PMR3523 Controle Moderno

Apoio à Aula

Filtro de Kalman

$$\dot{x} = Ax + Bu + F\underline{v}$$

$$y = Cx + \underline{w}$$

$v \rightarrow$ média 0

w covariância conhecida

v, w SÃO RUÍDOS BRANCOS GAUSSIANOS COM DENSIDADE ESPECTRAL CONHECIDA

OBTER ESTIMADOR ÓTIMO

$$\underbrace{\dot{\hat{x}} = A\hat{x} + B.u}_{\text{MODELO}} + \overbrace{\underbrace{\hat{K}}_{\text{MATRIZ DE OBSERVAÇÃO (KALMAN)}}\underbrace{(\overset{\text{SENSOR}}{y} - \overset{\text{ESTIMADO PELO MODELO}}{C\hat{x}})}_{\text{INOVAÇÃO}}}^{\text{SENSOR}}$$

$\hat{K} \rightarrow$ MATRIZ QUE MINIMIZA A COVARIÂNCIA DO ERRO

$$e = x - \hat{x} \rightarrow E\{e(t).e'(t) / y(\tau); \tau \leq t\}$$

SENDO $V(t)$ MATRIZ DE COVARIÂNCIA DE $v(t)$ →
RUIDO DO PROCESSO

$$V(t) = \begin{pmatrix} E(v_1,v_1) & E(v_1,v_2) & \cdots \\ \vdots & E(v_2,v_2) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \rightarrow$$

MATRIZ COVARIÂNCIA RUIDO DE MODELO OU PROCESSO

$$E(v_i,v_j) = 0 \; p/ \; i \neq j$$

(DIAGONAL)

$W(t)$ → MATRIZ COVARIÂNCIA RUIDO MEDIDA

$$W(t) = \begin{pmatrix} E(w_1,w_1) & \cdots \\ \vdots & \ddots \end{pmatrix} \rightarrow$$

COVARIANCIA DOS RUIDOS DOS SENSORES

EM GERAL $E(w_i,w_j) = 0 \; p/ \; i \neq j$

(DATASHEET DOS SENSORES)

$$\left[\begin{array}{l} \hat{K} = \hat{P} . C' . W^{-1} \\ \dot{\hat{P}} = A\hat{P} + \hat{P}A - \hat{P}C'W^{-1}C.\hat{P} + FVF' \end{array}\right.$$

A, C → SISTEMA

W, V → COVARIÂNCIA RUIDOS (CONHECIDO OU PARAMETRO DE TUNING)

P → COVARIANCIA ERRO ESTIMAÇÃO (CALCULADO)

FILTRO KALMAN EM REGIME

APOS TEMPO $\dot{\hat{P}} \to 0$

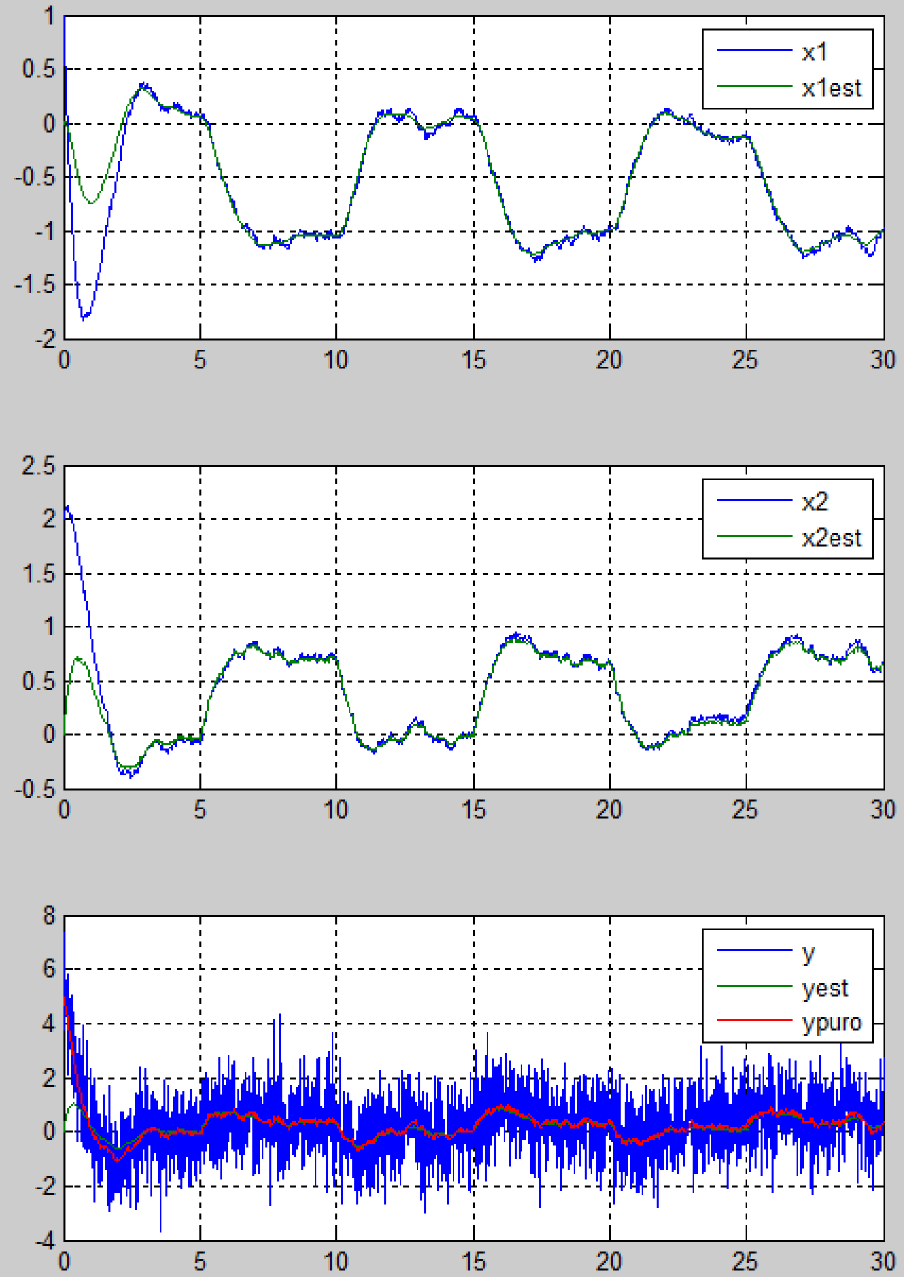
$$0 = A\hat{P} + \hat{P}A - \hat{P}C'W^{-1}C\hat{P} + FVF'$$

