

## CONDUCTION WITH PHASE CHANGE: MOVING BOUNDARY PROBLEMS

### 6.1 Introduction

There are many applications in which a material undergoes phase change such as in melting, freezing, casting, ablation, cryosurgery and soldering. Conduction with phase change is characterized by a moving interface separating two phases. Such problems are usually referred to as *moving boundary* or *free boundary* problems. They are inherently transient during interface motion. The motion and location of the interface are unknown a priori and thus must be determined as part of the solution. Since material properties change following phase transformation and since a discontinuity in temperature gradient exists at the interface, it follows that each phase must be assigned its own temperature function. Furthermore, changes in density give rise to motion of the liquid phase. If the effect of motion is significant, the heat equation of the liquid phase must include a convective term. However, in most problems this effect can be neglected.

A moving front which is undergoing a phase change is governed by a boundary condition not encountered in previous chapters. This condition, which is based on conservation of energy, is non-linear. Because of this non-linearity there are few exact solutions to phase change problems.

In this chapter we will state the heat conduction equations for one-dimensional phase change problems. The interface boundary condition will be formulated and its non-linear nature identified. The governing equations will be cast in dimensionless form to reveal the important parameters governing phase change problems. A simplified model based on quasi-

steady state approximation will be described. Solutions to exact problems will be presented.

### 6.2 The Heat Equations

Fig. 6.1 shows a region which is undergoing phase change due to some action at one of its boundaries. The temperature distribution in the two-phase region is governed by two heat equations, one for the solid phase and one for the liquid phase. We make the following assumptions:

- (1) Properties of each phase are uniform and remain constant.
- (2) Negligible effect of liquid phase motion due to changes in density.
- (3) One-dimensional conduction.
- (4) No energy generation.

Based on these assumptions, eq. (1.8) gives

$$\frac{\partial^2 T_s}{\partial x^2} = \frac{1}{\alpha_s} \frac{\partial T_s}{\partial t} \quad 0 < x < x_i, \quad (6.1)$$

and

$$\frac{\partial^2 T_l}{\partial x^2} = \frac{1}{\alpha_l} \frac{\partial T_l}{\partial t} \quad x > x_i, \quad (6.2)$$

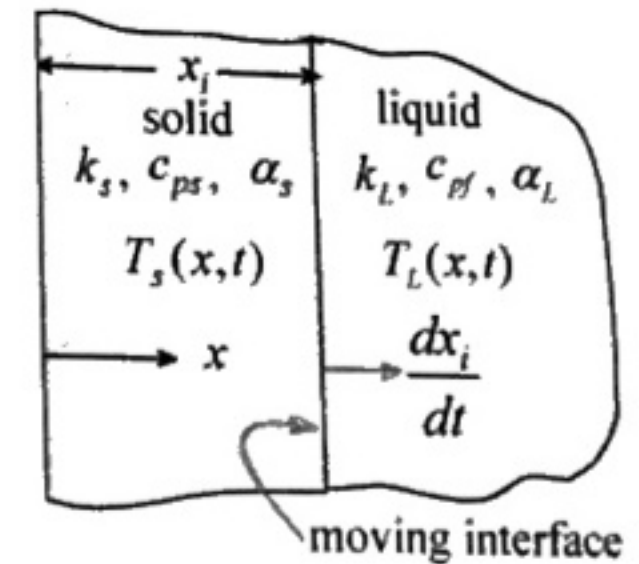


Fig. 6.1

where the subscripts  $L$  and  $s$  refer to liquid and solid, respectively.

### 6.3 Moving Interface Boundary Conditions

Continuity of temperature and conservation of energy give two boundary conditions at the solid-liquid interface. Mathematical description of these conditions follows.

- (1) **Continuity of temperature.** We assume that the material undergoes a phase change at a fixed temperature. Continuity of temperature at the interface requires that

$$T_s(x_i, t) = T_l(x_i, t) = T_f, \quad (6.3)$$

where

$$T(x, t) = \text{temperature variable}$$

$T_f$  = fusion (melting or freezing) temperature  
 $x_i = x_i(t)$  = interface location at time  $t$

(2) **Energy equation.** Fig. 6.2 shows a liquid element which is undergoing phase change to solid. A one-dimensional solid-liquid interface located at  $x = x_i(t)$  is assumed to move in the positive  $x$ -direction. Consider an element  $dx_L$  in the liquid adjacent to the interface. The element has a fixed mass  $\delta m$ . During a time interval  $dt$  the element undergoes phase change to solid. Because material density changes as it undergoes phase transformation, the thickness of the element changes to  $dx_S$ . Consequently, the interface advances a distance  $dx_i = dx_S$ . Application of conservation of energy to the element as it transforms from liquid to solid during time  $dt$ , gives

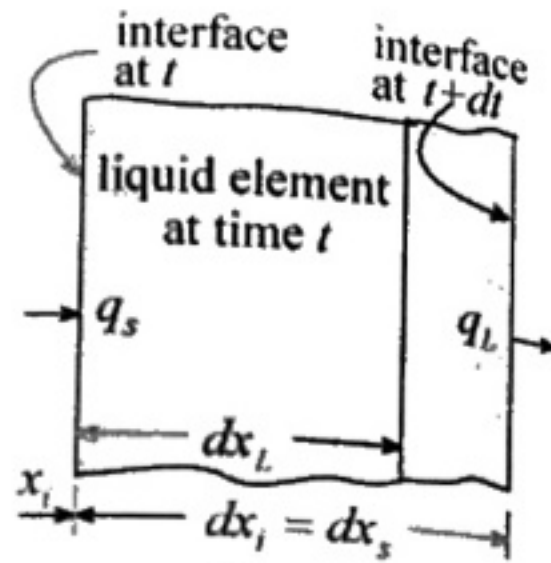


Fig. 6.2

$$\delta E_{in} - \delta E_{out} = \delta E, \tag{a}$$

where

$\delta E_{in}$  = energy added during time  $dt$

$\delta E_{out}$  = energy removed during time  $dt$

$\delta E$  = energy change of element during time  $dt$

The energy added by conduction through the solid phase is

$$\delta E_{in} = q_s dt, \tag{b}$$

where

$q_s$  = rate of heat conducted to the element through the solid phase

Energy is removed from the element by conduction and in the form of work done by the element due to changes in volume. These two components of energy are given by

$$\delta E_{out} = q_L dt + pdV, \tag{c}$$

where

$dV$  = change in volume of element from liquid to solid during  $dt$

$p$  = pressure

$q_L$  = rate of heat conducted from the element through the liquid phase

The change in energy of the element is given by

$$\delta E = (\hat{u}_s - \hat{u}_L) \delta m, \tag{d}$$

$\delta m$  = mass of element

$\hat{u}$  = internal energy per unit mass

Substituting (b), (c) and (d) into (a)

$$(q_s - q_L) dt = (\hat{u}_s - \hat{u}_L) \delta m + pdV. \tag{e}$$

Applying Fourier's law

$$q_s = -k_s A \frac{\partial T_s(x_i, t)}{\partial x}, \text{ and } q_L = -k_L A \frac{\partial T_L(x_i, t)}{\partial x}, \tag{f}$$

where

$A$  = surface area of element normal to  $x$

$k$  = thermal conductivity

$x$  = coordinate

$x_i = x_i(t)$  = interface location at time  $t$

The mass of the element  $\delta m$  is given by

$$\delta m = \rho_s A dx_i, \tag{g}$$

where  $\rho$  is density. The change in volume  $dV$  of the element is related to its mass and to changes in density as

$$dV = \left( \frac{1}{\rho_s} - \frac{1}{\rho_L} \right) \delta m. \tag{h}$$

Substituting (f), (g) and (h) into (e) and assuming that pressure remains constant

$$-k_s \frac{\partial T_s(x_i, t)}{\partial x} + k_L \frac{\partial T_L(x_i, t)}{\partial x} = \rho_s \left[ \left( \hat{u}_s + \frac{p}{\rho_s} \right) - \left( \hat{u}_L + \frac{p}{\rho_L} \right) \right] \frac{dx_i}{dt}. \tag{i}$$

However

$$\left(\hat{u}_l + \frac{p}{\rho_l}\right) - \left(\hat{u}_s + \frac{p}{\rho_s}\right) = \hat{h}_l - \hat{h}_s = \mathcal{L}, \quad (j)$$

where

$\hat{h}$  = enthalpy per unit mass  
 $\mathcal{L}$  = latent heat of fusion

Substituting (j) into (i)

$$k_s \frac{\partial T_s(x_i, t)}{\partial x} - k_l \frac{\partial T_l(x_i, t)}{\partial x} = \rho_s \mathcal{L} \frac{dx_i}{dt}. \quad (6.4)$$

Eq. (6.4) is the interface energy equation. It is valid for both solidification and melting. However, for melting  $\rho_s$  is replaced by  $\rho_l$  for melting.

