UNIVERSIDADE DE SÃO PAULO · ESCOLA POLITÉCNICA

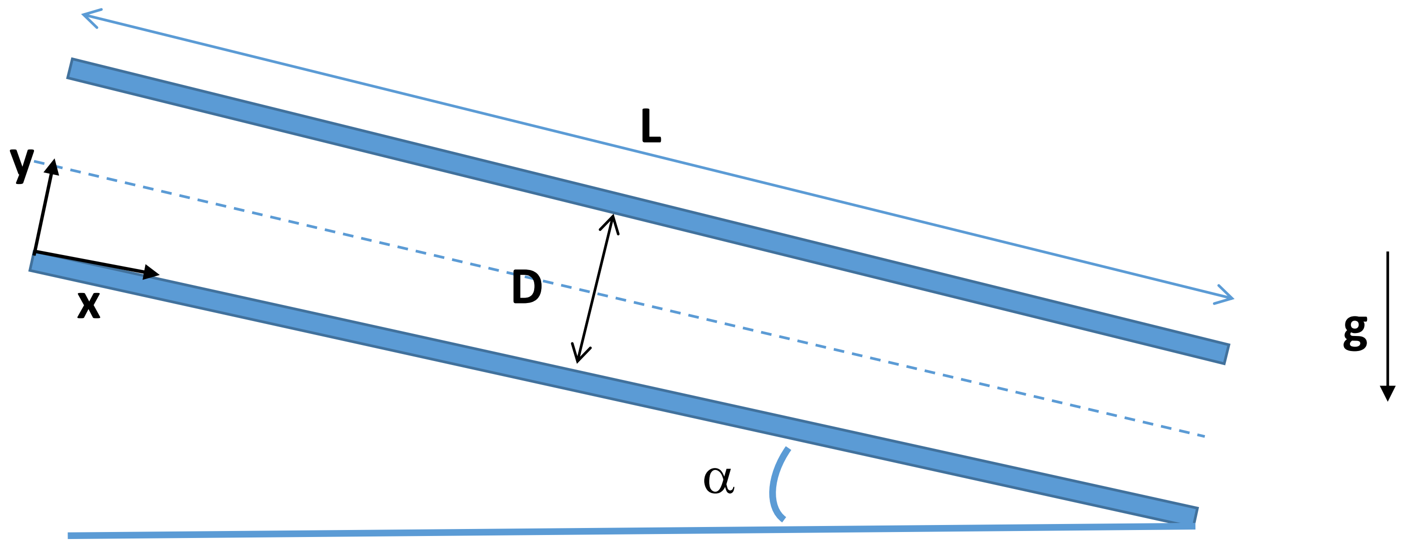
PQI 3202: FENÔMENOS DE TRANSPORTE I

Tensão em fluidos – Navier-Stokes

Exercícios - 1 e 2

Exercício 1

Duas placas grandes e planas estão dispostas paralelamente e separadas por uma distância D. O sistema está inclinado, sen α (vide figura abaixo). Entre as placas escoa um líquido viscoso, viscosidade μ e densidade ρ . As placas são fixas. O escoamento pode ser considerado laminar, newtoniano, incompressível e em regime permanente. As placas são bastante extensas e têm área S e comprimento L, sendo L >> D. As pressões nas bordas são iguais à P atm. O campo gravitacional é na direção vertical.



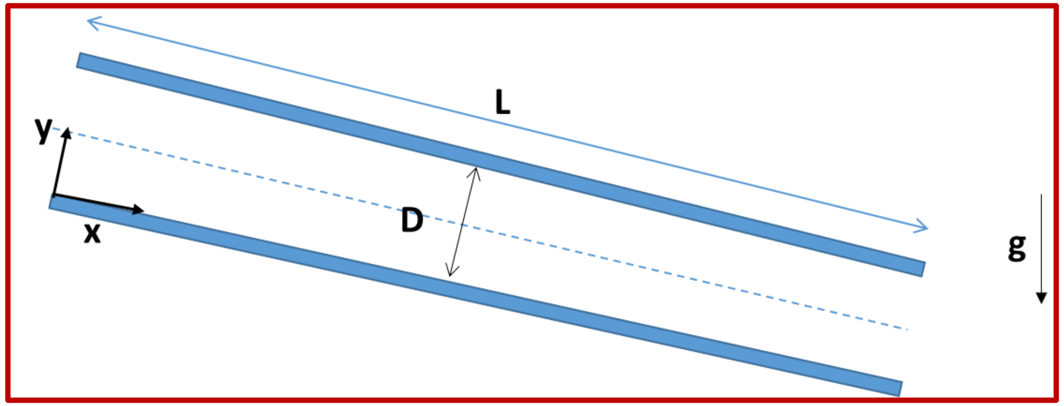
Eq. continuidade:

$$\frac{\partial \rho}{\partial t} + div(\rho\vec{v})=0$$

HIPÓTESES: { ESCOAMENTO DESENVOLVIDO
REGIME PERMANENTE
BIDIMENSIONAL (x,y)
μ e ρ ctes

R.P.