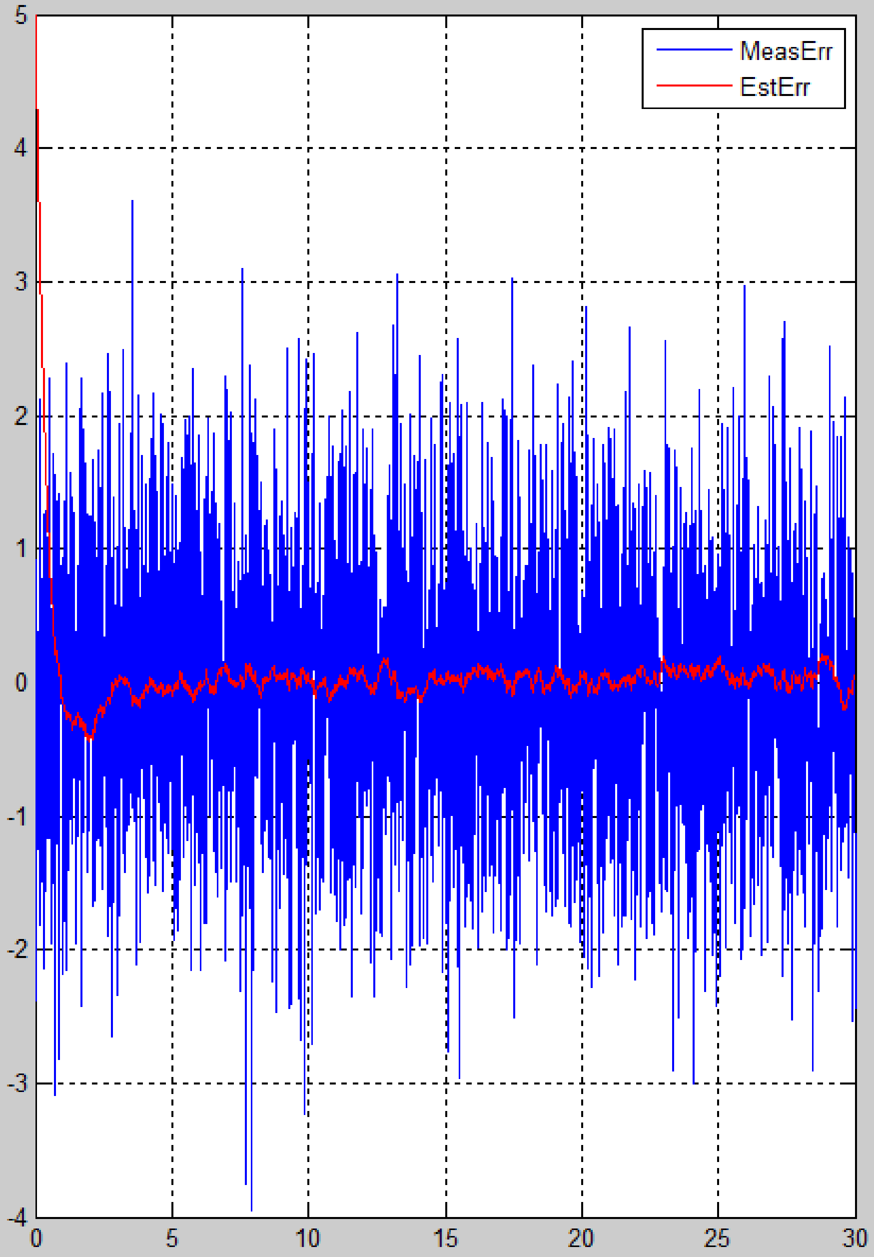
↳ ALGÉBRICA

Ex)

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}2 & 3\\ 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}u + \overbrace{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}^{F}\underbrace{\begin{pmatrix}w_1\\ w_2\end{pmatrix}}_{\text{RUIDOS PROCESSO}}$$

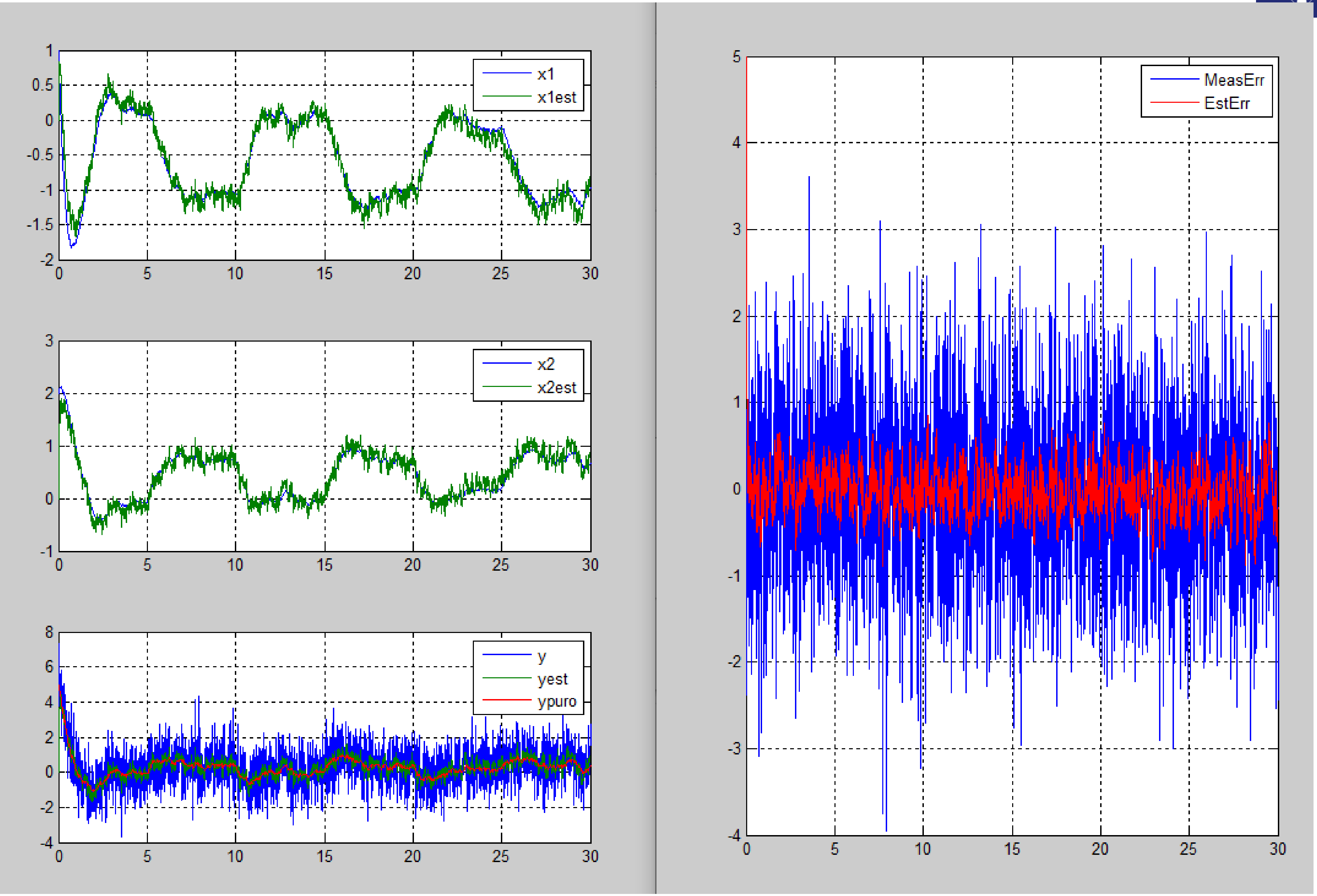
$$y = \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \underbrace{v}_{\text{RUIDO SENSOR}}$$