**(3) Convection at the interface.** There are problems where the liquid phase is not stationary. Examples include solidification associated with forced or free convection over a plate or inside a tube. For such problems equation (6.4) must be modified to include the effect of fluid motion. This introduces added mathematical complications. An alternate approach is to replace the temperature gradient in the liquid phase with Newton's law of cooling. Thus

$$k_s \frac{\partial T_s(x_i, t)}{\partial x} \pm h(T_f - T_\infty) = \rho_s \mathcal{L} \frac{dx_i}{dt}, \quad (6.5)$$

where

$h$  = heat transfer coefficient  
 $T_\infty$  = temperature of the liquid phase far away from the interface

The plus sign in eq. (6.5) is for solidification and the minus sign is for melting.

### 6.4 Non-linearity of the Interface Energy Equation

A solution to a phase change conduction problem must satisfy the energy condition of eq. (6.4) or eq. (6.5). Careful examination of these equations shows that they are non-linear. The non-linearity is caused by the dependence of the interface velocity  $dx_i/dt$  on the temperature gradient. To reveal the non-linear nature of equations (6.4) and (6.5), we form the total derivative of  $T_s$  in eq. (6.3)

$$\frac{\partial T_s(x_i, t)}{\partial x} dx + \frac{\partial T_s(x_i, t)}{\partial t} dt = 0.$$

Dividing through by  $dt$  and noting that  $dx = dx_i$ , gives  $dx_i/dt$

$$\frac{dx_i}{dt} = - \frac{\partial T_s(x_i, t) / \partial t}{\partial T_s(x_i, t) / \partial x}. \quad (6.6)$$

When this result is substituted into eq. (6.4) we obtain

$$k_s \left[ \frac{\partial T_s(x_i, t)}{\partial x} \right]^2 - k_l \frac{\partial T_l(x_i, t)}{\partial x} \frac{\partial T_s(x_i, t)}{\partial x} = -\rho_s \mathcal{L} \frac{\partial T_s(x_i, t)}{\partial t}. \quad (6.7)$$

Note that both terms on the left side of eq. (6.7) are non-linear. Similarly, substituting eq. (6.6) into eq. (6.5) shows that it too is non-linear.

### 6.5 Non-dimensional Form of the Governing Equations: Governing Parameters

To identify the governing parameters in phase change problems, the two heat equations, (6.1) and (6.2) and the interface energy equation (6.4) are cast in non-dimensional form. The following dimensionless quantities are defined:

$$\theta_s = \frac{T_s - T_f}{T_f - T_o}, \quad \theta_l = \frac{k_l T_l - T_f}{k_s T_f - T_o}, \quad \xi = \frac{x}{L}, \quad \tau = Ste \frac{\alpha_s}{L^2} t, \quad (6.8)$$

where  $L$  is a characteristic length and  $T_o$  is a reference temperature.  $Ste$  is the *Stefan number* which is defined as

$$Ste = \frac{c_{ps}(T_f - T_o)}{\mathcal{L}}, \quad (6.9)$$

where  $c_{ps}$  is the specific heat of the solid phase. Substituting eq. (6.8) into equations (6.1), (6.2) and (6.4) gives

$$\frac{\partial^2 \theta_s}{\partial \xi^2} = Ste \frac{\partial \theta_s}{\partial \tau}, \quad (6.10)$$

$$\frac{\alpha_l}{\alpha_s} \frac{\partial^2 \theta_l}{\partial \xi^2} = Ste \frac{\partial \theta_l}{\partial \tau}, \quad (6.11)$$

and

$$\frac{\partial \theta_s(\xi, t)}{\partial \xi} - \frac{\partial \theta_l(\xi, t)}{\partial \xi} = \frac{d\xi}{d\tau} \quad (6.12)$$

Examination of the dimensionless governing equations shows that two parameters, the ratio of thermal diffusivities and the Stefan number, govern phase change problems. It is worth noting the following:

- (1) By including the ratio  $(k_l/k_s)$  in the definition of  $\theta_l$ , this parameter is eliminated from the interface energy equation.
- (2) A phase change problem with convection at its boundary will introduce the Biot number as an additional parameter.
- (3) The Stefan number, defined in eq. (6.9), represents the ratio of the sensible heat to the latent heat. Sensible heat,  $c_{ps}(T_f - T_o)$ , is the energy removed from a unit mass of solid at the fusion temperature  $T_f$  to lower its temperature to  $T_o$ . Latent heat,  $\mathcal{L}$ , is the energy per unit mass which is removed (solidification) or added (melting) during phase transformation at the fusion temperature. Note that the Stefan number in eq. (6.9) refers to the solid phase since it is defined in terms of the specific heat of the solid  $c_{ps}$ . The definition for the liquid phase is

$$Ste = \frac{c_{pl}(T_o - T_f)}{\mathcal{L}} \quad (6.13)$$

Sensible heat,  $c_{pl}(T_o - T_f)$ , is the energy added to a unit mass of liquid at fusion temperature  $T_f$  to raise its temperature to  $T_o$ .

### 6.6 Simplified Model: Quasi-Steady Approximation

Because of the non-linearity of the interface energy equation, there are few exact solutions to conduction problems with phase change. An approximation which makes it possible to obtain solutions to a variety of problems is based on a *quasi-steady* model. In this model the Stefan number is assumed small compared to unity. A small Stefan number corresponds to sensible heat which is small compared to latent heat. To appreciate the significance of a small Stefan number, consider the limiting case of a material whose specific heat is zero, i.e.  $Ste = 0$ . Such a material has infinite thermal diffusivity. This means that thermal effects propagate with infinite speed and a steady state is reached instantaneously as the interface moves. Alternatively, a material with infinite latent heat

( $Ste = 0$ ), has a stationary interface. Thus the interface moves slowly for a small Stefan number and the temperature distribution at each instant corresponds to that of steady state. In practice, quasi-steady approximation is justified for  $Ste < 0.1$ . Setting  $Ste = 0$  in equations (6.10) and (6.11) gives

$$\frac{\partial^2 \theta_s}{\partial \xi^2} = 0, \quad (6.14)$$

$$\frac{\partial^2 \theta_l}{\partial \xi^2} = 0. \quad (6.15)$$

Note that the interface energy equation (6.12) is unchanged in this model and that temperature distribution and interface motion are time dependent. What is simplified in this approximation are the governing equations and their solutions.

#### Example 6.1: Solidification of a Slab at the Fusion Temperature $T_f$

A slab of thickness  $L$  is initially at the fusion temperature  $T_f$ . One side of the slab is suddenly maintained at constant temperature  $T_o < T_f$  while the opposite side is kept at  $T_f$ . A solid-liquid interface forms and moves towards the opposite face. Use a quasi-steady state model to determine the time needed for the entire slab to solidify.

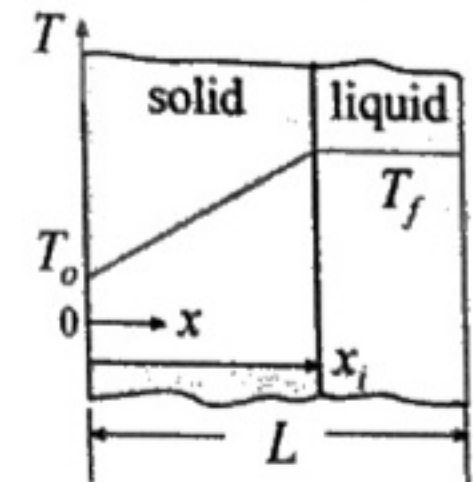


Fig. 6.3

(1) **Observations.** (i) Physical consideration requires that the liquid phase remains at a uniform temperature equal to the fusion temperature  $T_f$ . (ii) Solidification starts at time  $t = 0$ . (iii) The time it takes the slab to solidify is equal to the time needed for the interface to traverse the slab width  $L$ . Thus the problem reduces to determining the interface motion which is governed by the interface energy condition.

(2) **Origin and Coordinates.** Fig. 6.3 shows the origin and the coordinate axes.

(3) **Formulation.**

(i) **Assumptions.** (1) One-dimensional conduction, (2) constant properties of the liquid and solid phases, (3) no changes in fusion temperature and (4) quasi-steady state,  $Ste < 0.1$ .