$$\cancel{\frac{\partial \rho}{\partial t}}^{\,0} + div\, \rho \vec{v} = 0$$



→ Escoamento incompressível → $div\, \vec{v} = 0$

→ Bidimensional: $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$

→ Desenvolvido → $L/D >> 1$

"Longe" da borda → $\boxed{\frac{\partial v_x}{\partial x} = 0}$

$$\frac{\partial v_y}{\partial y} = -\frac{\partial v_x}{\partial x} = 0 \Longrightarrow \boxed{v_y = 0}$$

HIPÓTESES: ESCOAMENTO DESENVOLVIDO
REGIME PERMANENTE
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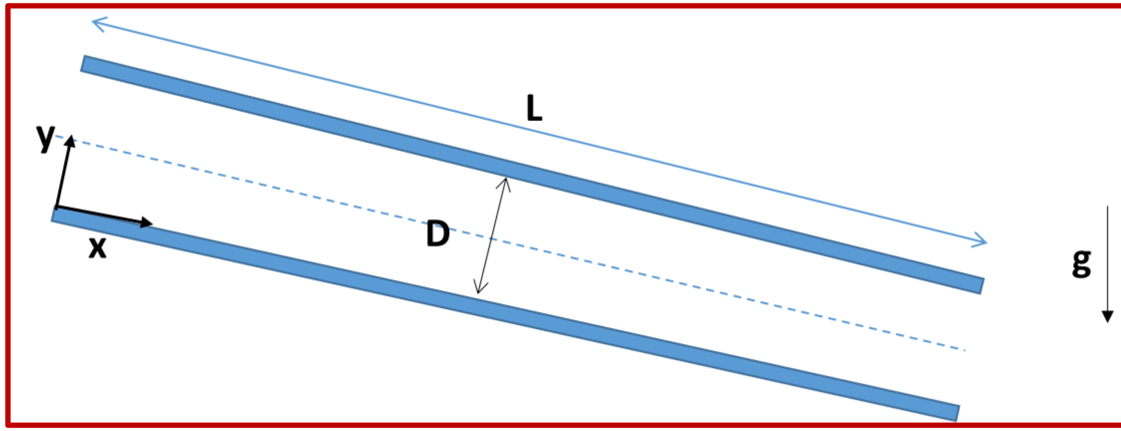


$$\rho \frac{D\vec{v}}{Dt} = \rho \frac{\partial \vec{v}}{\partial t} + \rho\, \vec{v}.\mathrm{gr\vec{a}d}\vec{v} = \rho \vec{g} - \mathrm{gr\vec{a}d}\, p + \mu\, \mathrm{lap}\, \vec{v}$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) + \rho g_y,$$

Naviet-Stokes: direção y

$$\frac{\partial \vec{v}}{\partial t} = 0 \qquad \vec{v}.\mathrm{gr\vec{a}d}\,\vec{v} = 0$$

v_y e $v_z = 0$ e $\frac{\partial v_x}{\partial x} = 0$

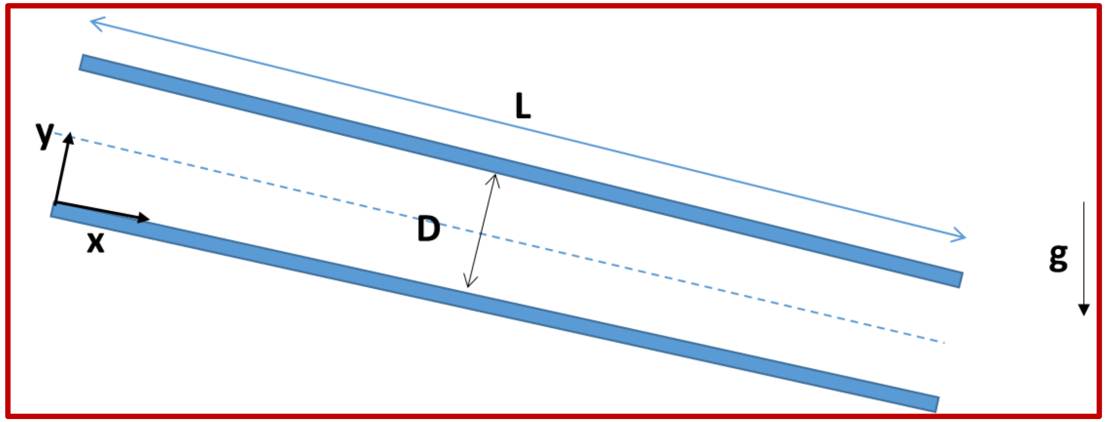
$$0 = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\underset{0}{\underset{\parallel}{\frac{\partial^2 v_x}{\partial x^2}}} + \underset{0}{\underset{\parallel}{\frac{\partial^2 v_y}{\partial y^2}}} + \underset{0}{\underset{\parallel}{\frac{\partial^2 v_y}{\partial z^2}}}\right)$$

$$g_y = -g \cos\alpha$$

$$\frac{\partial p}{\partial y} = \rho g_y = -\rho g \cos\alpha$$

$$p = -\rho g \cos\alpha\, y + \underbrace{f_1(x, z)}_{\to f_1(x)}$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z.$$

