Quantum Theory of Many-Body systems in Condensed Matter (4302112) 2020

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Today's class: *Electron-phonon RPA and Cooper instability*

- Combining e-ph and e-e interactions: attractive interaction.
- RPA for the effective electron-electron interaction.
- Cooper instability: simplified model.
- Estimate of the critical temperature.

Review: Effective electron-electron interaction

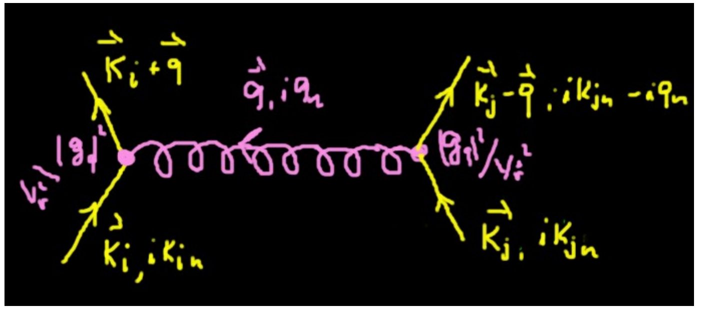
Electron-phonon Hamiltonian:

$$\hat{H} = \hat{H}_0 + \hat{H}_{\mathrm{el-ph}} \quad \begin{cases} \hat{H}_0 = \sum_{\mathbf{k},\sigma} \epsilon_k \hat{c}^\dagger_{\mathbf{k},\sigma} \hat{c}_{\mathbf{k},\sigma} + \sum_{\mathbf{q}} \Omega_{\mathbf{q}} \left(\hat{b}^\dagger_{\mathbf{q}} \hat{b}_{\mathbf{q}} + 1/2 \right) \\ \hat{H}_{\mathrm{el-ph}} = \frac{1}{V_{\mathbf{r}}} \sum_{\mathbf{k},\sigma,\mathbf{q}} g_{\mathbf{q}} \hat{c}^\dagger_{\mathbf{k}+\mathbf{q},\sigma} \hat{c}_{\mathbf{k},\sigma} \hat{A}_{\mathbf{q}} \end{cases}$$

Matsubara Green's function: $\mathcal{G}_\sigma(\mathbf{k},\tau) = -\left\langle \mathcal{T}_\tau \left[\hat{c}_{\mathbf{k},\sigma}(\tau) \hat{c}^\dagger_{\mathbf{k},\sigma}(0) \right] \right\rangle$

Perturbative expansion:

$$\mathcal{G}_\sigma(\mathbf{k},\tau) = -\frac{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta d\tau_1 \ldots d\tau_n \left\langle \mathcal{T}_\tau \left[\hat{H}_1(\tau_1) \ldots \hat{H}_1(\tau_n) (\hat{c}_{\mathbf{k},\sigma})_I(\tau) (\hat{c}^\dagger_{\mathbf{k},\sigma})_I(0) \right] \right\rangle}{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta d\tau_1 \ldots d\tau_n \left\langle \mathcal{T}_\tau \left[\hat{H}_1(\tau_1) \ldots \hat{H}_1(\tau_n) \right] \right\rangle}$$



with

$$\hat{H}_1(\tau_i) = \frac{1}{2} \int_0^\beta d\tau_j \sum_{\substack{\mathbf{k}_1\sigma_1,\mathbf{k}_2\sigma_2 \\ \mathbf{q}}} \frac{1}{V_{\mathbf{r}}^2} |g_{\mathbf{q}}|^2 D^{(0)}(\mathbf{q}, \tau_i - \tau_j) \hat{c}^\dagger_{\mathbf{k}_1+\mathbf{q},\sigma_1}(\tau_j) \hat{c}^\dagger_{\mathbf{k}_2-\mathbf{q},\sigma_2}(\tau_i) \hat{c}_{\mathbf{k}_2,\sigma_2}(\tau_i) \hat{c}_{\mathbf{k}_1,\sigma_1}(\tau_j)$$