(ii) **Governing Equations.** For quasi-steady state the conduction equations, expressed in dimensional form, are

$$\frac{\partial^2 T_s}{\partial x^2} = 0, \quad (a)$$

$$\frac{\partial^2 T_l}{\partial x^2} = 0. \quad (b)$$

(iii) **Boundary and Initial Conditions.**

$$(1) T_s(0, t) = T_o$$

$$(2) T_s(x_i, t) = T_f$$

$$(3) T_l(x_i, t) = T_f$$

$$(4) T_l(L, t) = T_f$$

Interface energy condition is

$$(5) k_s \frac{\partial T_s(x_i, t)}{\partial x} - k_l \frac{\partial T_l(x_i, t)}{\partial x} = \rho_s \mathcal{L} \frac{dx_i}{dt}.$$

The initial conditions are

$$(6) T_l(x, 0) = T_f$$

$$(7) x_i(0) = 0$$

(4) **Solution.** Direct integration of (a) and (b) gives

$$T_s(x, t) = Ax + B, \quad (c)$$

and

$$T_l(x, t) = Cx + D, \quad (d)$$

where  $A$ ,  $B$ ,  $C$ , and  $D$  are constants of integration. These constants can be functions of time. Application of the first four boundary conditions gives

$$T_s(x, t) = T_o + (T_f - T_o) \frac{x}{x_i(t)}, \quad (e)$$

and

$$T_l(x, t) = T_f. \quad (f)$$

As anticipated, the liquid temperature remains constant. The interface location is determined from condition (5). Substituting (e) and (f) into condition (5) gives

$$k_s \frac{T_f - T_o}{x_i} - 0 = \rho_s \mathcal{L} \frac{dx_i}{dt}.$$

Separating variables

$$x_i dx_i = \frac{k_s (T_f - T_o)}{\rho_s \mathcal{L}} dt.$$

Integration gives

$$x_i^2 = \frac{2k_s (T_f - T_o)}{\rho_s \mathcal{L}} t + C_1.$$

Initial condition (7) gives  $C_1 = 0$ . Thus

$$x_i(t) = \sqrt{\frac{2k_s (T_f - T_o)}{\rho_s \mathcal{L}} t}. \quad (6.16a)$$

In terms of the dimensionless variables of eq. (6.8), this result can be expressed as

$$\xi_i = \sqrt{2\tau}. \quad (6.16b)$$

The time  $t_o$  needed for the entire slab to solidify is obtained by setting  $x_i = L$  in eq. (6.16a)

$$t_o = \frac{\rho_s \mathcal{L} L^2}{2k_s (T_f - T_o)}. \quad (6.17a)$$

Since total solidification corresponds to  $\xi_i = 1$ , eq. (6.16b) gives

$$\tau_o = 1/2. \quad (6.17b)$$

(5) **Checking.** *Dimensional check:* eqs. (6.16a) and (6.17a) are dimensionally consistent.

*Limiting check:* (i) If  $T_o = T_f$ , no solidification takes place and the time needed to solidify the slab should be infinite. Setting  $T_o = T_f$  in eq.

(6.17a) gives  $t_o = \infty$ . (ii) If the slab is infinitely wide, solidification time should be infinite. Setting  $L = \infty$  in eq. (6.17a) gives  $t_o = \infty$ .

**(6) Comments.** (i) Initial condition (6) is not used in the solution. This is inherent in the quasi-steady state model. Since time derivatives of temperature are dropped in this model, there is no opportunity to satisfy initial conditions. Nevertheless, the solution to the temperature distribution in the liquid phase does satisfy initial condition (6). (ii) The solution confirms the observation that the liquid phase must remain at  $T_f$ . Thus analysis of the liquid phase is unnecessary. (iii) Since the liquid phase remains at the fusion temperature, fluid motion due to density changes or convection plays no role in the solution.

**Example 6.2: Melting of Slab with Time Dependent Surface Temperature**

The solid slab shown in Fig. 6.4 is initially at the fusion temperature  $T_f$ . The side at  $x = 0$  is suddenly maintained at a time dependent temperature above  $T_f$ , given by

$$T_L(0, t) = T_o \exp \beta t,$$

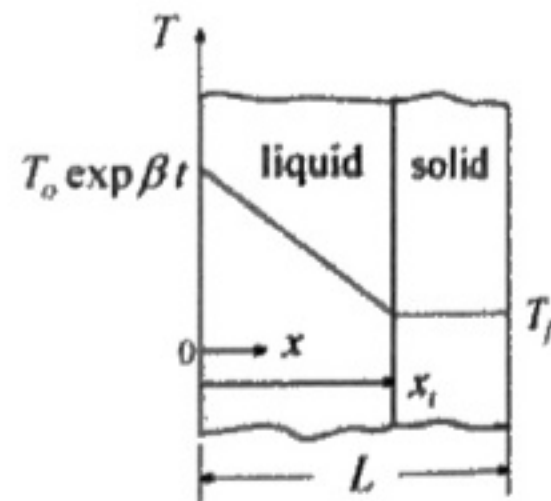
where  $T_o$  and  $\beta$  are constant. A liquid-solid interface forms at  $x = 0$  and moves towards the opposite side. Use a quasi-steady state model to determine the interface location and the time it takes for the slab to melt.

**(1) Observations.** (i) Based on physical consideration, the solid phase remains at uniform temperature equal to the fusion temperature  $T_f$ . (ii) Melting starts at time  $t = 0$ . (iii) The time it takes the slab to melt is equal to the time needed for the interface to traverse the slab width  $L$ .

**(2) Origin and Coordinates.** Fig. 6.4 shows the origin and the coordinate axes.

**(3) Formulation.**

**(i) Assumptions.** (1) One-dimensional conduction, (2) constant properties of the liquid phase, (3) no changes in fusion temperature, (4) neglect motion of the liquid phase and (5) quasi-steady state ( $Ste < 0.1$ ).



**Fig. 6.4**

**(ii) Governing Equations.** For quasi-steady state the liquid phase conduction equation, expressed in dimensional form, is

$$\frac{\partial^2 T_L}{\partial x^2} = 0. \tag{a}$$

Since no heat can transfer to the solid phase, its temperature remains constant. Thus

$$T_s(x, t) = T_f. \tag{b}$$

**(iii) Boundary and Initial Conditions.** The boundary conditions are

$$(1) T_L(0, t) = T_o \exp \beta t$$

$$(2) T_L(x_i, t) = T_f$$

Substituting (b) into eq. (6.4) gives the interface energy condition

$$(3) -k_L \frac{\partial T_L(x_i, t)}{\partial x} = \rho_L \mathcal{L} \frac{dx_i}{dt}$$

Note that since this is a melting problem,  $\rho_s$  in eq. (6.4) is replaced by  $\rho_L$ . The initial condition of the interface is

$$(4) x_i(0) = 0$$

**(4) Solution.** Integration of (a) gives

$$T_L(x, t) = Ax + B.$$

Application of boundary conditions (1) and (2) gives

$$T_L(x, t) = \frac{T_f - T_o \exp \beta t}{x_i} x + T_o \exp \beta t. \tag{c}$$

Substituting (c) into condition (3) and separating variables

$$-\frac{k_L}{\rho_L \mathcal{L}} (T_f - T_o \exp \beta t) dt = x_i dx_i. \tag{d}$$

Integrating and using initial condition (4)

$$-\frac{k_L}{\rho_L \mathcal{L}} \int_0^t (T_f - T_o \exp \beta t) dt = \int_0^{x_i} x_i dx_i,$$

or

$$-\frac{k_l}{\rho_l L} [T_f t - (T_o / \beta) \exp(\beta t) + (T_o / \beta)] = \frac{x_i^2}{2}$$

Solving this result for  $x_i(t)$

$$x_i(t) = \sqrt{\frac{2k_l}{\rho_l L} [(T_o / \beta) \exp(\beta t) - T_f t - (T_o / \beta)]} \quad (6.18)$$

To determine the time  $t_o$  it takes for the slab to melt, set  $x_i = L$  in eq. (6.18)

$$L = \sqrt{\frac{2k_l}{\rho_l L} [(T_o / \beta) \exp(\beta t_o) - T_f t_o - (T_o / \beta)]} \quad (6.19)$$

This equation can not be solved for  $t_o$  explicitly. A trial and error procedure is needed to determine  $t_o$ .

**(5) Checking. Dimensional check:** Eq. (6.18) is dimensionally consistent.

**Limiting check:** For the special case of constant surface temperature at  $x = 0$ , results for this example should be identical with the solidification problem of Example 6.1 with the properties of Example 6.1 changed to those of the liquid phase. Setting  $\beta = 0$  in the time dependent surface temperature gives  $T(0, t) = T_o$ , which is the condition for Example 6.1. However, since  $\beta$  appears in the denominator in eq. (6.18), direct substitution of  $\beta = 0$  can not be made. Instead, the term  $\exp \beta t$  in eq. (6.18) is expanded for small values of  $\beta t$  before setting  $\beta = 0$ . Thus, for small  $\beta t$  eq. (6.18) is written as

$$x_i(t) = \sqrt{\frac{2k_l}{\rho_l L} [(T_o / \beta) (1 + (\beta t) / 1! + \dots) - T_f t - (T_o / \beta)]}$$

or, for  $\beta = 0$

$$x_i(t) = \sqrt{\frac{2k_l}{\rho_l L} (T_o - T_f) t} \quad (6.20)$$

This result agrees with eq. (6.16a).

**(6) Comments.** Time dependent boundary conditions do not introduce mathematical complications in the quasi-steady state model.