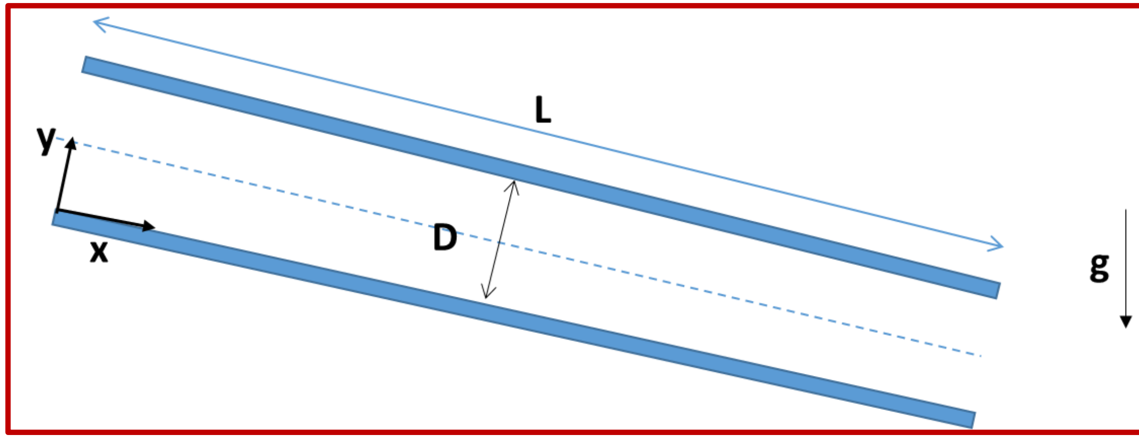


<u>direção z</u>

$$\rho g_z - \frac{\partial p}{\partial z} = 0 \quad ; \quad g_z = 0$$

$$\frac{\partial p}{\partial z} = 0$$

$$\rho\frac{D\vec{v}}{Dt} = \rho\frac{\partial\vec{v}}{\partial t} + \rho\,\vec{v}.\mathrm{gr\vec{a}d}\,\vec{v} = \rho\vec{g} - \mathrm{gr\vec{a}d}\,p + \mu\,\mathrm{lap}\,\vec{v}$$



$$p(x,y) = -\rho g \cos\alpha\, y + f_1(x)$$

$$p(x=0, y=D) = -\rho g \cos\alpha\, D + f_1(0) = P_{atm}$$

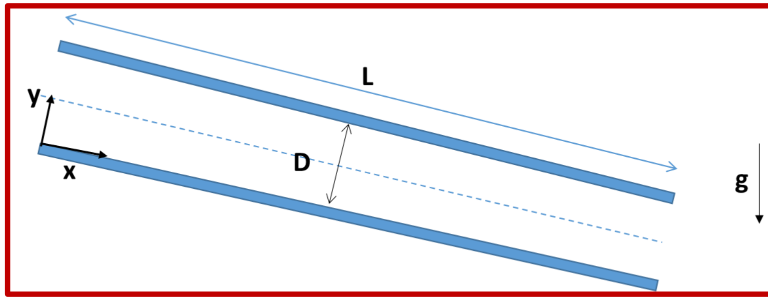
$$p(x=L, y=D) = -\rho g \cos\alpha\, D + f_1(L) = P_{atm}$$

$$f_1(0) = f_1(D) = f_1(x) = P_{atm} + \rho g \cos\alpha\, D$$

$$p(x,y) = -\rho g \cos\alpha . y + P_{atm} + \rho g \cos\alpha\, D$$

$$\boxed{p(x,y) = P_{atm} + \rho g \cos\alpha\,(D-y)}$$

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)+\rho g_x,$$



direção x

$$0=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 v_x}{\partial y^2}+\underbrace{\frac{\partial^2 v_x}{\partial x^2}}_{=0}+\underbrace{\frac{\partial^2 v_x}{\partial z^2}}_{=0}\right)$$

$$g_x = g\,\mathrm{sen}\,\alpha$$

$$\frac{\partial p}{\partial x}=?$$

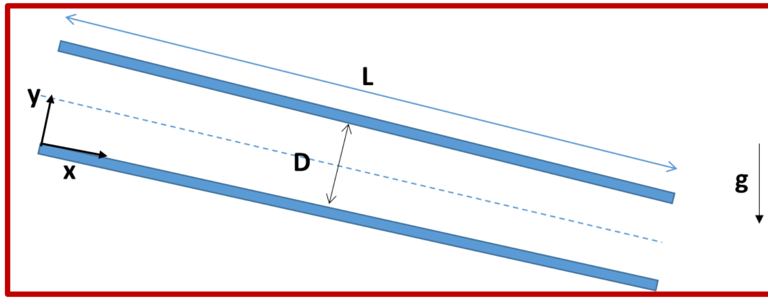
$$\frac{\partial p}{\partial x}=\frac{\partial(-\rho g\cos\alpha\, y)}{\partial x}+f_1'(x)=f_1'(x)$$

$x=0$ e $x=L \rightarrow p = p_{atm}$ em $y=D$

$f_1'(x)=0 \rightarrow P\ \text{cte} = P_{atm}\big|_{y=D}$

$$\left.\begin{array}{l}\frac{\partial p}{\partial x}=\frac{\partial(-\rho g\cos\alpha\, y)}{\partial x}+f_1'(x)=f_1'(x)\\ x=0 \text{ e } x=L \rightarrow p=p_{atm} \text{ em } y=D\\ f_1'(x)=0 \rightarrow P\ \text{cte}=P_{atm}\big|_{y=D}\end{array}\right\}\ \frac{\partial p}{\partial x}=0$$

