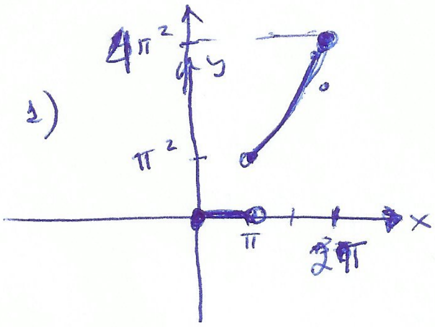
Exemplo: Seja $f(x) = \begin{cases} x^2, & \pi < x \le 2\pi \\ 0, & 0 \le x < \pi \end{cases}$

1) a série de cossenos ?

1) 

$L = 2\pi \longrightarrow$

$T = 4\pi$.

Série de cossenos: $a_0 = \frac{2}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} 0 + \frac{1}{\pi}\int_\pi^{2\pi} x^2\,dx =$

$$a_0 = \frac{1}{\pi}\,\frac{x^3}{3}\Big|_\pi^{2\pi} = \frac{1}{3\pi}\left[8\pi^3 - \pi^3\right] = \frac{7\pi^2}{3}.$$

$$a_n = \frac{2}{2\pi}\int_0^{2\pi} f(x)\cos\left(\frac{n\pi x}{2\pi}\right)dx = \frac{1}{\pi}\int_\pi^{2\pi} x^2\cos\left(\frac{nx}{2}\right)dx$$

$$\int x^2\cos\left(\frac{nx}{2}\right)dx = x^2\, sen\left(\frac{nx}{2}\right)\cdot\frac{2}{n} - \int \frac{4x}{n}\, sen\left(\frac{nx}{2}\right)dx =$$

$u = x^2 \quad dv = \cos\left(\frac{nx}{2}\right)$

$du = 2x\,dx \quad v = sen\left(\frac{nx}{2}\right)\cdot\frac{2}{n}$

$$= \frac{2}{n}x^2\, sen\left(\frac{nx}{2}\right) - \frac{4}{n}\left[-x\,\frac{2}{n}\cos\left(\frac{nx}{2}\right) + \int \frac{2}{n}\cos\left(\frac{nx}{2}\right)dx\right] =$$

$u = x \quad dv = sen\left(\frac{nx}{2}\right)$

$du = dx \quad v = -\cos\left(\frac{nx}{2}\right)\cdot\frac{2}{n}$

$$= \frac{2}{n}x^2\, sen\left(\frac{nx}{2}\right) + \frac{8}{n^2}x\cos\left(\frac{nx}{2}\right) - \frac{8}{n^2}\, sen\left(\frac{nx}{2}\right)\cdot\frac{2}{n} =$$

$$= \frac{2}{n}\, sen\left(\frac{nx}{2}\right)\left[x^2 - \frac{8}{n^2}\right] + \frac{8}{n^2}x\cos\left(\frac{nx}{2}\right)$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{2}{n}\, sen\left(\frac{nx}{2}\right)\left(\frac{x^2n^2 - 8}{n^2}\right) + \frac{8}{n^2}x\cos\left(\frac{nx}{2}\right)\right]\Bigg|_\pi^{2\pi} =$$

$$= \frac{1}{\pi}\left[\frac{8}{n^2}(2\pi)\cos(n\pi) - \frac{2}{n}\, sen\left(\frac{n\pi}{2}\right)\left(\frac{\pi^2n^2-8}{n^2}\right)\right] =$$

$$= \frac{16}{n^2}\cos(n\pi) - \frac{2}{\pi n}\left(\frac{\pi^2n^2-8}{n^2}\right) sen\left(\frac{n\pi}{2}\right) =$$

$$= \frac{16}{n^2}(-1)^n - \frac{2}{\pi n}\left(\frac{\pi^2n^2-8}{n^2}\right)(-1)^{n+1} = \frac{16}{n^2}(-1)^n + \frac{2}{n\pi}\left(\frac{\pi^2n^2-8}{n^2}\right)(-1)^n$$