### 6.7 Exact Solutions

#### 6.7.1 Stefan's Solution

One of the earliest published exact solutions to phase change problems is credited to Stefan who published his work in 1891 [1]. He considered solidification of a semi-infinite liquid region shown in Fig. 6.5. The liquid is initially at the fusion temperature  $T_f$ . The surface at  $x = 0$  is suddenly maintained at temperature  $T_o < T_f$ . Solidification begins instantaneously at  $x = 0$ . We wish to determine the temperature distribution and interface location. Since no heat can transfer to the liquid phase, its temperature remains constant throughout. Thus

$$T_l(x, t) = T_f \quad (a)$$

The governing equation in the solid phase is given by eq. (6.1)

$$\frac{\partial^2 T_s}{\partial x^2} = \frac{1}{\alpha_s} \frac{\partial T_s}{\partial t} \quad (b)$$

The boundary conditions are

- (1)  $T_s(0, t) = T_o$
- (2)  $T_s(x_i, t) = T_f$

Substituting (a) into eq. (6.4) gives

$$(3) k_s \frac{\partial T_s(x_i, t)}{\partial x} = \rho_s L \frac{dx_i}{dt}$$

The initial condition is

- (4)  $x_i(0) = 0$

Equation (b) is solved by similarity transformation. We assume that the two independent variables  $x$  and  $t$  can be combined into a single variable  $\eta = \eta(x, t)$ . Following the use of this method in solving the problem of transient conduction in a semi-infinite region, the appropriate similarity variable is

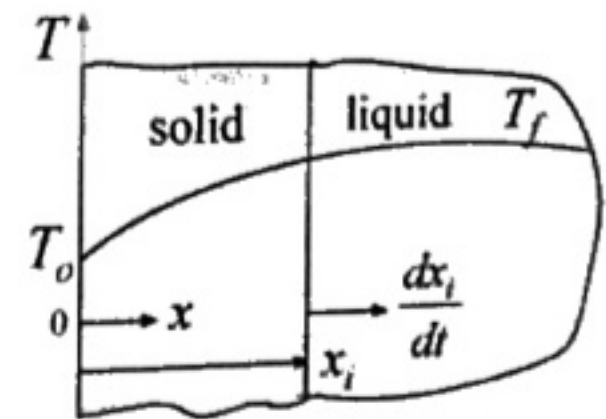


Fig. 6.5

$$\eta = \frac{x}{\sqrt{4\alpha_s t}} \quad (6.21)$$

We postulate that the solution to (b) can be expressed as

$$T_s = T_s(\eta) \quad (c)$$

Using eq. (6.21) and (c), equation (b) transforms to

$$\frac{d^2 T_s}{d\eta^2} + 2\eta \frac{dT_s}{d\eta} = 0 \quad (d)$$

Thus the governing partial differential equation (b) is transformed into an ordinary differential equation. Note that (d) is identical to eq. (4.33) which describes transient conduction in a semi-infinite region without phase change. Following the procedure used to solve eq. (4.33), the solution to (d) is

$$T_s = A \operatorname{erf} \eta + B \quad (6.22)$$

Applying boundary condition (2) and using eq. (6.21) give

$$T_f = A \operatorname{erf} \frac{x_i}{\sqrt{4\alpha_s t}} + B \quad (e)$$

Since  $T_f$  is constant, it follows that the argument of the error function in (e) must also be constant. Thus we conclude that

$$x_i \propto \sqrt{t}$$

Let

$$x_i = \lambda \sqrt{4\alpha_s t} \quad (f)$$

where  $\lambda$  is a constant to be determined. Note that this solution to the interface location satisfies initial condition (4). Applying boundary condition (1) to eq. (6.22) and noting that  $\operatorname{erf} 0 = 0$ , gives

$$B = T_o \quad (g)$$

Boundary condition (2), (g) and eq. (6.22) give the constant  $A$

$$A = \frac{T_f - T_o}{\operatorname{erf} \lambda} \quad (h)$$

Substituting (g) and (h) into eq. (6.22) gives the temperature distribution in the solid phase

$$T_s(x, t) = T_o + \frac{T_f - T_o}{\operatorname{erf} \lambda} \operatorname{erf} \eta \quad (6.23)$$

Finally, interface energy condition (3) is used to determine the constant  $\lambda$ . Substituting (f) and eq. (6.23) into condition (3) gives

$$k_s \frac{T_f - T_o}{\operatorname{erf} \lambda} \left[ \frac{d}{d\eta} (\operatorname{erf} \eta) \frac{\partial \eta}{\partial x} \right]_{x_i} = \rho_s \ell \frac{\lambda \sqrt{4\alpha_s}}{2\sqrt{t}}$$

The derivative of the error function is given by eq. (4.34). Substituting into the above and using eq. (6.21) gives

$$\lambda (\exp \lambda^2) \operatorname{erf} \lambda = \frac{(T_f - T_o) c_{ps}}{\sqrt{\pi} \ell} \quad (6.24)$$

where  $c_{ps}$  is specific heat of the solid. Equation (6.24) gives the constant  $\lambda$ . However, since eq. (6.24) can not be solved explicitly for  $\lambda$ , a trial and error procedure is required to obtain a solution. Note that  $\lambda$  depends on the material as well as the temperature at  $x = 0$ .

It is interesting to examine Stefan's solution for small values of  $\lambda$ . To evaluate eq. (6.24) for small  $\lambda$ , we note that

$$\exp \lambda^2 = 1 + \frac{\lambda^2}{1!} + \frac{\lambda^4}{2!} + \dots \approx 1,$$

and

$$\operatorname{erf} \lambda = \frac{2}{\sqrt{\pi}} \left( \lambda - \frac{\lambda^3}{3 \times 1!} + \frac{\lambda^5}{5 \times 2!} + \dots \right) \approx \frac{2}{\sqrt{\pi}} \lambda.$$

Substituting into eq. (6.24) gives

$$\lambda = \sqrt{\frac{c_{ps} (T_f - T_o)}{2\ell}}, \quad \text{for small } \lambda \quad (6.25)$$

Thus, according to eq. (6.13), a small  $\lambda$  corresponds to a small Stefan number. Substituting eq. (6.25) into (f) gives



$$x_i = \sqrt{\frac{2k_s(T_f - T_o)}{\rho_s \ell}} t, \text{ for small } \lambda \text{ (small } Ste). \quad (6.26)$$

This result is identical to eq. (6.16a) of the quasi-steady state model.

### 6.7.2 Neumann's Solution: Solidification of Semi-Infinite Region

Neumann [1] solved the more general problem of phase change in a semi-infinite region which is not initially at the fusion temperature. Thus Stefan's solution is a special case of Neumann's problem. Although Neumann's solution was presented in his lectures in the 1860's, the work was not published until 1912.

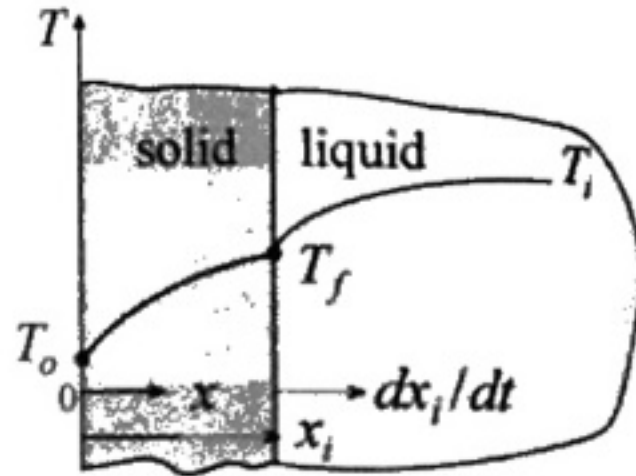


Fig. 6.6

Consider the solidification of a semi-infinite region shown in Fig. 6.6. Initially the region is at a uniform temperature  $T_i$  which is above the solidification temperature  $T_f$ . The surface at  $x = 0$  is suddenly maintained at a constant temperature  $T_o < T_f$ . A solid-liquid interface forms instantaneously at  $x = 0$  and propagates through the liquid phase. Since  $T_i \neq T_f$ , heat is conducted through the liquid towards the interface. Neumann's solution to this problem is based on the assumption that the temperature distribution is one-dimensional, properties of each phase are uniform and remain constant and that motion of the liquid phase is neglected. Thus the governing equations are

$$\frac{\partial^2 T_s}{\partial x^2} = \frac{1}{\alpha_s} \frac{\partial T_s}{\partial t} \quad 0 < x < x_i, \quad (6.1)$$

and

$$\frac{\partial^2 T_l}{\partial x^2} = \frac{1}{\alpha_l} \frac{\partial T_l}{\partial t} \quad x > x_i. \quad (6.2)$$

The boundary conditions are

$$(1) T_s(0, t) = T_o$$

$$(2) T_s(x_i, t) = T_f$$

$$(3) T_l(x_i, t) = T_f$$

$$(4) T_l(\infty, t) = T_i$$

The interface energy equation is

$$(5) k_s \frac{\partial T_s(x_i, t)}{\partial x} - k_l \frac{\partial T_l(x_i, t)}{\partial x} = \rho_s \ell \frac{dx_i}{dt}$$