$R=0{,}01$

$Q=\begin{pmatrix} 0{,}01 & 0 \\ 0 & 0{,}01 \end{pmatrix}$

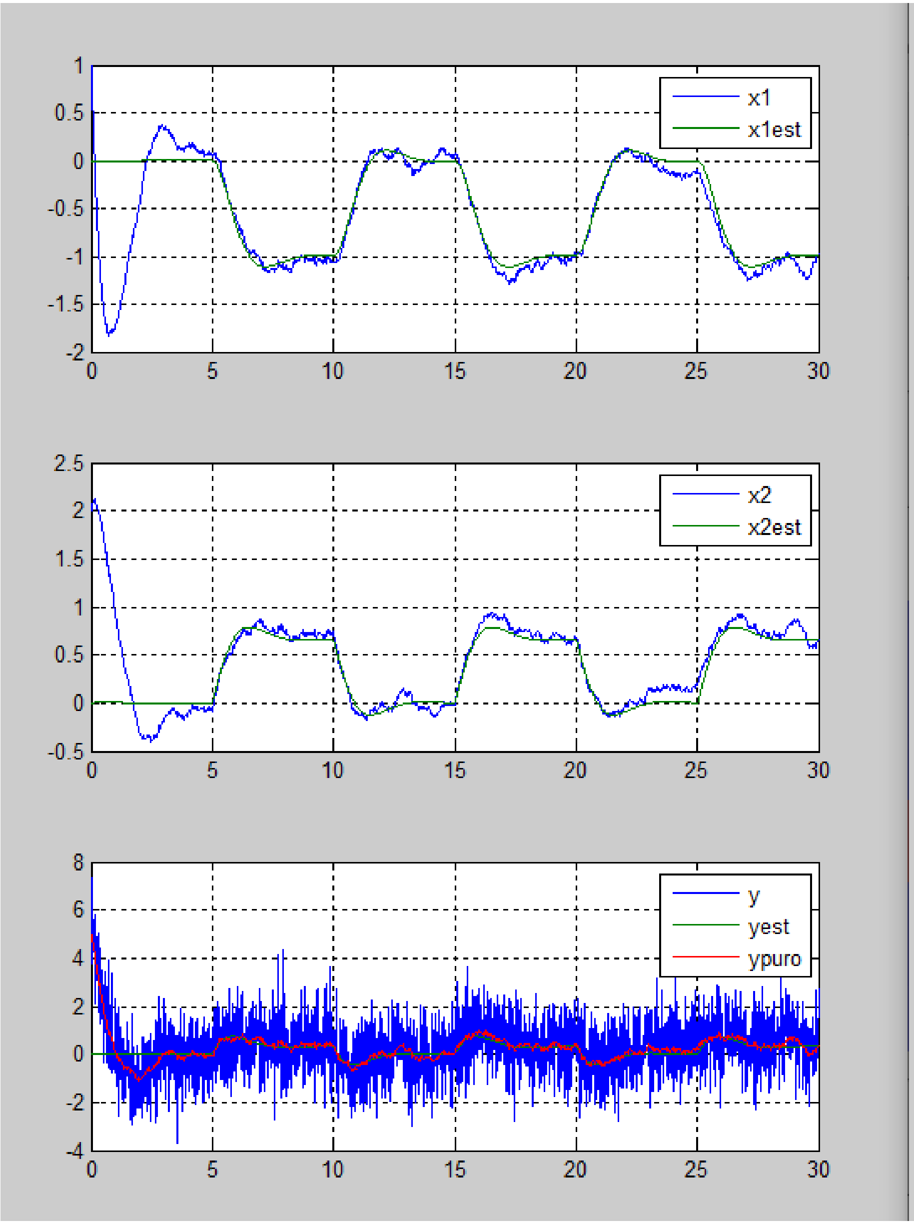


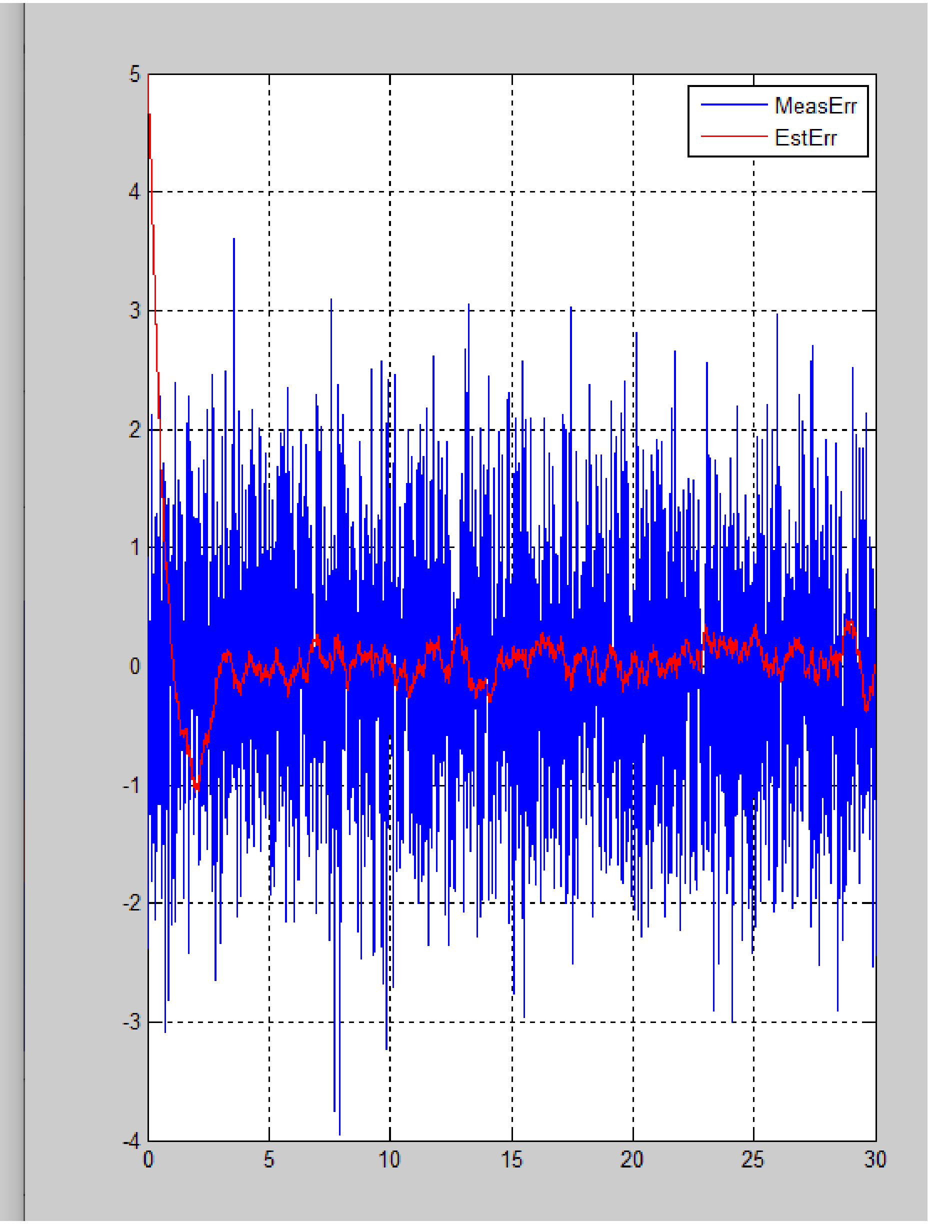
$R = 0{,}0001$

$$Q = \begin{pmatrix} 0{,}01 & 0 \\ 0 & 0{,}01 \end{pmatrix}$$

COMO R = PEQUENO O FK PASSA A "CONFIAR" MAIS NO SENSOR, E MENOS NO MODELO.

A SAÍDA DO FK PASSA A ACOMPANHAR MAIS AS MEDIDAS DO SENSOR





$R = 1$

$Q = \begin{pmatrix} 0{,}01 & 0 \\ 0 & 0{,}01 \end{pmatrix}$

COMO R=GRANDE, FK CONFIA MAIS NO MODELO, PRATICAMENTE IGNORA MEDIDA DO SENSOR

Syntax

```
[kest,L,P] = kalman(sys,Qn,Rn,Nn)
[kest,L,P] = kalman(sys,Qn,Rn,Nn,sensors,known)
[kest,L,P,M,Z] = kalman(sys,Qn,Rn,...,type)
```

SYS=SS(A,[B G],C,[D H])

Description

`kalman` designs a Kalman filter or Kalman state estimator given a state-space model of the plant and the process and measurement noise covariance data. The Kalman estimator provides the optimal solution to the following continuous or discrete estimation problems.