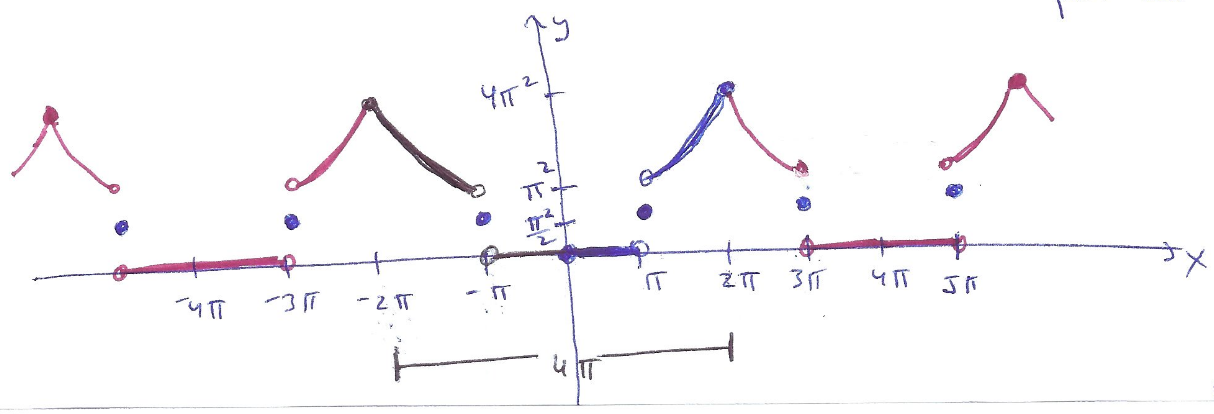
$$= (-1)^n\left[\frac{16}{n^2} + \frac{2}{n^3\pi}(\pi^2n^2-8)\right]$$

Série de cossenos:

$$\frac{7\pi^2}{6} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{16}{n^2} + \frac{2}{n^3\pi}(\pi^2 n^2 - 8)\right] \cos\left(\frac{n\pi x}{2\pi}\right)$$

$$= \begin{cases} x^2, & \pi < x \leq 2\pi \\ 0, & 0 \leq x < \pi \\ 0, & -\pi < x < 0 \\ x^2, & -2\pi < x < -\pi \\ \frac{0+\pi^2}{2} = \frac{\pi^2}{2}, & x = \pm\pi, \pm 3\pi, \dots \pm(2k+1)\pi \end{cases}$$

nos outros $x \in \mathbb{R}$, converge para a extensão par de f.



$S(\pi) = \frac{\pi^2}{2}$; $\quad S(2\pi) = 4\pi^2$

$S\left(\frac{3\pi}{2}\right) = \left(\frac{3\pi}{2}\right)^2 = \frac{9\pi^2}{4}$

$S(100\pi) = S(36\pi) = S(20\pi) = S(4\pi) = S(0) = 0.$

$100 = a + 64$ $\quad 36\pi = a\pi + 16\pi \rightarrow a = 20.$

$S(102\pi) = S(38\pi) = S(22\pi) = S(6\pi) = S(2\pi) = 4\pi^2.$

$102\pi = a\pi + 64\pi \rightarrow a = 38$ $\quad 22\pi = a\pi + 16\pi \rightarrow a = 6$

$S\left(\frac{35\pi}{2}\right) = S\left(\frac{3\pi}{2}\right) = \left(\frac{3\pi}{2}\right)^2 = \frac{9\pi^2}{4}.$

$\frac{35}{2}\pi = a\pi + 16\pi$

$S\left(\frac{103\pi}{2}\right) = S\left(-\frac{25}{2}\pi + 64\pi\right) = S\left(-\frac{25}{2}\pi\right) = S\left(\frac{7}{2}\pi\right) = S\left(-\frac{\pi}{2}\right) = 0.$