The initial conditions are

$$(6) T_l(x, 0) = T_i$$

$$(7) x_i(0) = 0$$

Following the procedure used to solve Stefan's problem, the similarity method is applied to solve equations (6.1) and (6.2). The appropriate similarity variable is

$$\eta = x / \sqrt{4\alpha_s t}. \quad (6.21)$$

We assume that the solutions to eq. (6.1) and eq. (6.2) can be expressed as

$$T_s = T_s(\eta), \quad (a)$$

and

$$T_l = T_l(\eta). \quad (b)$$

Using eq. (6.21), (a) and (b), equations (6.1) and (6.2) transform to

$$\frac{d^2 T_s}{d\eta^2} + 2\eta \frac{dT_s}{d\eta} = 0 \quad 0 < \eta < \eta_i, \quad (c)$$

and

$$\frac{d^2 T_l}{d\eta^2} + 2\frac{\alpha_s}{\alpha_l} \eta \frac{dT_l}{d\eta} = 0 \quad \eta > \eta_i, \quad (d)$$

where

$$\eta_i = x_i / \sqrt{4\alpha_s t}. \quad (e)$$

Solutions to (c) and (d) are

$$T_s = A \operatorname{erf} \eta + B, \quad (f)$$

and

$$T_L = C \operatorname{erf} \sqrt{\alpha_s / \alpha_l} \eta + D. \quad (g)$$

Applying boundary condition (2)

$$T_f = A \operatorname{erf} \frac{x_i}{\sqrt{4\alpha_s t}} + B.$$

Thus

$$x_i \propto \sqrt{t}.$$

Let

$$x_i = \lambda \sqrt{4\alpha_s t}. \quad (h)$$

Using eq. (6.21) and (h), conditions (1) to (6) are transformed to

$$(1) T_s(0) = T_o$$

$$(2) T_s(\lambda) = T_f$$

$$(3) T_L(\lambda) = T_f$$

$$(4) T_L(\infty) = T_i$$

$$(5) k_s \frac{dT_s(\lambda)}{d\eta} - k_l \frac{dT_L(\lambda)}{d\eta} = 2\rho_s \alpha_s \lambda \mathcal{L}$$

$$(6) T_L(\infty) = T_i$$

Note that (h) satisfies condition (7) and that conditions (4) and (6) are identical. In the transformed problem the interface appears stationary at  $\eta = \lambda$ . Fig. 6.7 shows the temperature distribution of the transformed problem. Boundary conditions (1)-(4) give the four constants of integration,  $A$ ,  $B$ ,  $C$  and  $D$ . Solutions (f) and (g) become

$$T_s(x, t) = T_o + \frac{(T_f - T_o)}{\operatorname{erf} \lambda} \operatorname{erf} \frac{x}{\sqrt{4\alpha_s t}}, \quad (6.27)$$

and

$$T_L(x, t) = T_i + \frac{(T_f - T_i)}{1 - \operatorname{erf} \sqrt{(\alpha_s / \alpha_l)} \lambda} \left(1 - \operatorname{erf} \frac{x}{\sqrt{4\alpha_l t}}\right). \quad (6.28)$$

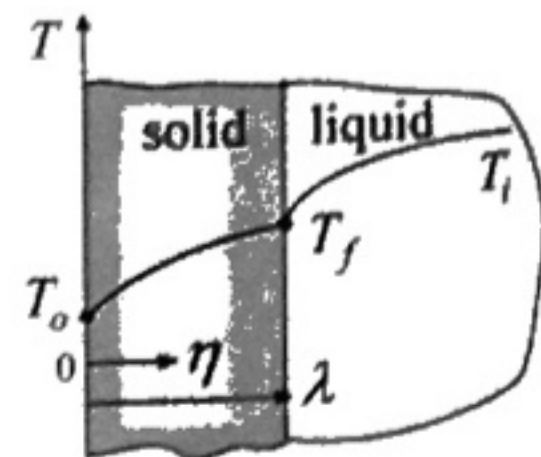


Fig. 6.7

The constant  $\lambda$  appearing in the interface solution (h) and in equations (6.27) and (6.28) is still unknown. All boundary conditions are satisfied except the interface energy condition. Substituting equations (6.27) and (6.28) into condition (5) gives an equation for  $\lambda$

$$\frac{\exp(-\lambda^2)}{\operatorname{erf} \lambda} - \frac{\sqrt{\alpha_s} k_l}{\alpha_l k_s} \frac{T_i - T_f}{T_f - T_o} \frac{\exp(-\lambda^2 \alpha_s / \alpha_l)}{1 - \operatorname{erf}(\sqrt{\alpha_s / \alpha_l} \lambda)} = \frac{\sqrt{\pi} \mathcal{L} \lambda}{c_{ps}(T_f - T_o)} \quad (6.29)$$

Note that Stefan's solution is a special case of Neumann's solution. It can be obtained from Neumann's solution by setting  $T_i = T_f$  in equations (6.27)-(6.29).

### 6.7.3 Neumann's Solution: Melting of Semi-infinite Region

The same procedure can be followed to solve the corresponding melting problem. In this case the density  $\rho_s$  in interface condition (5) above is replaced by  $\rho_l$ . The solutions to the interface location and temperature distribution are

$$x_i = \lambda \sqrt{4\alpha_l t}, \quad (6.30)$$

$$T_L(x, t) = T_o + \frac{(T_f - T_o)}{\operatorname{erf} \lambda} \operatorname{erf} \frac{x}{\sqrt{4\alpha_l t}}, \quad (6.31)$$

and

$$T_s(x, t) = T_i + \frac{(T_f - T_i)}{1 - \operatorname{erf} \sqrt{(\alpha_l / \alpha_s)} \lambda} \left(1 - \operatorname{erf} \frac{x}{\sqrt{4\alpha_s t}}\right), \quad (6.32)$$

where  $\lambda$  is given by

$$\frac{\exp(-\lambda^2)}{\operatorname{erf} \lambda} - \frac{\sqrt{\alpha_l} k_s}{\alpha_s k_l} \frac{T_f - T_i}{T_o - T_f} \frac{\exp(-\alpha_l \lambda^2 / \alpha_s)}{1 - \operatorname{erf}(\sqrt{\alpha_l / \alpha_s} \lambda)} = \frac{\sqrt{\pi} \mathcal{L} \lambda}{c_{pl}(T_o - T_f)} \quad (6.33)$$

It is important to recognize that for the same material at the same  $|T_i - T_f|$  and  $|T_o - T_f|$ , values of  $\lambda$  for solidification and for melting are not identical.

### 6.8 Effect of Density Change on the Liquid Phase

Although we have taken into consideration property change during phase transformation, we have neglected the effect of liquid motion resulting from density change. A material that expands during solidification causes the liquid phase to move in the direction of interface motion as shown in Fig. 6.8. The heat conduction equation for a fluid moving with a velocity  $U$  is obtained from eq. (1.7). Assuming constant properties and one-dimensional conduction, eq. (1.7) gives

$$\alpha_L \frac{\partial^2 T_L}{\partial x^2} = \frac{\partial T_L}{\partial t} + U \frac{\partial T_L}{\partial x} \quad (6.34)$$

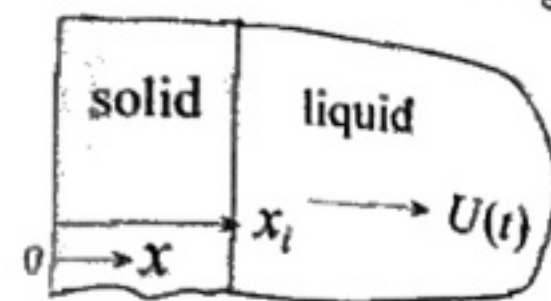


Fig. 6.8

To formulate an equation for the liquid velocity  $U$ , consider phase change in which the liquid phase is not restrained from motion by an external rigid boundary. Fig. 6.9 shows a liquid element  $dx_L$  adjacent to the interface at  $x_i$ . During a time interval  $dt$  the element solidifies and expands to  $dx_s$ . The interface moves a distance  $dx_i$  given by

$$dx_i = dx_s \quad (a)$$

The expansion of the element causes the liquid phase to move with a velocity  $U$  given by

$$U = \frac{dx_i - dx_L}{dt} \quad (b)$$

Conservation of mass for the element yields

$$\rho_L dx_L = \rho_s dx_s = \rho_s dx_i,$$

or

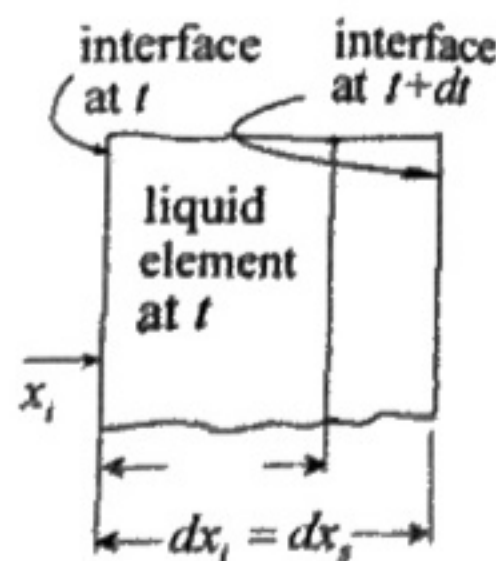


Fig. 6.9

$$dx_L = \frac{\rho_s}{\rho_L} dx_i \quad (c)$$

Substituting (c) into (b)

$$U = \left(1 - \frac{\rho_s}{\rho_L}\right) \frac{dx_i}{dt} \quad (6.35)$$

Since interface velocity  $dx_i/dt$  is not constant, it follows that the velocity  $U$  is time dependent. Substituting eq. (6.35) into eq. (6.34) gives

$$\alpha_L \frac{\partial^2 T_L}{\partial x^2} - \left(1 - \frac{\rho_s}{\rho_L}\right) \frac{dx_i}{dt} \frac{\partial T_L}{\partial x} = \frac{\partial T_L}{\partial t} \quad (6.36)$$

The solution to Neumann's problem, taking into consideration liquid phase motion, can be obtained by applying the similarity transformation method.