Continuous-Time Estimation

Given the continuous plant

$$\dot{x} = Ax + Bu + Gw \qquad \text{(state equation)} \quad *$$
$$y = Cx + Du + \underline{Hw} + v \qquad \text{(measurement equation)}$$

w → RUIDO PROCESSO } CONTRÁRIO À
v → " SENSOR } NOTAÇÃO LIVRO!

H = 0 (em geral)

with known inputs u, white process noise w, and white measurement noise v satisfying

$$E(w) = E(v) = 0, \quad \underline{E(ww^T) = Q}, \quad \underline{E(vv^T) = R}, \quad \underline{E(wv^T) = N} \rightarrow N=0$$

$$* \begin{cases} \dot{x} = Ax + [B \,\vdots\, G]\begin{pmatrix} u \\ w \end{pmatrix} \\ y = Cx + [D \,\vdots\, H]\begin{pmatrix} u \\ w \end{pmatrix} \end{cases}$$

ENTRADA EXTENDIDA $\begin{pmatrix} u \\ w \end{pmatrix}$

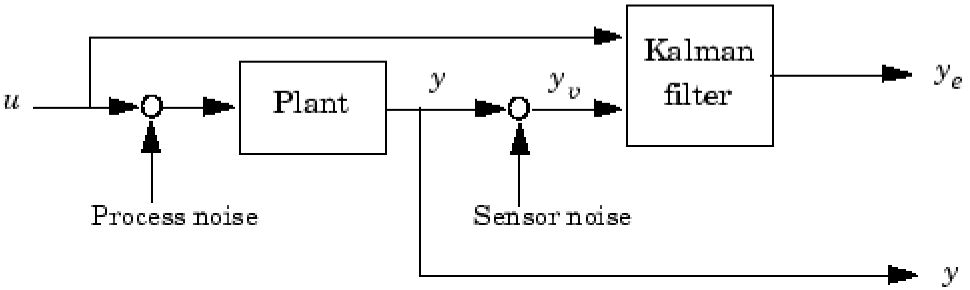
construct a state estimate $\hat{x}(t)$ that minimizes the steady-state error covariance

$$P = \lim_{t\to\infty} E(\{x-\hat{x}\}\{x-\hat{x}\}^T)$$

The optimal solution is the Kalman filter with equations

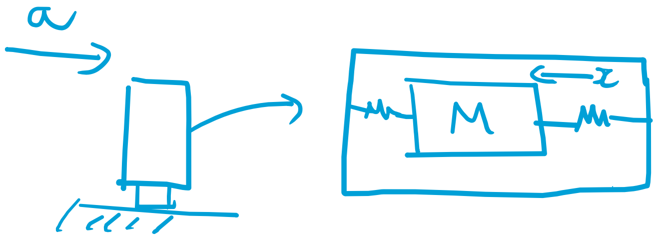
SISTEMA Kest

$$\left[\begin{aligned} &\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x} - Du) \\ &\begin{bmatrix} \hat{y} \\ \hat{x} \end{bmatrix} = \begin{bmatrix} C \\ I \end{bmatrix}\hat{x} + \begin{bmatrix} D \\ 0 \end{bmatrix}u \end{aligned}\right.$$



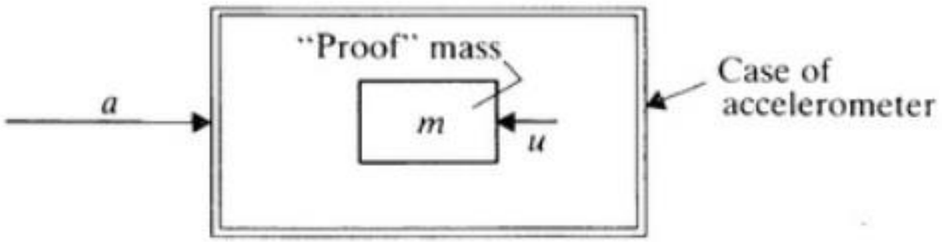
The filter gain L is determined by solving an algebraic Riccati equation to be

$$\hat{K} \left[\; L = (PC^T + \overline{N})\overline{R}^{-1}\right.$$



Acelerômetro

REF. NÃO INERCIAL



$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u + \underline{a} \\ y = x_1 + w \end{cases}$$

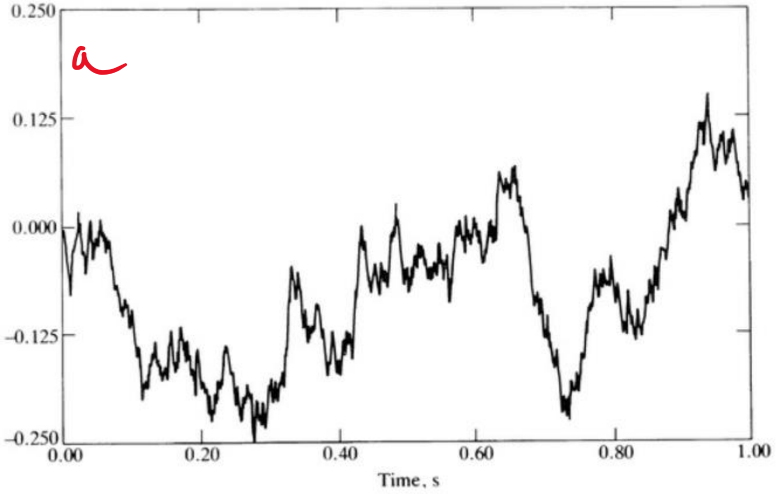
$$\begin{cases} \dot{x} = v \\ \dot{v} = F + \underline{a} \\ y = x + w \end{cases} \rightarrow$$

LVDT
OTICO
PIEZO

$\dot{a} = v$

← NOVO ESTADO

RUIDO
v



Objetivo – medir a (acel. externa)

- a é considerado um processo de Wierner

$$\dot{a} = v \xrightarrow[\text{Sistema estendido}]{} \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u + x_3 \\ \dot{x}_3 = v \end{cases}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \underline{u} + \overset{\underline{\underline{a}}}{\boxed{x_3}}$$

$$\dot{x}_3 = \underline{v}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad F = \begin{bmatrix} 0 \\ 0 \\ \underline{1} \end{bmatrix}$$

$$C = [1 \quad 0 \quad 0]$$

V – covariância de v (ruído do processo de Wierner)
W – covariância de w (ruído do sensor)

Let the optimum covariance matrix be

$$\hat{P} = \begin{bmatrix} p_1 & p_2 & p_3 \\ p_2 & p_4 & p_5 \\ p_3 & p_5 & p_6 \end{bmatrix}$$

Then the components of P satisfy:

$$\dot{p}_1 = 0 = 2p_2 - \frac{p_1^2}{W}$$

$$\dot{p}_2 = 0 = p_3 + p_4 - \frac{p_1 p_2}{W}$$

$$\dot{p}_3 = 0 = p_5 - \frac{p_1 p_3}{W}$$

$$\dot{p}_4 = 0 = 2p_5 - \frac{p_2^2}{W}$$

$$\dot{p}_5 = 0 = p_6 - \frac{p_2 p_3}{W}$$

$$\dot{p}_6 = 0 = -\frac{p_3^2}{W} + V$$

ARE dinâmica

$$p_1 = 2V^{1/6}W^{5/6}$$

$$p_2 = 2V^{1/3}W^{1/3}$$

$$p_4 = 3V^{1/2}W1/2$$

$$p_3 = V^{1/2}W^{1/2}$$

$$p_5 = 2V^{2/3}W^{1/3}$$

$$p_6 = 2V^{5/6}W^{1/6}$$

ARE estática

Filtro de Kalman

$$\dot{\hat{x}} = A\hat{x} + Bu + \uparrow \hat{K}\,\boxed{(\underline{y - C\hat{x}})}$$

$$\uparrow \underline{\hat{K}} = \hat{P}C'W^{-1} = \begin{bmatrix} p_1/W \\ p_2/W \\ p_3/W \end{bmatrix} = \begin{bmatrix} 2(\underline{V/W})^{1/6} \\ 2(V/W)^{1/3} \\ (V/W)^{1/2} \end{bmatrix}$$