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)+\rho g_x,$$



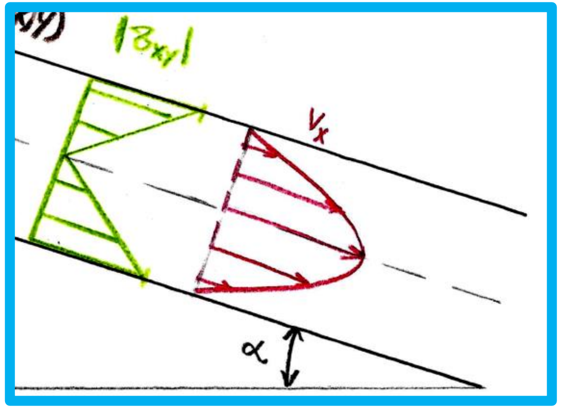
$$\rho g_x + \mu \frac{\partial^2 v_x}{\partial y^2} = 0 = \rho g \sin\alpha + \mu \frac{d^2 v_x}{dy^2}$$

$$\mu \frac{d^2 v_x}{dy^2} = -\rho g \sin\alpha$$

$$\frac{dv_x}{dy} = -\frac{\rho g}{\mu}\sin\alpha \; y + C_1$$

$$v_x = -\frac{\rho g}{\mu}\sin\alpha \frac{y^2}{2} + C_1 y + C_2$$

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)+\rho g_x,$$



$$v_x = -\frac{\rho g}{\mu}\,\text{sen}\,\alpha\,\frac{y^2}{2} + C_1 y + C_2$$

$$\begin{cases} p/\ y=0 \longrightarrow v_x = 0 \\ p/\ y=D \longrightarrow v_x = 0 \end{cases}$$

alternativa $y = D/2 \longrightarrow \dfrac{dv_x}{dy} = 0$

$$\boxed{v_x = \frac{\rho g\,\text{sen}\,\alpha}{2\mu}\left[Dy - y^2\right]}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right),$$
$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v_y}{\partial z}+\frac{\partial v_z}{\partial y}\right),$$
$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial v_z}{\partial x}+\frac{\partial v_x}{\partial z}\right).$$

$$\tau_{xy} = \tau_{yx} = \mu\left[\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right]$$

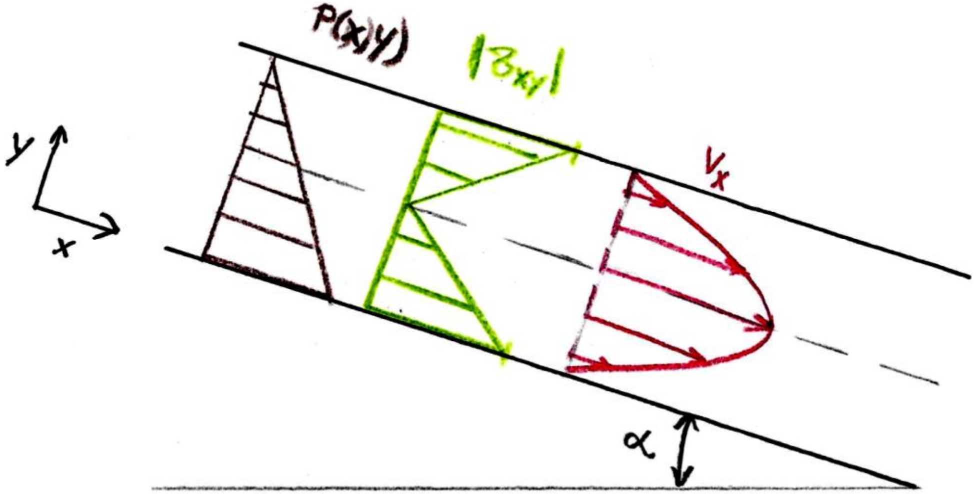
$$\tau_{xy} = \mu \cdot \frac{\rho g\,\text{sen}\,\alpha}{2\mu}\left[D - 2y\right]$$

$$\tau_{xy} = \frac{\rho g\,\text{sen}\,\alpha}{2}\left[D - 2y\right]$$

$$\tau_{xy} = \frac{\rho g \sin\alpha}{2}\left[D - 2y\right]$$

$$\left.(F_y)_{cisal}\right|_{y=0} = \frac{\rho g \sin\alpha}{2} \cdot D \cdot A = \left.(F_y)_{cis.}\right|_{y=D}$$

$$\left.F_y\, cisal\right|_{placas} = \rho g \sin\alpha \cdot D \cdot A = m g \sin\alpha$$



