n-1}x)$$

$$\int \varphi_n \, d\mu \le \underbrace{\int \varphi_1 \, d\mu + \int \varphi_1 \circ f \, d\mu + \cdots + \int \varphi_1 \circ f^{n-1} d\mu}_{n \int \varphi_1 \, d\mu}$$

$\Rightarrow$ if $\varphi_1 \in L^1(\mu)$ then $\varphi_n \in L^1(\mu)$ $\forall n$.

Corollary 1 (Ergodic theorem of Birkhoff)

let $\varphi: M \to \mathbb{R}$ be a μ-integrable function then

$$\tilde{\varphi}(x) = \lim_{n \to +\infty} \frac{1}{n} \left(\varphi(x) + \varphi(f(x)) + \cdots + \varphi(f^{n-1}x) \right) \text{ exists}$$

μ-a.e $x \in M$

$$\int \varphi_n(x) = n \int \varphi \, d\mu \qquad \boxed{\frac{1}{n} \int \varphi_n} = \int \varphi \, d\mu$$

ⓐ $\tilde{\varphi}$ is f-invariant

ⓑ $\int \tilde{\varphi} \, d\mu = \int \varphi \, d\mu$.

Corollary 2 (Furstenberg - Kesten)

Let $(M, \mathcal{B}, \mu)$ be a probability space.

$\boxed{\varphi_n = \log \| A^n(x) \|}$

$\varphi_1 = \log \| A(x) \|$

$f : M \to M$ measure preserving map ($f_* \mu = \mu$)

$A : M \to GL(d)$ be a measurable funct° s.t

$\boxed{\varphi_n = \log m(A^n(x))}$

$\log^+ \| A \| ; \log^+ \| A^{-1} \| \in L^1(\mu)$

$\int \log^+ \| A(x) \| \, d\mu(x) < +\infty \qquad \int \log^+ \| (A(x))^{-1} \| \, d\mu < +\infty$

Then:

ⓐ $\lambda_+(x) = \lim_n \frac{1}{n} \log \| A^n(x) \|$ exits for μ-a.e $x \in M$.

ⓑ $\lambda_-(x) = \lim_n \frac{1}{n} \log \| (A^n(x))^{-1} \|^{-1}$ exits μ.a.e $x \in M$.

Moreover λ_+ and λ_- satisfy:

<< λ_+, λ_- in terms of invariant measures on projective space. >>

(a) $\lambda_\pm \circ f = \lambda_\pm \quad \mu$-a.e

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(b) $\begin{cases} \int \lambda_+ \, d\mu = \inf_n \frac{1}{n} \int \log \|A^n\| \, d\mu. \\ \int \lambda_- \, d\mu = \sup_n \frac{1}{n} \int \log \|A^{-n}\| \, d\mu. \end{cases}$

$A \longmapsto \int \lambda_+ d\mu$

$f: M \to M \quad \mu \quad \left| \begin{array}{l} \forall \varphi: M \to \mathbb{R} \\ \varphi \circ f = \varphi \ \mu\text{-a.e} \end{array} \right.$ is constant μ-a.e.

Rmk: If (f, μ) is ergodic then $\lambda_\pm$ are constant.

$M = X^{\mathbb{N}} \quad X = \{A_1, \dots, A_N\} \quad f: M \to M$ shift map

$p = (p_1, \dots, p_N) \quad \mu = p^{\mathbb{N}} \quad (f, \mu)$ - Bernoulli shift

ergodic.