$\dot{a} = v \rightarrow V$

V ← SINAL

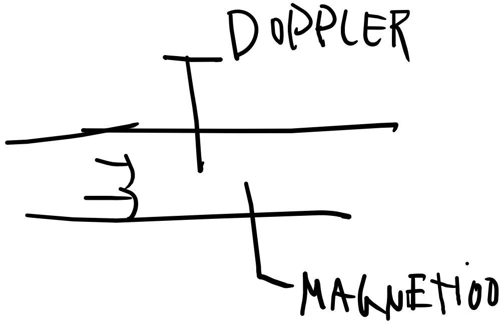
W ← RUIDO SENSOR

Since V is an indicator of the randomness of the acceleration that the instrument is trying to measure and W is an indicator of the random noise in making the measurement, the ratio V/W can be regarded as a "signal-to-noise ratio" and it is seen that the filter gains all increase with increasing signal-to-noise ratio, which seems reasonable.

FUSÃO SENSORIAL

- VÁRIOS SENSORES MEDINDO MESMA GRANDEZA

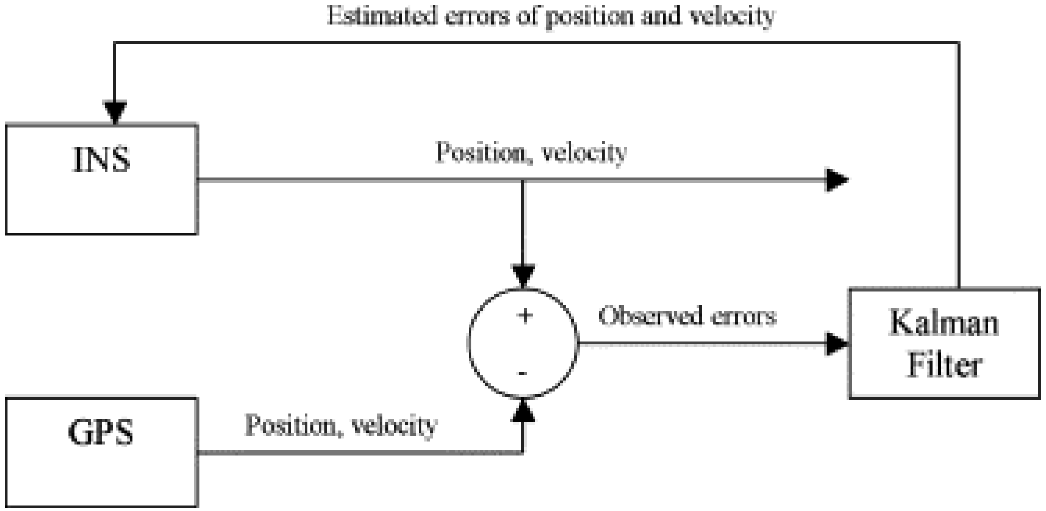
$$C = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$



→ 2 SENSORES MEDINDO "x_1"

$$W = \text{RUIDO SENSOR} = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}$$

FUSÃO SENSORIAL



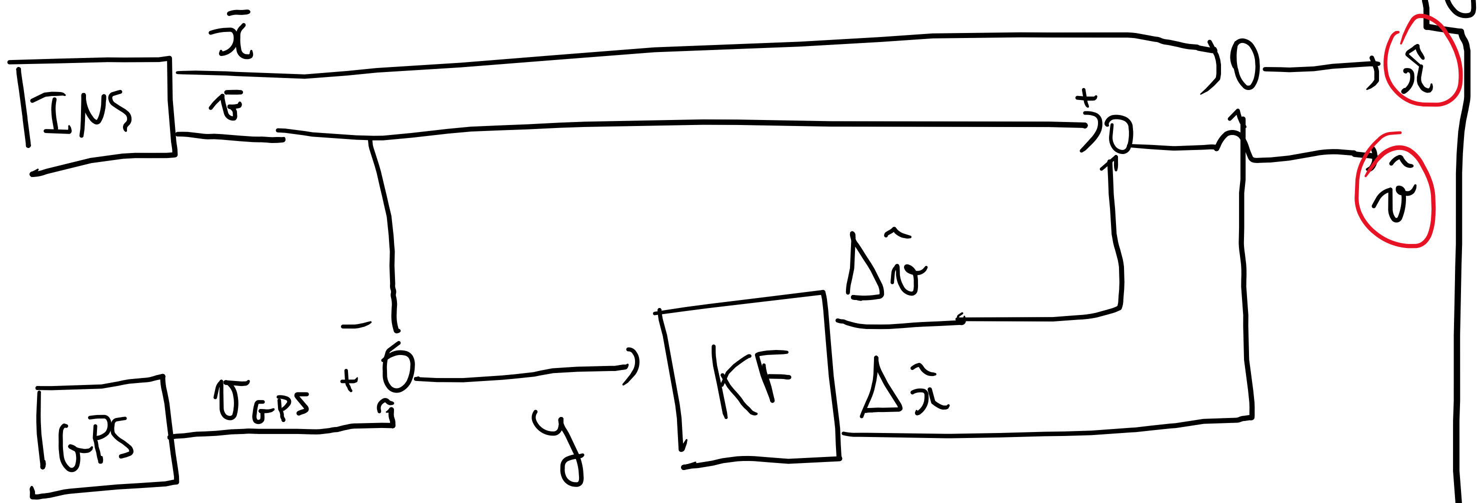
$y = v_{GPS} - \bar{v}$ = MEDIDA RUIDOSA DO ERRO DE VELOCIDADE

$$\begin{pmatrix}\underline{\Delta\dot{x}}\\ \underline{\Delta\dot{v}}\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}\Delta x\\ \Delta v\end{pmatrix} + 0.u + F.v$$

$$y = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}\Delta x\\ \Delta v\end{pmatrix}$$

$$F = \begin{pmatrix}0\\ 1\end{pmatrix}$$

MONTAR FK A PARTIR DESTE MODELO



FK-DISCRETO

MODELO DISCRETO

$$\boxed{x_k = A\cdot x_{k-1} + B\cdot u_k + w_{k-1}}$$

→ F = I

$$\dot{x} = Ax + Bu$$

$$\frac{x_k - x_{k-1}}{\Delta t} = Ax_{k-1} + Bu_k$$

$$\boxed{x_k = \underbrace{[A\Delta t + I]}_{A_{DISC}} x_{k-1} + \underbrace{B\Delta t}_{B_{DISC}} u_k}$$

$$p(w) \sim N(0,Q)$$

Q is called *Process Noise Covariance*. **It represents the** *uncertainty* **in the process or model.**