$$\boxed{p(x,y) = P_{atm} + \rho g \cos\alpha\,(D - y)}$$

$$P|_{y=0} = -\rho g \cos\alpha \cdot 0 + P_{atm} + \rho g \cos\alpha D$$

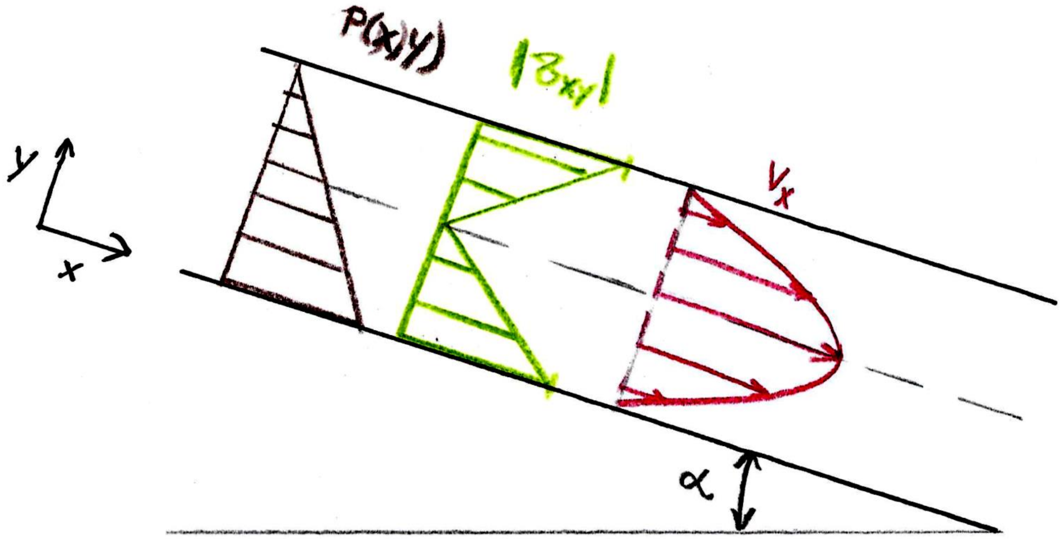
$$\begin{cases} P = -\rho g \cos\alpha\, y + C_4 \\ y = D \text{ e } x = 0 \rightarrow P = P_{atm} = -\rho g \cos\alpha D + C_4 \\ C_4 = P_{atm} + \rho g \cos\alpha D \end{cases}$$

$$F_y|_{normal}|_{y=0} = (P_{atm} + \rho g \cos\alpha D) A$$

$$= P_{atm} \cdot A + m g \cos\alpha$$

$$F_y|_{normal}|_{y=D} = P_{atm} \cdot A$$

$$\rho\vec{g} = \begin{bmatrix} \rho g_x \\ \rho g_y \\ \rho g_z \end{bmatrix} = \begin{bmatrix} \rho g \operatorname{sen}\alpha \\ -\rho g \cos\alpha \\ 0 \end{bmatrix}$$



$$\overrightarrow{\mathrm{grad}}\, p = \begin{bmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 \\ -\rho g \cos\alpha \\ 0 \end{bmatrix}$$

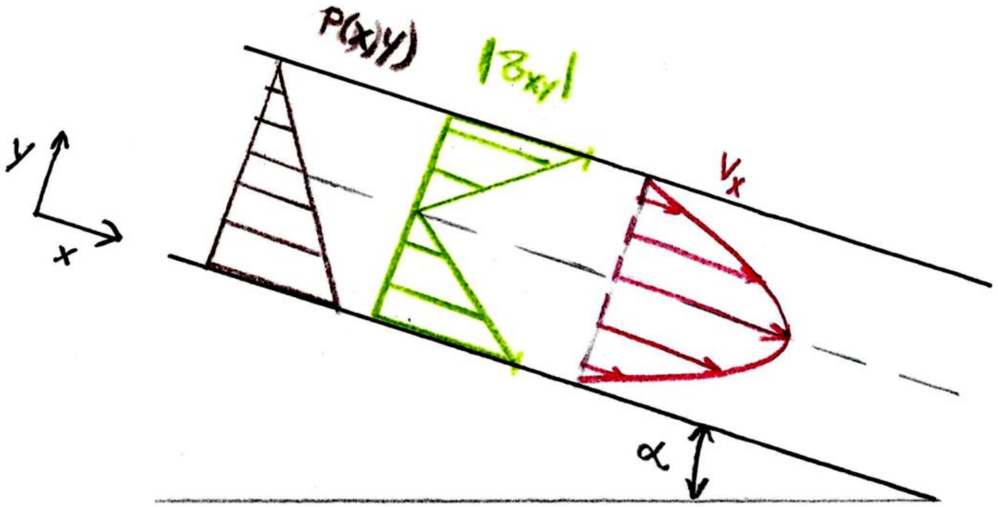
$$\vec{\tau} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\rho g \operatorname{sen}\alpha}{2}(D-2y) & 0 \\ \frac{\rho g \operatorname{sen}\alpha}{2}(D-2y) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{div}\,\vec{\tau} = \begin{bmatrix} -\rho g \operatorname{sen}\alpha \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{V} = \frac{1}{WD}\int_0^W\int_0^D v_x \, dy \, dz = \frac{W\rho g\,\mathrm{sen}\,\alpha}{WD \cdot 2\mu}\int_0^D (Dy - y^2)\,dy$$

$$\bar{V} = \frac{W\rho g\,\mathrm{sen}\,\alpha}{W \cdot D\, 2\mu}\left(\frac{Dy^2}{2} - \frac{y^3}{3}\right)_0^D = \frac{W\rho g\,\mathrm{sen}\,\alpha}{12\mu \cdot WD} D^3$$

$$\boxed{\bar{V} = \frac{2}{3} V_{MAX}} \qquad \boxed{V_{MAX} = \frac{\rho g\,\mathrm{sen}\,\alpha}{8\mu} D^2}$$