Combining e-ph and e-e interactions

Electron-phonon Hamiltonian: $\hat{H} = \hat{H}_0 + \hat{H}_{\mathrm{el-el(ph)}} + \hat{H}_{\mathrm{el-el}}$

$$\begin{cases} \hat{H}_0 = \sum_{\mathbf{k},\sigma} \epsilon_k \hat{c}^\dagger_{\mathbf{k},\sigma}\hat{c}_{\mathbf{k},\sigma} + \sum_{\mathbf{q}} \Omega_{\mathbf{q}} \left(\hat{b}^\dagger_{\mathbf{q}}\hat{b}_{\mathbf{q}} + 1/2\right) \\ \hat{H}_{\mathrm{el-el}} = \frac{1}{2V_{\mathbf{r}}} \sum_{\substack{\mathbf{k}_1\sigma_1,\mathbf{k}_2\sigma_2 \\ \mathbf{q}}} V(\mathbf{q})\hat{c}^\dagger_{\mathbf{k}_1+\mathbf{q},\sigma_1}\hat{c}^\dagger_{\mathbf{k}_2-\mathbf{q},\sigma_2}\hat{c}_{\mathbf{k}_2,\sigma_2}\hat{c}_{\mathbf{k}_1,\sigma_1} \\ \hat{H}_{\mathrm{el-el(ph)}}(\tau_1) = \frac{1}{2V_{\mathbf{r}}}\int_0^\beta d\tau_2 \sum_{\substack{\mathbf{k}_1\sigma_1,\mathbf{k}_2\sigma_2 \\ \mathbf{q}}} \frac{|g_{\mathbf{q}}|^2}{V_{\mathbf{r}}} D^{(0)}(\mathbf{q},\tau_i-\tau_j)\hat{c}^\dagger_{\mathbf{k}_1+\mathbf{q},\sigma_1}(\tau_2)\hat{c}^\dagger_{\mathbf{k}_2-\mathbf{q},\sigma_2}(\tau_1)\hat{c}_{\mathbf{k}_2,\sigma_2}(\tau_1)\hat{c}_{\mathbf{k}_1,\sigma_1}(\tau_2) \end{cases}$$

Jellium model:

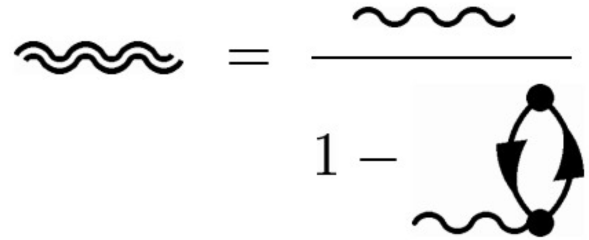
$$\frac{1}{V_{\mathbf{r}}}|g^{\mathrm{jell}}_{\mathbf{q}}|^2 = \frac{1}{2}V(\mathbf{q})\Omega$$

$$V_{\mathrm{eff}}(\mathbf{q},iq_n) = V(\mathbf{q}) + \frac{|g_{\mathbf{q}}|^2}{V_{\mathbf{r}}}D^{(0)}(\mathbf{q},iq_n) \;\Rightarrow\; V_{\mathrm{eff}}(\mathbf{q},iq_n) = V(\mathbf{q})\frac{(iq_n)^2}{(iq_n)^2-\Omega^2}$$

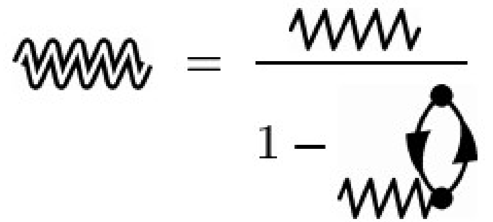
Attractive for ω<Ω!!

RPA renormalization

$$V^{\mathrm{RPA}}(\vec{q}, iq_n) = \frac{V(\vec{q})}{1 - \Pi^{(0)}(\vec{q}, iq_n)V(\vec{q})}$$



$$V^{\mathrm{RPA}}_{\mathrm{eff}}(\mathbf{q}, iq_n) = \frac{V_{\mathrm{eff}}(\mathbf{q}, iq_n)}{1 - \Pi^{(0)}(\vec{q}, iq_n)V_{\mathrm{eff}}(\mathbf{q}, iq_n)}$$



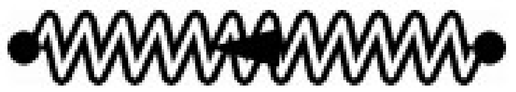
$$V^{\mathrm{RPA}}_{\mathrm{eff}}(\mathbf{q}, iq_n) = V^{\mathrm{RPA}}(\mathbf{q}, iq_n) + \frac{|g^{\mathrm{RPA}}_{\mathbf{q}}|^2}{V_{\mathbf{r}}} D^{\mathrm{RPA}}(\mathbf{q}, iq_n)$$

$$V^{\mathrm{RPA}}_{\mathrm{eff}}(\mathbf{q}, iq_n) = V^{\mathrm{RPA}}(\mathbf{q}, iq_n) \frac{(iq_n)^2}{(iq_n)^2 - \tilde{\Omega}_q^2}$$