$-\frac{25}{2}\pi = a\pi - 16\pi$

$\frac{7}{2}\pi = a\pi + 4\pi$

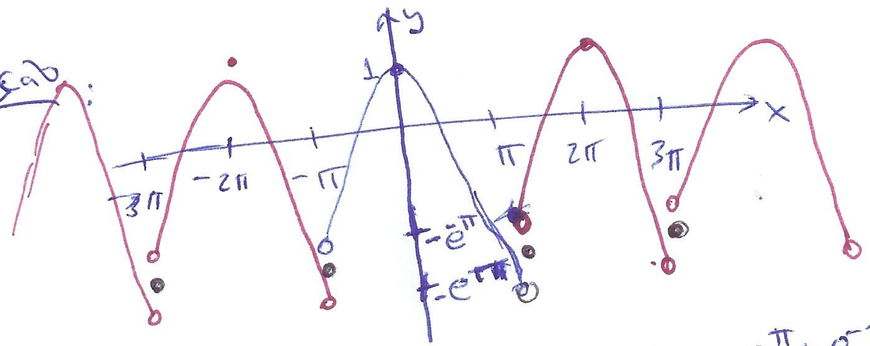
$S(67\pi) = S(3\pi + 64\pi) = S(3\pi) = S(-\pi) = \frac{\pi^2}{2}.$

Exemplo: 2) sabe-se que para as constantes $a_n, n \geq 0$ e $b_n, n \geq 1$, são tais que

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \operatorname{sen} nx) = e^x \cos x \quad \text{em} \quad -\pi < x < \pi.$$

então calcule a soma da série para $S(\pi)$, $S(15\pi)$, $S(\frac{15\pi}{2})$, $S(2\pi)$.

Solução:



$T = 2\pi$

$L = \pi$.

$$S(\pi) = \frac{-e^{\pi} + (-e^{-\pi})}{2} = -\frac{e^{\pi} + e^{-\pi}}{2}$$

$$S(15\pi) = S(a\pi + 14\pi) = S(\pi) = -\frac{e^{\pi} + e^{-\pi}}{2}.$$

$$S\left(\frac{15\pi}{2}\right) = S\left(\frac{7}{2}\pi + 4\pi\right) = S\left(\frac{7\pi}{2}\right) = S(a\pi + 4\pi)$$

$$= S\left(-\frac{\pi}{2}\right) = e^{-\pi/2} \cos\left(-\frac{\pi}{2}\right) = 0.$$

$$S(2\pi) = S(0\pi + 2\pi) = S(0\pi) = S(0) = 1.$$

Exemplo: Sabe-se que

$$x(\pi - x) = \frac{\pi^2}{6} - \left(\cos 2x + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \dots\right),\ 0 < x < \pi.$$

Então: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$.

De fato: segue-se da relação dada, substituindo $x = \frac{\pi}{2}$:

$$\frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{\pi^2}{6} - \left(-1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \dots\right)$$

$$\frac{\pi}{2}\left(\frac{\pi}{2}\right) - \frac{\pi^2}{6} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\pi^2\left(\frac{1}{4} - \frac{1}{6}\right) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \Rightarrow$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\pi^2 \frac{2}{24} = -\frac{\pi^2}{12}.$$

Exemplo: Calcule $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

Solução: Usamos o exemplo anterior com identidade de Parseval.

$$\frac{2}{\pi}\int_0^{\pi} f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$\Rightarrow \frac{\pi^2}{6} = \frac{a_0}{2} \Rightarrow a_0 = \frac{\pi^2}{3},\quad e,\quad a_{2n} = -\frac{1}{n^2},\quad a_{2n+1} = 0,\ b_n = 0.$$

$$\Rightarrow \frac{2}{\pi}\int_0^{\pi} x^2(\pi - x)^2\,dx = \frac{\pi^4}{18} + \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Mas: $\int_0^{\pi} x^2(\pi^2 - 2\pi x + x^2)\,dx = \frac{x^3}{3}\pi^2 - \pi\frac{x^4}{2} + \frac{x^5}{5}\Big|_0^{\pi} = \pi^5\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \frac{\pi^5}{30}$.