$$D_e = 4 r_H = 4 \cdot \frac{A_{esc}}{P_{er}} = \frac{4WD}{W + D} \simeq 2D$$

$$Re = \frac{\rho \bar{V} D_e}{\mu} = \frac{2\rho \bar{V} D}{\mu} = \frac{\frac{4}{3}\rho V_{MAX} D}{\mu}$$

$$\boxed{f} = \frac{2 \cdot \tau_W}{\frac{1}{2}\rho \bar{V}^2} = \frac{2\rho g\,\mathrm{sen}\,\alpha \frac{D}{2}}{\frac{1}{2}\rho \bar{V}^2} = \boxed{\frac{48}{Re}}$$

$$lwf = \frac{2 f L \bar{V}^2}{D_e} = \frac{2 \cdot 48}{Re} \frac{L \bar{V}^2}{2D}$$

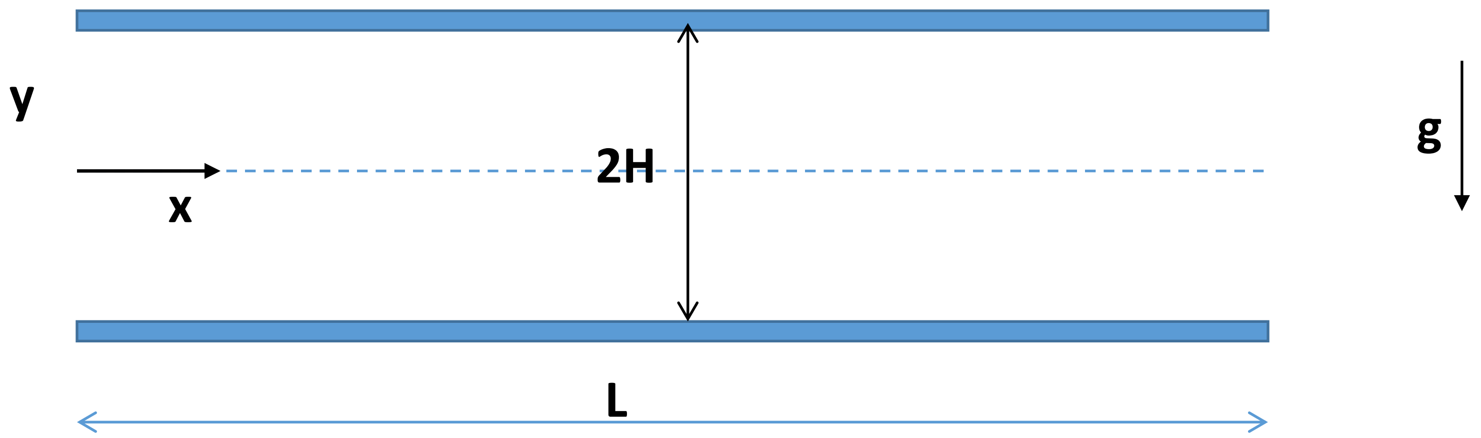
$$lwf = \frac{48\mu}{\rho \bar{V} 4D} \cdot \frac{L \bar{V}^2}{D} = \frac{12\mu}{\rho}\frac{L}{D^2}\bar{V} =$$

$$lwf = \frac{12\mu}{\rho}\frac{L}{D^2} \cdot \frac{\rho g\,\mathrm{sen}\,\alpha}{12\mu} D^2$$

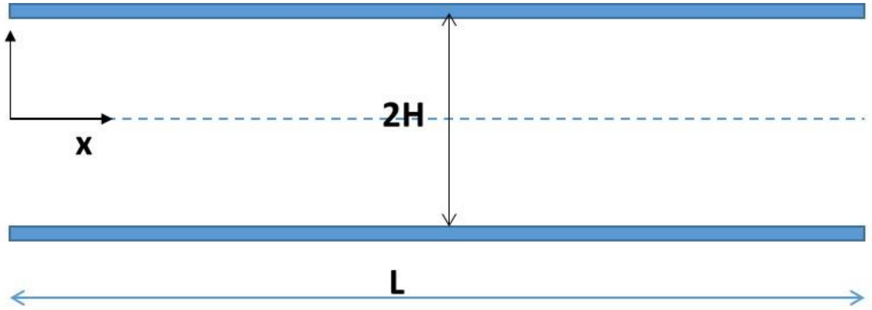
$$\boxed{lwf = L g\,\mathrm{sen}\,\alpha}$$

Exercício 2

Considere o escoamento de um fluido com viscosidade e densidades conhecidas. O escoamento ocorre entre duas placas paralelas, fixas e extensas. A pressão no ponto (0,0) é P0 e a pressão no ponto (L, 0) é PL.