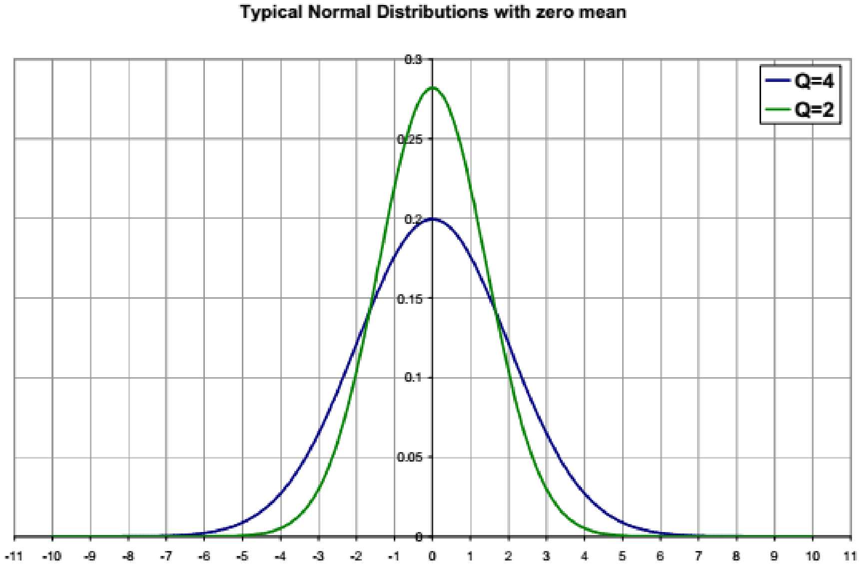


Figure 4 – Typical Normal Distribution with Zero Mean

SENSOR

$$\boxed{z_k = H \cdot x_k + v_k}$$

$$p(v) \sim N(0, R)$$

R is called ***Measurement Noise Covariance***. It represents the ***uncertainty*** in the measurement.

$$y = Cx + \cancel{D}.u + v$$

C=H

0

RUÍDO

- **Prediction**

The first step of the Kalman filter operation is called the **Prediction** step, or **Time Update** step, or **State Estimate Extrapolation**. The Kalman filter is going to predict the state of the system based on the current state and the model. Using the state dynamic model presented in equation 2.1.1 the Kalman filter determines the *a priori* estimate during this prediction step. We have the following prediction equation:

$$\boxed{\hat{x}_k^- = A \cdot \hat{x}_{k-1} + B \cdot u_k} \qquad (2.2.1)$$

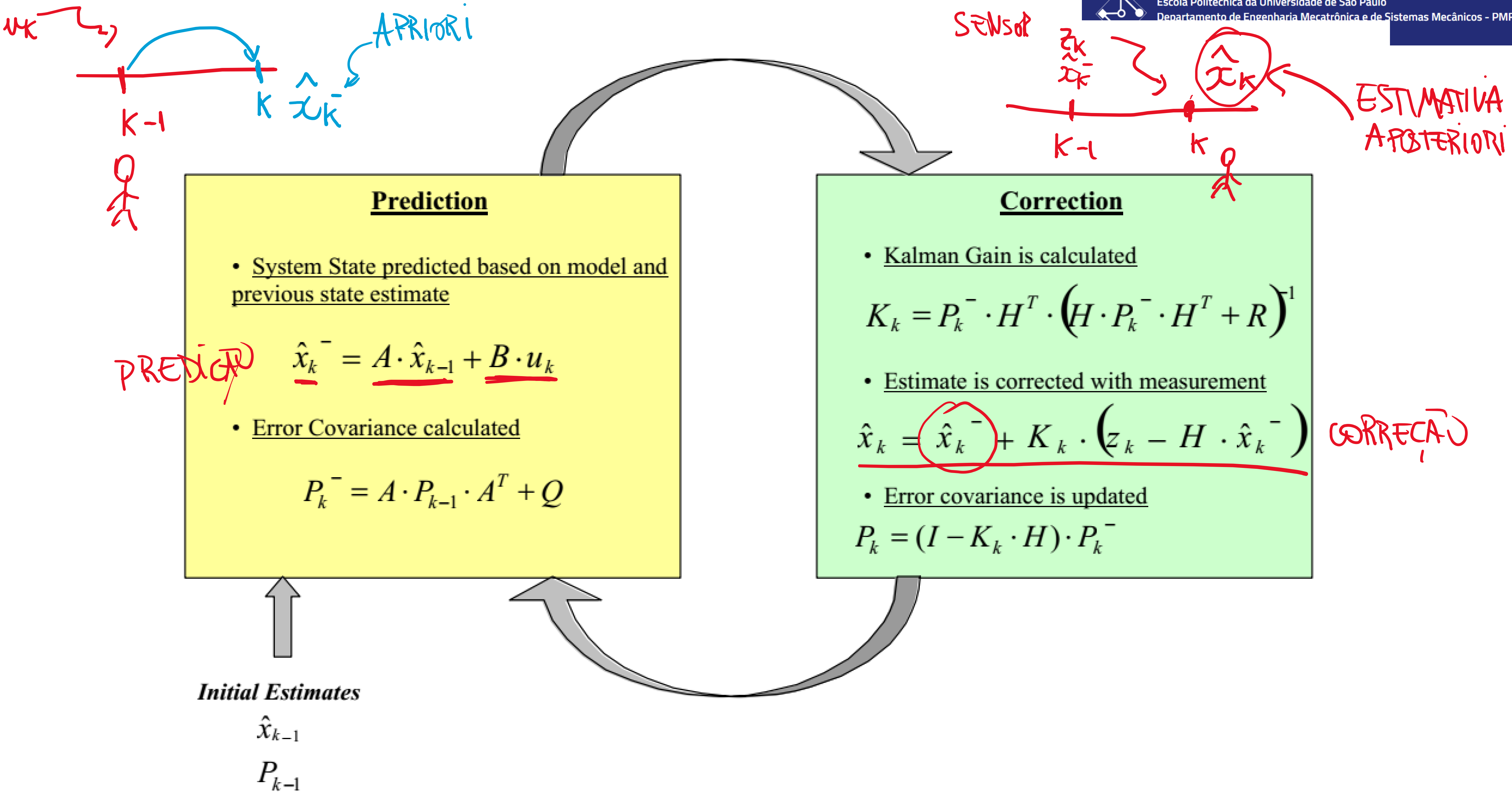
In addition at this step the Kalman filter projects what is called the ***error covariance*** P_k^- The error covariance can be considered as the uncertainty of this first prediction of the state.

$$\boxed{P_k^- = A \cdot P_{k-1} \cdot A^T + Q} \qquad (2.2.2)$$

Correction

The next step is called the **Correction** step, or **Measurement Update** step, or **State Estimate Observational Update**. The Kalman filter is going to correct or update its first prediction obtained at step 1 based on the measurement received from the Position Reference System. Please note that at this stage the Kalman gain K_k is calculated. We will come back in paragraph 2.3. on how this gain is computed. The result of this second step is a new estimated state of the system, the *a posteriori* state estimate (as defined above). We can see from the formula below that this *a posteriori* state estimate is in fact the *a priori* state estimate plus a correction factor which is proportional to the difference between the measurement and the measurement prediction. This is why we call this second step the **correction** step. Please note that this difference between the measurement and the estimated measurement is called **innovation** or **residual**. This is an important part of the Kalman filter. We will use the terminology **residual** in the rest of the paper.

$$\boxed{\hat{x}_k = \underbrace{\hat{x}_k^-}_{\text{a priori state estimate obtained at step 1}} + \underbrace{K_k \cdot \left(z_k - H \cdot \hat{x}_k^-\right)}_{\text{Correction = Kalman gain } K_k \text{ multiplied by } \underline{\text{residual}} \text{ [difference between measurement and measurement prediction]}}} \qquad (2.2.3)$$