Logo $\frac{2}{\pi}\frac{\pi^5}{30} = \frac{\pi^4}{18} + \sum_1^{\infty} \frac{1}{n^4} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{15} - \frac{\pi^4}{18} = \pi^4\left(\frac{18-15}{18(15)}\right)$

$$= \frac{3\pi^4}{18(15)} = \frac{\pi^4}{18(5)} = \frac{\pi^4}{90}$$

Lista 3:

IX) $f(x) = \int_0^x \frac{\operatorname{sen} t}{t}\, dt \ , \quad f(0) = 0.$

$$g(x) = \int_0^x e^{-t^2}\, dt$$

$$f'(x) = \frac{\sin x}{x} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \quad \forall x \in \mathbb{R} - \{0\}.$$

$$f'(x) = \sum_0^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!} \Rightarrow f(x) = \int_0^x \sum_{n=0}^{\infty} (-1)^n \frac{\xi^{2n}}{(2n+1)!}\, d\xi =$$

$$= \sum_0^{\infty} (-1)^n \frac{1}{(2n+1)!} \int_0^x \xi^{2n}\, d\xi = \sum_0^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{\xi^{2n+1}}{2n+1} \Big|_0^x .$$

$$\Rightarrow f(x) = \sum_0^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{x^{2n+1}}{2n+1}, \quad \text{com } x \in \mathbb{R}.$$

2) $g'(x) = e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_0^{\infty} \frac{(-1)^n x^{2n}}{n!} \Rightarrow$

$$g(x) = \int_0^x \sum_0^{\infty} (-1)^n \frac{\xi^{2n}}{n!}\, d\xi = \sum_0^{\infty} \frac{(-1)^n}{n!} \int_0^x \xi^{2n}\, d\xi =$$

$$= \sum_0^{\infty} \frac{(-1)^n}{n!} \frac{\xi^{2n+1}}{2n+1} \Big|_0^x = \sum_0^{\infty} \frac{(-1)^n}{n!(2n+1)} (x^{2n+1})$$

$$\Rightarrow g(x) = \sum_0^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1}, \quad \text{com } x \in \mathbb{R}.$$

Resposta: b).

IV) $A = 1 + 2x + 3x^2 + \dots + (n+1)x^n + \dots$

$B = \frac{x^4}{4} + \frac{x^8}{8} + \frac{x^{12}}{12} + \dots$

Analise a convergencia dessas series de potencia:

$$A = \sum_{n=0}^{\infty} (n+1)x^n = \left(\sum_{n=0}^{\infty} x^{n+1}\right)' = \left(\sum_{n=0}^{\infty} x^n - 1\right)' = \left(\sum_{n=0}^{\infty} x^n\right)'$$

$$\underset{|x|<1}{\underset{\downarrow}{=}} \left(\frac{1}{1-x}\right)' = -\frac{(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}.$$

$$B = \sum_{n=0}^{\infty} \frac{x^{4n}}{4n} = \frac{1}{4}\sum_{0}^{\infty} \frac{(x^4)^n}{n} \underset{\downarrow}{=} -\frac{1}{4}\ln(1-x^4).$$

pois $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$, $|x|<1$.

Resposta: d).

II) $f_1(x) = \frac{1}{x^2+25}$, $f_2(x) = \text{arctg}(2x)$

a expansão em serie de potencia ao redor de $a=0$.

$$f_1(x) = \frac{1}{25+x^2} = \frac{1}{25\left(1+\frac{x^2}{25}\right)} = \frac{1}{25}\frac{1}{1+\left(\frac{x}{5}\right)^2} = \frac{1}{25}\frac{1}{1-\left(-\left(\frac{x}{5}\right)^2\right)} =$$

$$\underset{\left|-\left(\frac{x}{5}\right)^2\right|<1}{\underset{\downarrow}{=}} \frac{1}{25}\sum_{n=0}^{\infty}\left(-\left(\frac{x^2}{25}\right)\right)^n = \frac{1}{25}\sum_{0}^{\infty}(-1)^n\frac{x^{2n}}{25^n} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{5^{2n+2}}$$

em $|x|^2 < 25 \iff |x| < 5$.

$$f_2(x) = \text{arctg}(2x) \underset{|2x|\leq 1}{\underset{\downarrow}{=}} \sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n+1}}{2n+1} = \sum_{0}^{\infty}(-1)^n\frac{2^{2n+1}x^{2n+1}}{2n+1}$$

$\iff |x| \leq \frac{1}{2}$.

Resposta: d).