HIPÓTESES:
- ESCOAMENTO DESENVOLVIDO
- REGIME PERMANENTE
- BIDIMENSIONAL (x,y)
- μ e ρ ctes



Eq. continuidade: $\underbrace{\cancel{\frac{\partial \rho}{\partial t}}}_{RP} + div\, \rho \vec{V}^{\,0} = 0 \Rightarrow$ incompressível: $div\, \vec{V} = 0$

$$\frac{\partial \rho}{\partial t} + div(\rho \vec{v}) = 0$$

$$\underbrace{\cancel{\frac{\partial V_x}{\partial x}}}_{DESENV} + \frac{\partial V_y}{\partial y} + \underbrace{\cancel{\frac{\partial V_z}{\partial z}}}_{BIDI} = 0 \qquad \frac{\partial V_y}{\partial y} = 0$$

$V_y = cte$, p/ $y = \pm H \rightarrow V_y = 0 \Rightarrow \boxed{V_y = 0}$

$V_y = 0 \qquad \frac{\partial V_x}{\partial x} = 0 \qquad V_z = 0$

$$\rho \frac{D\vec{v}}{Dt} = \rho \frac{\partial \vec{v}}{\partial t} + \rho\, \vec{v}.gr\vec{a}d\vec{v} = \rho \vec{g} - gr\vec{a}d\, p + \mu\, lap\, \vec{v}$$

Navier-Stokes

$$\rho \frac{D\vec{V}^{\,0}}{Dt} = \rho \Big(\underbrace{\cancel{\frac{\partial \vec{V}}{\partial t}}}_{R.P} + \vec{V}.gr\vec{a}d\, \vec{V}^{\,0} \Big) = \rho \vec{g}^{\,0} - gr\vec{a}d\, p + \mu Lap\, \vec{V}$$

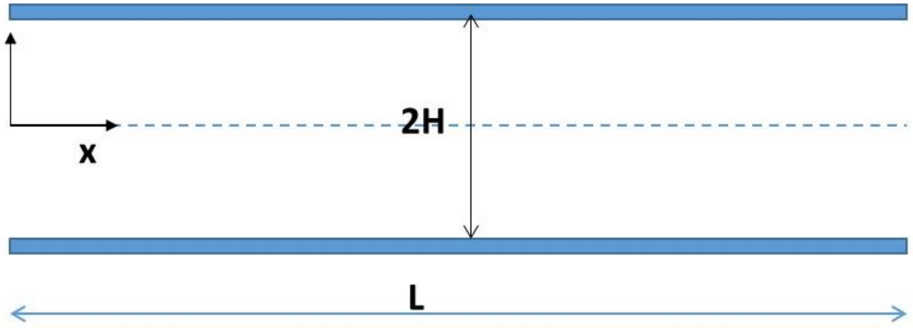
$0 \,\|$

$$\cancel{V_x \frac{\partial V_x}{\partial x}} + \cancel{V_y} \frac{\partial V_x}{\partial y} + \cancel{V_z} \frac{\partial V_z}{\partial z}$$

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)+\rho g_x,$$

$$\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)+\rho g_y,$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z.$$



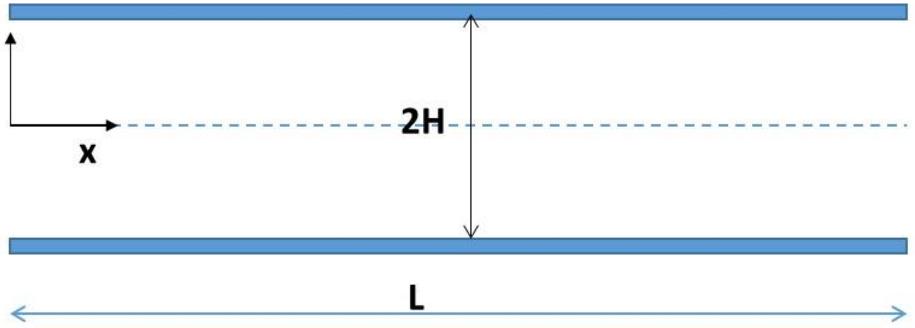
direção y:

$$\rho g_y - \frac{\partial p}{\partial y} + \mu \underbrace{\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right)}_{=0 \quad (v_y = 0)} = \rho \frac{Dv_y}{Dt} = 0$$

$$\frac{\partial p}{\partial y} = \rho g_y = -\rho g$$

$$p = -\rho g y + f_1(x,z)$$

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)+\rho g_x,$$
$$\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)+\rho g_y,$$
$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z.$$



direção z: $\rho g_z - \frac{\partial P}{\partial z} = 0$; $g_z = 0 \Rightarrow \frac{\partial P}{\partial z} = 0$

$f_1(x,z) = f_1(x)$ P não depende de z.

$$\boxed{P = -\rho g y + f_1(x)}$$

direção x: $\mu \frac{\partial^2 v_x}{\partial y^2} = \underbrace{\frac{\partial P}{\partial x}}_{f_1'(x)} - \rho g_x$ $\qquad g_x = 0$

$\mu \frac{\partial^2 v_x}{\partial y^2}$ → não é função de x $\Longrightarrow$ $f_1'(x)$ não é função de x