Prediction

- System State predicted based on model and previous state estimate

PREDIÇÃO

$$\underline{\hat{x}_k^-} = \underline{A \cdot \hat{x}_{k-1}} + \underline{B \cdot u_k}$$

- Error Covariance calculated

$$P_k^- = A \cdot P_{k-1} \cdot A^T + Q$$

Correction

- Kalman Gain is calculated

$$K_k = P_k^- \cdot H^T \cdot \left(H \cdot P_k^- \cdot H^T + R\right)^{-1}$$

- Estimate is corrected with measurement

$$\underline{\hat{x}_k = \boxed{\hat{x}_k^-} + K_k \cdot \left(z_k - H \cdot \hat{x}_k^-\right)}$$

CORREÇÃO

- Error covariance is updated

$$P_k = (I - K_k \cdot H) \cdot P_k^-$$

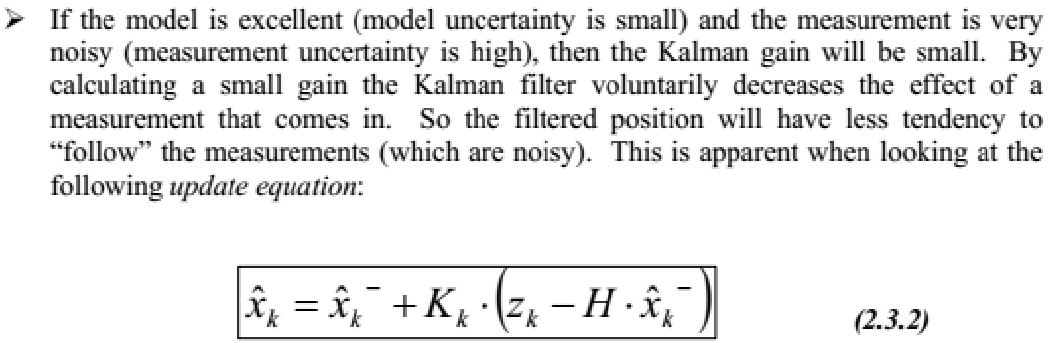
Initial Estimates

$\hat{x}_{k-1}$

P_{k-1}

$$\boxed{K_k = P_k^- \cdot H^T \cdot \left(H \cdot P_k^- \cdot H^T + R\right)^{-1}}$$

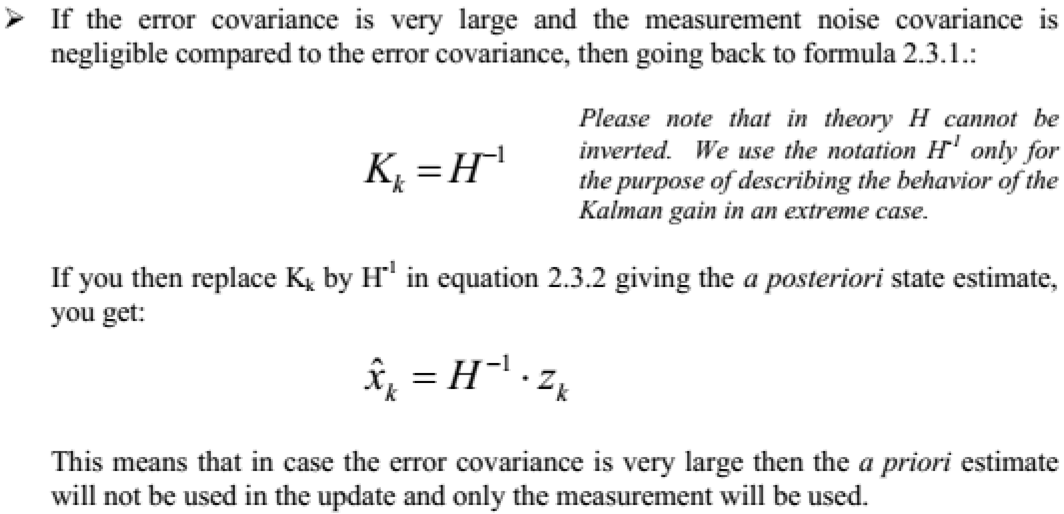
Modelo Bom



- If the model is excellent (model uncertainty is small) and the measurement is very noisy (measurement uncertainty is high), then the Kalman gain will be small. By calculating a small gain the Kalman filter voluntarily decreases the effect of a measurement that comes in. So the filtered position will have less tendency to "follow" the measurements (which are noisy). This is apparent when looking at the following *update equation*:

$$\boxed{\hat{x}_k = \hat{x}_k^- + K_k \cdot \left(z_k - H \cdot \hat{x}_k^-\right)} \qquad \textit{(2.3.2)}$$

Modelo Ruim



- If the error covariance is very large and the measurement noise covariance is negligible compared to the error covariance, then going back to formula 2.3.1.:

$$K_k = H^{-1}$$

Please note that in theory H cannot be inverted. We use the notation H^{-1} only for the purpose of describing the behavior of the Kalman gain in an extreme case.

If you then replace K_k by H^{-1} in equation 2.3.2 giving the *a posteriori* state estimate, you get:

$$\hat{x}_k = H^{-1} \cdot z_k$$

This means that in case the error covariance is very large then the *a priori* estimate will not be used in the update and only the measurement will be used.

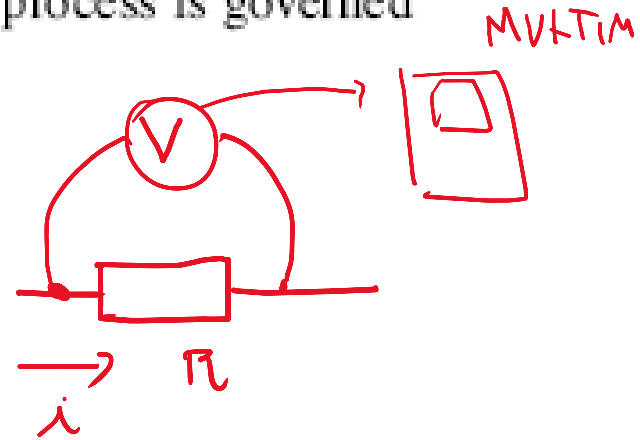
Filtro de Kalman

Estimação de uma constante

In this simple example let us attempt to estimate a scalar random constant, a voltage for example. Let's assume that we have the ability to take measurements of the constant, but that the measurements are corrupted by a 0.1 volt RMS *white* measurement noise (e.g. our analog to digital converter is not very accurate). In this example, our process is governed by the linear difference equation

$$x_k = Ax_{k-1} + Bu_k + w_k$$

$$x_k = x_{k-1} + w_k$$



with a measurement $z \in \Re^1$ that is

$$z_k = Hx_k + v_k$$

$$z_k = x_k + v_k$$

The state does not change from step to step so $A = 1$. There is no control input so $u = 0$. Our noisy measurement is of the state directly so $H = 1$. (Notice that we dropped the subscript k in several places because the respective parameters remain constant in our simple model.)