### 6.9 Radial Conduction with Phase Change

Exact analytic solutions to radial conduction with phase change can be constructed if phase transformation takes place in an infinite region. Examples include phase change around a line heat source or heat sink. Consider solidification due to a line heat sink shown in Fig. 6.10. The liquid is initially at  $T_i > T_f$ . Heat is suddenly removed along a line sink at a rate  $Q_o$  per unit length. Assuming constant properties in each phase and neglecting the effect of liquid motion, the heat conduction equations are

$$\frac{\partial^2 T_s}{\partial r^2} + \frac{1}{r} \frac{\partial T_s}{\partial r} = \frac{1}{\alpha_s} \frac{\partial T_s}{\partial t} \quad 0 < r < r_i(t), \quad (6.37)$$

$$\frac{\partial^2 T_L}{\partial r^2} + \frac{1}{r} \frac{\partial T_L}{\partial r} = \frac{1}{\alpha_L} \frac{\partial T_L}{\partial t} \quad r > r_i(t). \quad (6.38)$$

where  $r_i(t)$  is the interface location. The boundary condition at the center  $r = 0$  is based on the strength of the heat sink. It is expressed as

$$(1) \lim_{r \rightarrow 0} \left[ 2\pi r k_s \frac{\partial T_s}{\partial r} \right] = Q_o$$

The remaining boundary conditions are

- (2)  $T_s(r_i, t) = T_f$
- (3)  $T_L(r_i, t) = T_f$
- (4)  $T_L(\infty, t) = T_i$

The interface energy equation is

$$(5) \quad k_s \frac{\partial T_s(r_i, t)}{\partial r} - k_L \frac{\partial T_L(r_i, t)}{\partial r} = \rho_s L \frac{dr_i}{dt}$$

The initial temperature is

$$(6) \quad T_L(r, 0) = T_i$$

Assuming that solidification begins instantaneously, the interface initial condition is

$$(7) \quad r_i(0) = 0$$

Solution to this problem is based on the similarity method. The similarity variable is [1]

$$\eta = r^2 / 4\alpha_s t. \quad (6.39)$$

Assume that

$$T_s(x, t) = T_s(\eta),$$

and

$$T_L(x, t) = T_L(\eta).$$

Using eq. (6.39), equations (6.37) and (6.38) transform to

$$\frac{d^2 T_s}{d\eta^2} + \left(1 + \frac{1}{\eta}\right) \frac{dT_s}{d\eta} = 0, \quad (6.40)$$

and

$$\frac{d^2 T_L}{d\eta^2} + \left(\frac{\alpha_s}{\alpha_L} + \frac{1}{\eta}\right) \frac{dT_L}{d\eta} = 0. \quad (6.41)$$

These equations can be integrated directly. Separating variables in eq. (6.40) and integrating

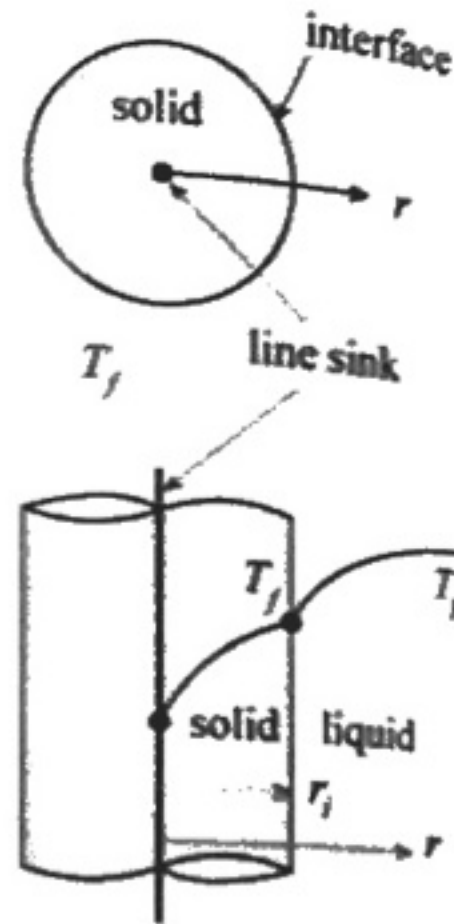


Fig. 6.10

$$\int^{T_s} \frac{d(dT_s / d\eta)}{dT_s / d\eta} = \int^{\eta} \left(-\frac{d\eta}{\eta} - d\eta\right),$$

$$\ln \frac{dT_s}{d\eta} = -\ln \eta - \eta + \ln A,$$

$$\ln \left[ \frac{1}{A} \eta \frac{dT_s}{d\eta} \right] = -\eta.$$

Rearranging, separating variables and integrating again

$$\int^{T_s} dT_s = A \int_{\eta}^{\infty} \frac{e^{-\eta}}{\eta} d\eta + B,$$

$$T_s = A \int_{\eta}^{\infty} \frac{e^{-\eta}}{\eta} d\eta + B. \quad (a)$$

where  $A$  and  $B$  are constants of integration. The choice of the upper limit in the above integral will be explained later. The same approach is used to solve eq. (6.41). The result is

$$T_L = C \int_{\eta}^{\infty} \frac{e^{-(\alpha_s / \alpha_L) \eta}}{\eta} d\eta + D, \quad (b)$$

where  $C$  and  $D$  are constants of integration. Solution (a) and boundary condition (1) give

$$\lim_{\eta \rightarrow 0} \left[ 2\pi r k_s \frac{dT_s}{d\eta} \frac{\partial \eta}{\partial r} \right] = \lim_{\eta \rightarrow 0} \left[ -2\pi k_s r A \frac{e^{-\eta}}{\eta} \frac{2r}{4\alpha_s t} \right] =$$

$$\lim_{\eta \rightarrow 0} \left[ -4\pi k_s A \frac{e^{-\eta}}{\eta} \right] = -4\pi k_s A = Q_o.$$

This gives  $A$  as

$$A = -\frac{Q_o}{4\pi k_s}. \quad (c)$$

To determine  $r_i(t)$  solution (a) is applied to boundary condition (2) to give

$$T_f = A \int_{\eta_i}^{\infty} \frac{e^{-\eta}}{\eta} d\eta + B, \quad (d)$$

where  $\eta_i$  is the value of  $\eta$  at the interface. It is determined by setting  $r = r_i$  in eq. (6.39)

$$\eta_i = r_i^2 / 4\alpha_s t. \quad (6.42)$$

Since  $T_f$  remains constant at all times, this result requires that  $\eta_i$  be independent of time. It follows from eq. (6.42) that  $r_i^2 \propto t$ . Thus we let

$$r_i^2 = 4\lambda\alpha_s t. \quad (6.43)$$

Note that this form of  $r_i$  satisfies initial condition (7). Substituting into eq. (6.42) gives

$$\eta_i = \lambda, \quad (e)$$

where  $\lambda$  is a constant to be determined. Substituting (c) and (e) into (d)

$$T_f = -\frac{Q_o}{4\pi k_s} \int_{\lambda}^{\infty} \frac{e^{-\eta}}{\eta} d\eta + B.$$

Solving for  $B$

$$B = T_f + \frac{Q_o}{4\pi k_s} \int_{\lambda}^{\infty} \frac{e^{-\eta}}{\eta} d\eta. \quad (f)$$

Applying condition (4) to solution (b) gives

$$T_i = C \int_{\infty}^{\infty} \frac{e^{-(\alpha_s/\alpha_l)\eta}}{\eta} d\eta + D = 0 + D,$$

or

$$D = T_i. \quad (g)$$

Boundary condition (3) and solution (b) give

$$C = \frac{T_f - T_i}{\int_{\lambda}^{\infty} \frac{e^{-(\alpha_s/\alpha_l)\eta}}{\eta} d\eta}. \quad (h)$$

Interface energy equation (5) gives an equation for  $\lambda$

$$\frac{Q_o}{4\pi} e^{-\lambda} - \frac{k_l(T_i - T_f)}{\int_{\lambda}^{\infty} \frac{e^{-(\alpha_s/\alpha_l)\eta}}{\eta} d\eta} e^{-(\alpha_s/\alpha_l)\lambda} = \lambda \rho_s \alpha_s \mathcal{L}. \quad (6.44)$$