$f'(x) = cte$

$$\mu \frac{d^2 v_x}{dy^2} = \frac{\partial p}{\partial x} = cte$$

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \rho g_x,$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) + \rho g_y,$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_z.$$

$$\frac{dv_x}{dy} = \frac{1}{\mu}\frac{\partial p}{\partial x} y + C_1$$

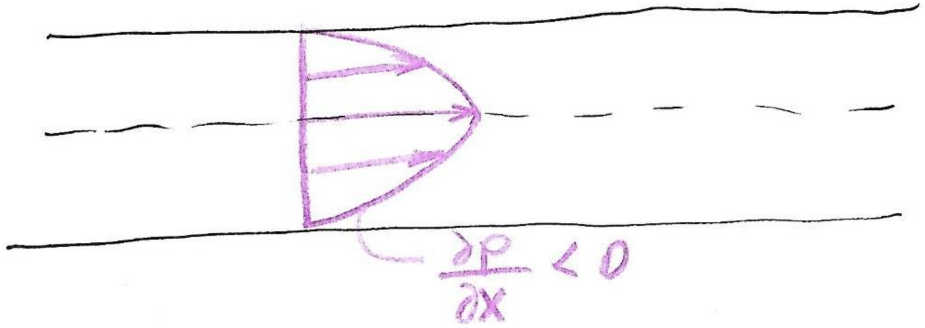
$$v_x = \frac{1}{\mu}\frac{\partial p}{\partial x}\frac{y^2}{2} + C_1 y + C_2$$

$$y = \pm H \longrightarrow v_x = 0$$

$$y = 0 \longrightarrow \frac{\partial v_x}{\partial y} = 0 \longleftarrow \text{simetria} \longrightarrow C_1 = 0$$

ou $\left.\begin{array}{l} y = H \rightarrow v_x = 0 \\ y = -H \rightarrow v_x = 0 \end{array}\right] \; C_1 = 0$

$$\boxed{v_x = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(y^2 - h^2\right)}$$



Pressão:

$$\frac{\partial p}{\partial x} = \underset{\text{desenvolvido}}{\text{cte}} = C_3 \Longrightarrow P = C_3 x + C_4 + f(y)$$

$$P = C_3 x + C_4 - \rho g y$$

$$(0,0) \longrightarrow x = 0, y = 0 \longrightarrow P = P_0$$

$$(L,0) \longrightarrow x = L, y = 0 \longrightarrow P = P_L$$

$$C_3 = \frac{P_L - P_0}{L} = \frac{\Delta P}{L} \quad \text{e} \quad C_4 = P_0$$

$$\boxed{P = -\rho g y + \frac{(P_L - P_0)}{L} x + P_0}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right),$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right),$$

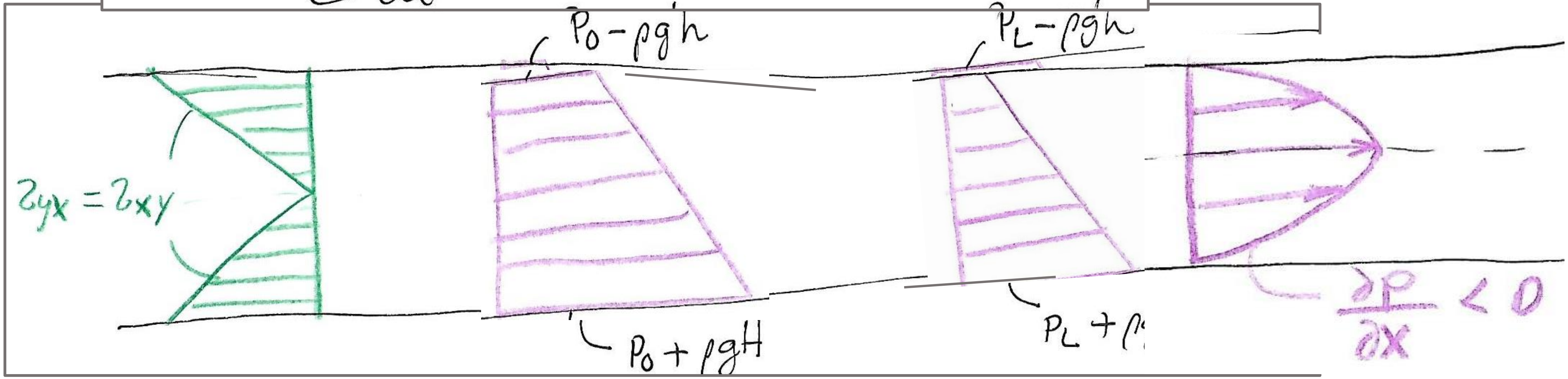
$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right).$$

Tensão cisalhamento:

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right)$$

$$\mu \frac{\partial V_x}{\partial y} = \frac{1}{2\mu}\frac{\partial p}{\partial x} \cdot 2y = \boxed{\frac{1}{\mu}\frac{\partial p}{\partial x} y}$$

$$\tau_{xy} = \underbrace{\frac{\partial p}{\partial x}}_{cte} y \qquad P = -\rho g y + \frac{\Delta P}{L} x + P_0$$



$$\langle v_x \rangle = \frac{1}{2HW} \cdot \int_{-H}^{H} \int_{0}^{W} a(y^2 - H^2)\, dz\, dy = \frac{W}{2HW} \cdot a \int_{-H}^{H} (y^2 - h^2)\, dy$$

$$a = \frac{1}{2\mu} \frac{\partial p}{\partial x}$$

$$\langle v_x \rangle = \frac{a}{2H} \left[\frac{y^3}{3} - H^2 y \right]_{-H}^{H} = -\frac{2}{3} H^2 a$$

$$\boxed{\langle v_x \rangle = -\frac{1}{3\mu} \frac{\partial p}{\partial x} H^2} = \frac{2}{3} v_{MÁX}$$