4.3.2 The Filter Equations and Parameters

Our time update equations are

$$\hat{x}_k^- = \hat{x}_{k-1},$$

$$P_k^- = P_{k-1} + Q,$$

and our measurement update equations are

$$\begin{aligned} K_k &= P_k^-(P_k^- + R)^{-1} \\ &= \frac{P_k^-}{P_k^- + R} \end{aligned}, \tag{4.32}$$

$$\hat{x}_k = \hat{x}_k^- + K_k(z_k - \hat{x}_k^-),$$

$$P_k = (1 - K_k)P_k^-.$$

10^{-5}

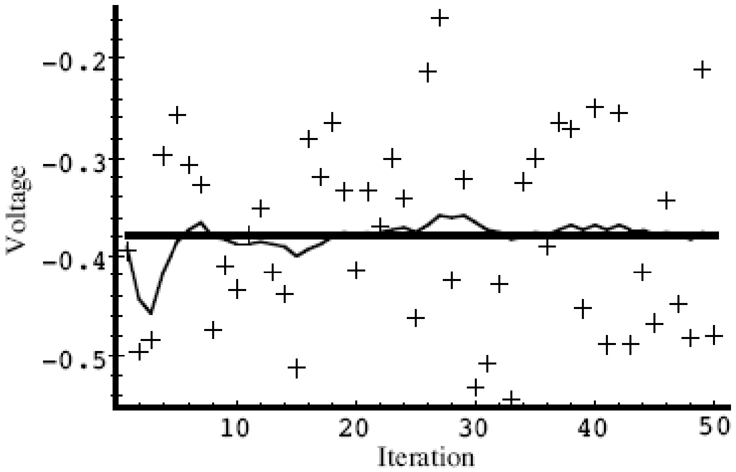
$$\boxed{x_{k+1} = x_k + \cancel{w}_k}$$

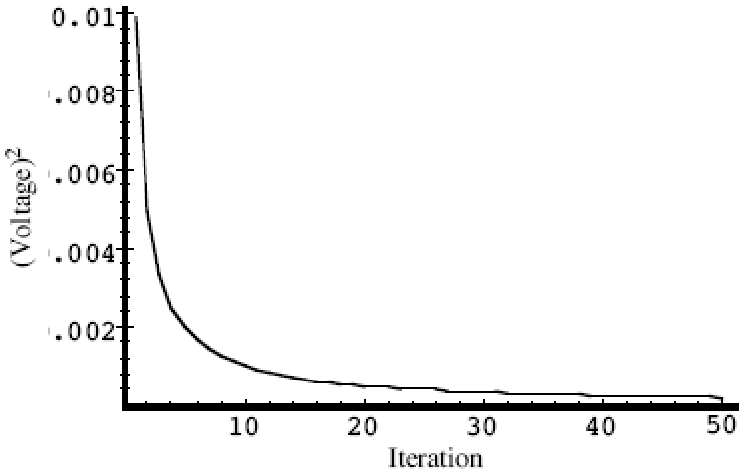
Presuming a very small process variance, we let $\underline{Q = 1e-5}$. (We could certainly let $Q = 0$ but assuming a small but non-zero value gives us more flexibility in "tuning" the filter as we will demonstrate below.) Let's assume that from experience we know that the

a scalar constant $z = -0.37727$

$$R = (0.1)^2 = \underline{0.01}$$

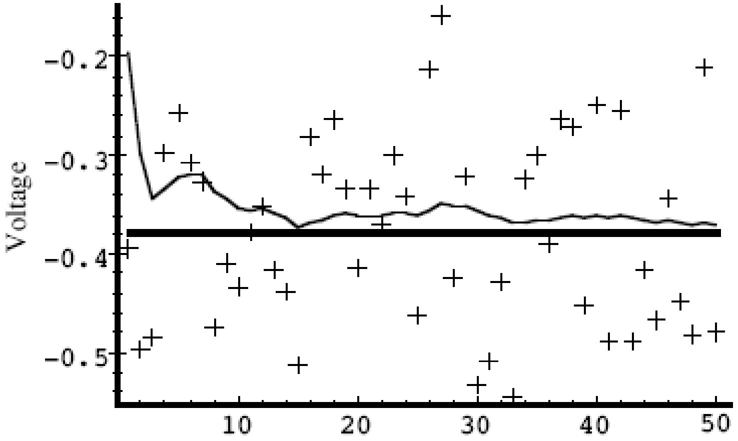
this is the “true” measurement error variance, we would expect the “best” performance





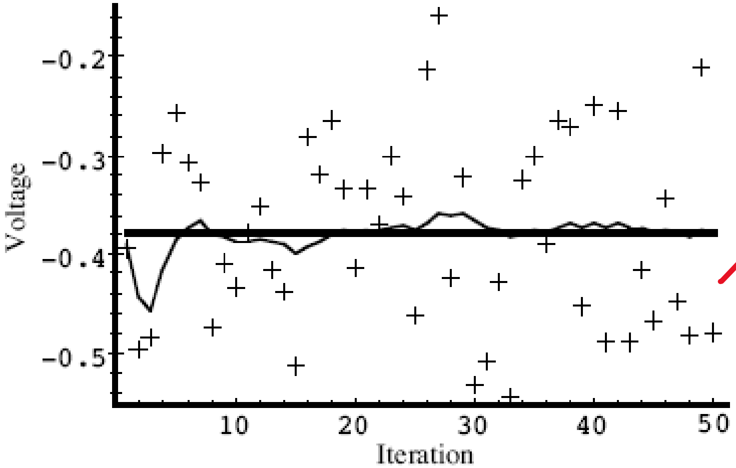
Caso 1 – ideal – R exato

a scalar constant $z = -0.37727$



Caso 2 – FK sintonizado com R 100 vezes maior – FK dará menos peso às medidas

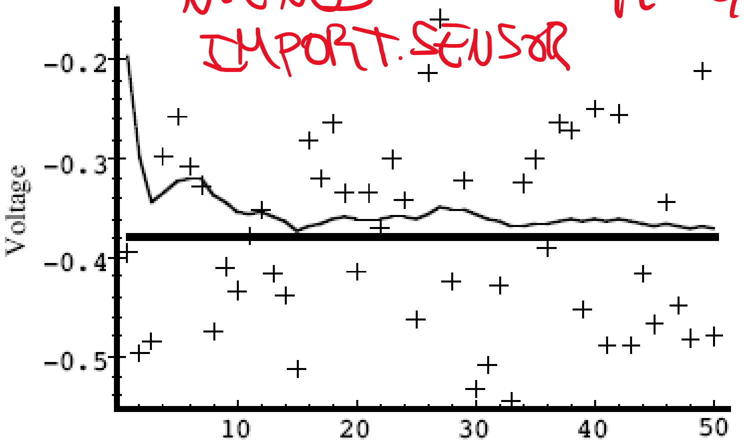
a scalar constant $z = -0.37727$



R=0,01
(CORRETO)

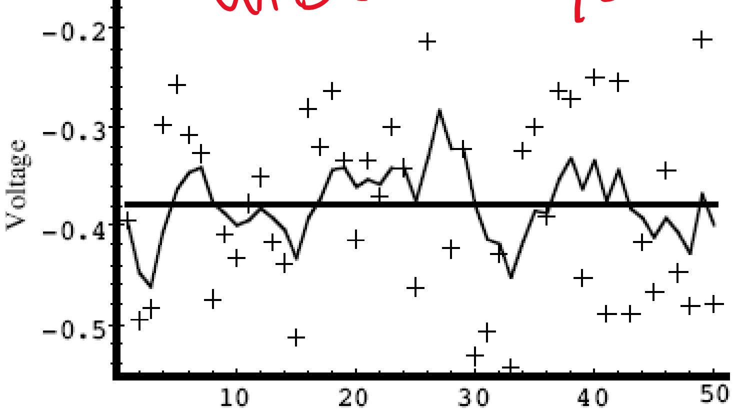
MENOS IMPORT. SENSOR

R=0,01×100



FK SINTONIZADO
C/ R MAIOR

MAIS IMPORT. P/ SENSOR



C/ R MENOR

Caso 2 – FK sintonizado com R 100 vezes menor – FK dará peso demais às medidas

Filtro de Kalman

Matlab