Thus all the required constants are determined. Before substituting the constants of integration into solutions (a) and (b) we will examine the integrals appearing in this solution. These integrals are encountered in other application and are tabulated in the literature [2]. The *exponential integral function*  $Ei(-x)$  is defined as

$$Ei(-x) = - \int_x^{\infty} \frac{e^{-v}}{v} dv. \quad (6.45)$$

This explains why in integrating equations (6.41) and (6.42) the upper limit in the integrals in (a) and (b) is set at  $\eta = \infty$ . Values of exponential integral function at  $x = 0$  and  $x = \infty$  are

$$Ei(0) = \infty, \quad Ei(\infty) = 0. \quad (6.46)$$

Using the definition of  $Ei(-x)$  and the constants given in (c), (f), (g) and (h), equations (a), (b) and (6.44), become

$$T_s(r, t) = T_f + \frac{Q_o}{4\pi k_s} [Ei(-r^2 / 4\alpha_s t) - Ei(-\lambda)], \quad (6.47)$$

$$T_l = T_i + \frac{T_f - T_i}{Ei(-\lambda\alpha_s / \alpha_l)} Ei(-r^2 / 4\alpha_l t), \quad (6.48)$$

and

$$\frac{Q_o}{4\pi} e^{-\lambda} + \frac{k_l(T_i - T_f)}{Ei(-\lambda\alpha_s / \alpha_l)} e^{-(\lambda\alpha_s / \alpha_l)} = \lambda \rho_s \alpha_s \mathcal{L}. \quad (6.49)$$

## 6.10 Phase Change in Finite Regions

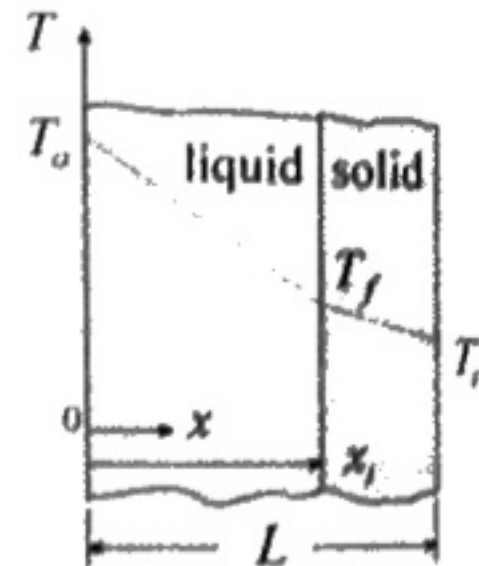
Exact analytic solutions to phase change problems are limited to semi-infinite and infinite regions. Solutions to phase change in finite slabs and inside or outside cylinders and spheres are not available. Such problems are usually solved approximately or numerically.

REFERENCES

- [1] Carslaw, H.S., and Jaeger, J.G., *Conduction of Heat in Solids*, 2nd edition, Oxford University Press, 1959.
- [2] Selby, S.M., *Standard Mathematical Tables*, The Chemical Rubber Co., Cleveland, Ohio, 1968.

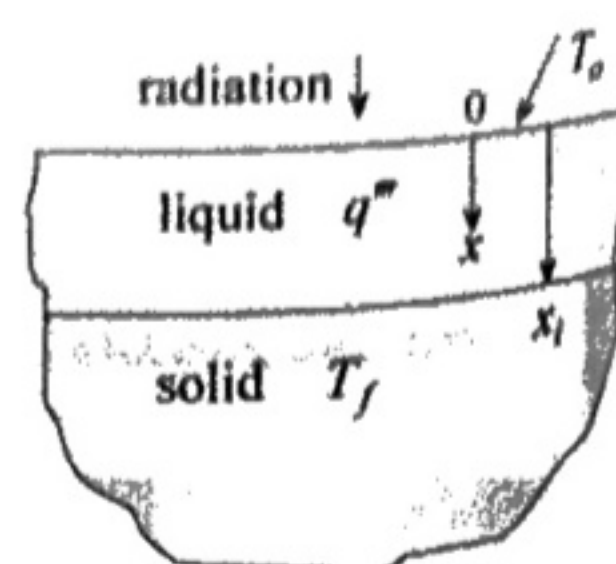
PROBLEMS

6.1 A slab of thickness  $L$  and fusion temperature  $T_f$  is initially solid at temperature  $T_i < T_f$ . At time  $t \geq 0$  one side is heated to temperature  $T_o > T_f$  while the other side is held at  $T_i$ . Assume that  $Ste < 0.1$ , determine the transient and steady state interface location.



6.2 Consider freezing of a deep lake which is initially at the fusion temperature  $T_f$ . During a sudden cold wave which lasted four weeks, air temperature dropped to  $-18^\circ\text{C}$ . Assume that lake surface temperature is approximately the same as air temperature. Justify using a quasi-steady model and determine the ice thickness at the end of four weeks. Ice properties are:  $c_{ps} = 2093 \text{ J/kg}\cdot^\circ\text{C}$ ,  $k_s = 2.21 \text{ W/m}\cdot^\circ\text{C}$ ,  $\mathcal{L} = 333,730 \text{ J/kg}$  and  $\rho_s = 916.8 \text{ kg/m}^3$ .

6.3 Radiation is beamed at a semi-infinite region which is initially solid at the fusion temperature  $T_f$ . The radiation penetrates the liquid phase resulting in a uniform energy generation  $q''$ . The surface at  $x=0$  is maintained at temperature  $T_o > T_f$ . Assume that  $Ste < 0.1$ , determine the transient and steady state interface location.



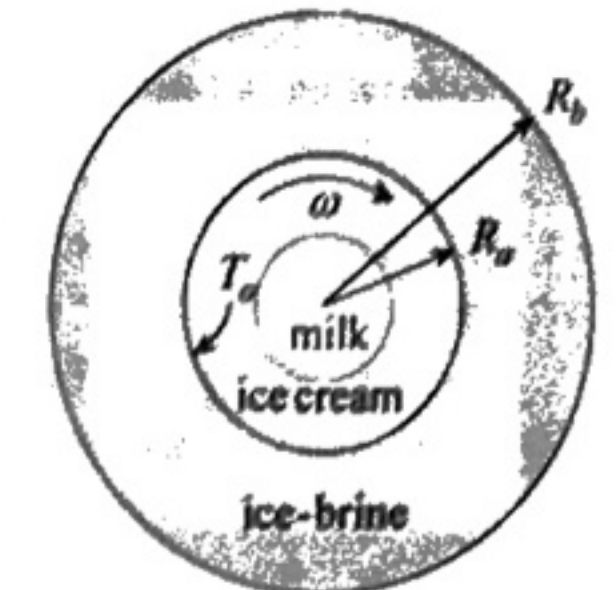
6.4 You decided to make ice during a cold day by placing water in a pan outdoors. Heat transfer from the water is by convection. Air temperature is  $-10^\circ\text{C}$  and the heat transfer coefficient is  $12.5 \text{ W/m}^2\cdot^\circ\text{C}$ . Initially the water is at the fusion temperature. How thick will the ice layer be after 7 hours? Justify using a quasi-steady model to obtain an approximate answer. Properties of ice are:  $c_{ps} = 2093 \text{ J/kg}\cdot^\circ\text{C}$ ,  $k_s = 2.21 \text{ W/m}\cdot^\circ\text{C}$ ,  $\mathcal{L} = 333,730 \text{ J/kg}$  and  $\rho_s = 916.8 \text{ kg/m}^3$ .

6.5 An old fashioned ice cream kit consists of two concentric cylinders of radii  $R_a$  and  $R_b$ . The inner cylinder is filled with milk and ice cream ingredients while the space between the two cylinders is filled with an ice-brine mixture. To expedite the process, the inner cylinder is manually rotated. Assume that the surface of the cylinder is at the brine temperature  $T_o$  and that  $Ste < 0.1$ . Assume further that the liquid is initially at the fusion temperature. Derive an expression for the total solidification time. Apply the result to the following case:  $R_a = 10 \text{ cm}$ ,  $R_b = 25 \text{ cm}$ ,  $T_o = -20^\circ\text{C}$ ,  $T_i = T_f$ . Assume that ice cream has the same properties as ice, given by:

$$c_{ps} = 2093 \text{ J/kg}\cdot^\circ\text{C}, k_s = 2.21 \text{ W/m}\cdot^\circ\text{C}, \mathcal{L} = 333,730 \text{ J/kg} \text{ and } \rho_s = 916.8 \text{ kg/m}^3.$$

6.6 Liquid at the fusion temperature  $T_f$  is contained between two concentric cylinders of radii  $R_a$  and  $R_b$ . At time  $t \geq 0$  the inner cylinder is cooled at a time dependent rate  $q'_o(t)$  per unit length. The outer cylinder is insulated.

- [a] Assuming that  $Ste < 0.1$ , derive an expression for the total solidification time.
- [b] Determine the solidification time for the special case of  $q'_o(t) = C/\sqrt{t}$ , where  $C$  is constant.



- 6.7 Liquid at the fusion temperature fills a thin walled channel of length  $L$  and variable cross section area  $A_c(x)$  given by

$$A_c(x) = A_0 e^{-\alpha x/L}$$

where  $A_0$  is constant and  $x$  is distance along the channel. At time  $t \geq 0$  the surface at  $x = 0$  is maintained at  $T_s < T_f$ . Assume that the channel is well insulated and neglect heat conduction through the wall. Use quasi-steady approximation to determine the interface location.