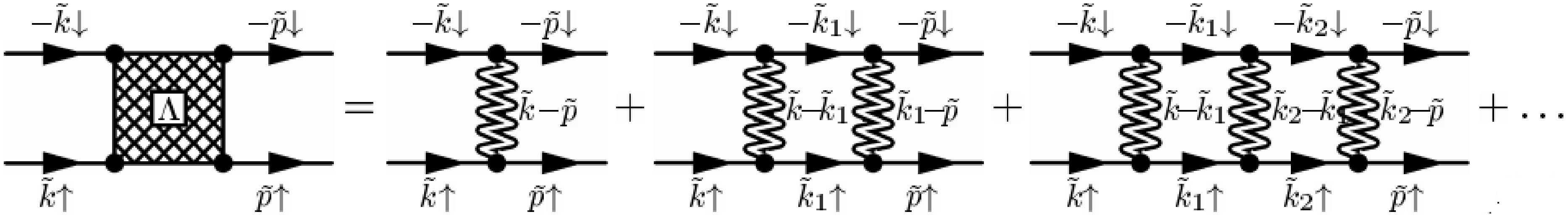
Pair scattering vertex

$$V_{\text{eff}}^{\text{RPA}}(\mathbf{q}, iq_n) = V^{\text{RPA}}(\mathbf{q}, iq_n)\frac{(iq_n)^2}{(iq_n)^2 - \tilde{\Omega}_q^2}$$

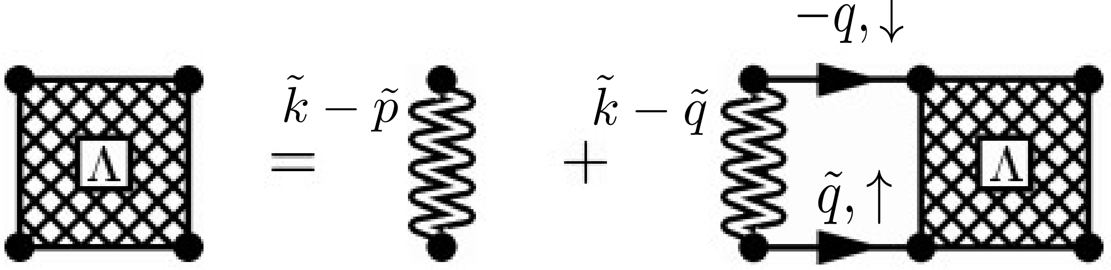
$$\tilde{q} \equiv (\mathbf{q}, iq_n)$$



Pair interaction (opposite spins):



Dyson's equation:



$$\Lambda(\tilde{k}-\tilde{p}) = -V_{\text{eff}}^{\text{RPA}}(\tilde{k}-\tilde{p}) - \frac{1}{V_{\mathbf{r}}\beta}\sum_{\tilde{q}} V_{\text{eff}}^{\text{RPA}}(\tilde{k}-\tilde{q})\mathcal{G}_{\uparrow}^{(0)}(\tilde{q})\mathcal{G}_{\downarrow}^{(0)}(-\tilde{q})\Lambda(\tilde{q}-\tilde{p})$$

Cooper instability

Simplification in the Debye model:

$$V_{\text{eff}}^{\text{RPA}}(\mathbf{q}, iq_n) \to -V\theta(iq_n - \omega_D)$$

$$\Lambda = \frac{V}{1 - \frac{V}{\beta}\sum_{iq_n}^{\omega_D} \frac{1}{V_{\mathbf{r}}} \sum_{\mathbf{q}} \mathcal{G}_{\uparrow}^{(0)}(\mathbf{q}, iq_n)\mathcal{G}_{\downarrow}^{(0)}(-\mathbf{q}, -iq_n)}$$

Matsubara sum:

$$VS(\beta) = \frac{V}{\beta}\sum_{iq_n}^{\omega_D} \frac{1}{V_{\mathbf{r}}} \sum_{\mathbf{q}} \frac{1}{(iq_n) - \epsilon_q} \frac{1}{(-iq_n) - \epsilon_q} \approx \frac{V\rho(\epsilon_F)}{2} \ln\left(\frac{2\omega_D}{\pi}\right)$$

Critical temperature:

$$k_B T_c \approx \hbar\omega_D e^{-2/(V\rho(\epsilon_F))}$$

$$\Lambda \to \infty$$

Fermions of opposite spins form a bound state