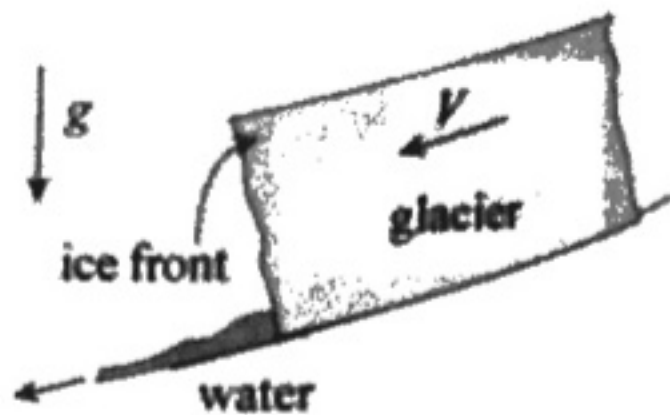


- 6.8 Consider two concentric spheres of radii  $R_1$  and  $R_2$ . The space between the spheres is filled with solid material at the fusion temperature  $T_f$ . At time  $t \geq 0$  the inner sphere is heated such that its surface temperature is maintained at  $T_0 > T_f$ . Determine the time needed for the solid to melt. Assume one-dimensional conduction and  $Ste < 0.1$ .
- 6.9 Consider a semi-infinite solid region at the fusion temperature  $T_f$ . The surface at  $x = 0$  is suddenly heated with a time dependent flux given by

$$q_0''(t) = \frac{C}{\sqrt{t}}$$

where  $C$  is constant. Determine the interface location for the case of a Stefan number which is large compared to unity.

- 6.10 A glacier slides down on an inclined plane at a rate of 1.2 m/year. The ice front is heated by convection. The heat transfer coefficient is  $h = 8 \text{ W/m}^2 \cdot ^\circ\text{C}$ . The average ambient temperature during April through September is  $\bar{T}_\infty = 10^\circ\text{C}$ . During the remaining months the temperature is below freezing. Melting ice at the front flows into an adjacent stream.



Because of continuous melting, the front recedes slowly. Assume that conduction through the ice at the front is small compared to convection. Determine the location of the front after 10 years. Ice properties are:

$$c_{ice} = 1460 \text{ J/kg} \cdot ^\circ\text{C} \quad k_i = 3.409 \text{ W/m} \cdot ^\circ\text{C}$$

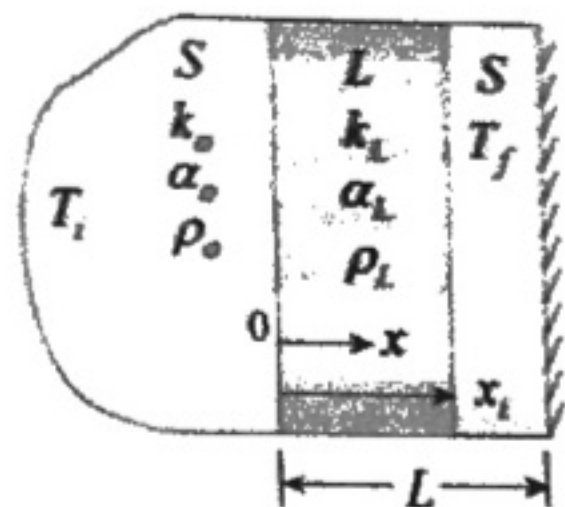
$$L = 334940 \text{ J/kg}$$

$$\rho_i = 928 \text{ kg/m}^3$$

- 6.11 A semi-infinite liquid column at the fusion temperature  $T_f$  is suddenly brought in contact with a semi-infinite solid at a uniform temperature  $T_i < T_f$ . The solid does not undergo phase transformation. The liquid column begins to solidify and grow. Determine the interface location for the case of a Stefan number which is large compared to unity.

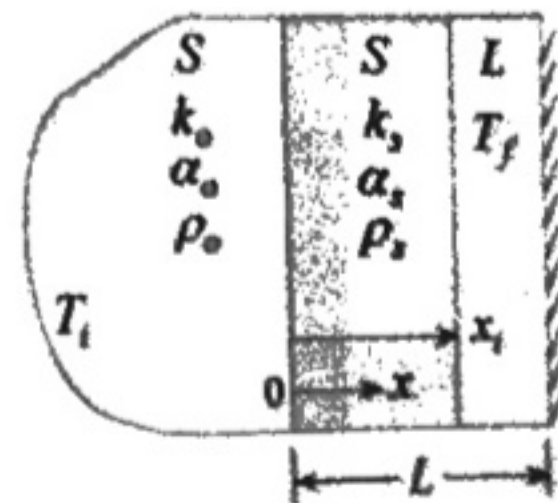


- 6.12 A slab of width  $L$  is initially solid at the fusion temperature  $T_f$ . The slab is brought into contact with a semi-infinite solid region which is initially at uniform temperature  $T_i > T_f$ . The solid region does not undergo phase change while the slab melts. Obtain an exact solution for the time needed for the entire slab to melt. Assume that the free surface of the slab is insulated.



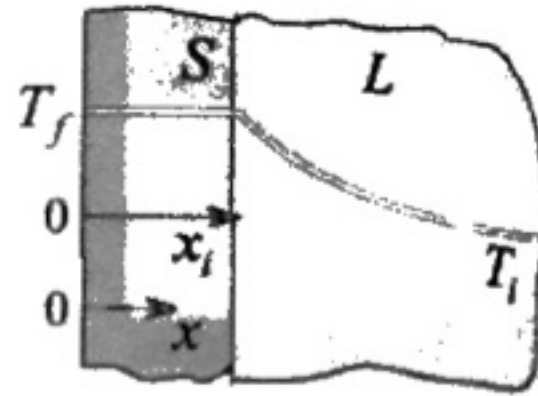
- 6.13 Solve Neumann's problem taking into consideration the effect of density change.

- 6.14 A slab of width  $L$  is initially liquid at the fusion temperature  $T_f$ . The slab is brought into contact with a semi-infinite solid region which is initially at uniform temperature  $T_i < T_f$ . The solid region does not



undergo phase change while the slab solidifies. Obtain an exact solution for the time needed for the entire slab to solidify. Assume that the free surface of the slab is insulated.

- 6.15 Under certain conditions the temperature of a liquid can be lowered below its fusion temperature without undergoing solidification. The liquid in such a state is referred to as supercooled. Consider a supercooled semi-infinite liquid region which is initially at uniform temperature  $T_i < T_f$ . The surface at  $x = 0$  is suddenly maintained at the fusion temperature  $T_f$ . Solidification begins immediately and a solid-liquid front propagates through the liquid phase. Note that the solid phase is at uniform temperature  $T_f$ . Determine the interface location.



- 6.16 Consider melting due to a line heat source in an infinite solid region. The solid is initially at  $T_i < T_f$ . Heat is suddenly added along a line source at a rate  $Q_o$  per unit length. Determine the interface location. Assume constant properties in each phase and neglect fluid motion due to density change.

## NON-LINEAR CONDUCTION PROBLEMS

### 7.1 Introduction

Non-linearity in conduction problems arises when properties are temperature dependent or when boundary conditions are non-linear. Surface radiation and free convection are typical examples of non-linear boundary conditions. In phase change problems the interface energy equation is non-linear.

Although the method of separation of variables has wide applicability, it is limited to linear problems. Various methods are used to solve non-linear problems. Some are exact and others are approximate. In this chapter we will examine the source of non-linearity and present three methods of solution. Chapters 8 and 9 deal with approximate techniques that are also applicable to non-linear problems.

### 7.2 Sources of Non-linearity

#### 7.2.1 Non-linear Differential Equations

Let us examine the following heat equation for one-dimensional transient conduction

$$\frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + q''' = \rho c_p \frac{\partial T}{\partial t} \quad (7.1)$$

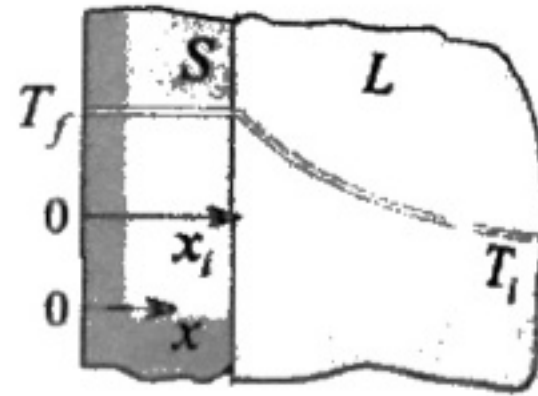
In this equation the density  $\rho$ , specific heat  $c_p$  and thermal conductivity  $k$  can be functions of temperature. Variation of  $\rho$  and/or  $c_p$  with temperature makes the transient term non-linear. Similarly, if  $k = k(T)$  the first term becomes non-linear. This is evident if we rewrite eq. (7.1) as

$$k \frac{\partial^2 T}{\partial x^2} + \frac{dk}{dT} \left[ \frac{\partial T}{\partial x} \right]^2 + q''' = \rho c_p \frac{\partial T}{\partial t} \quad (7.2)$$



undergo phase change while the slab solidifies. Obtain an exact solution for the time needed for the entire slab to solidify. Assume that the free surface of the slab is insulated.

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