Lista 14: Equação de ondas (extraídos do livro de Boyce e DiPrima)

Problems

Consider an elastic string of length L whose ends are held fixed. The string is set in motion with no initial velocity from an initial position $u(x, 0) = f(x)$. In each of Problems 1 through 4, carry out the following steps. Let $L = 10$ and $a = 1$ in parts (b) through (d).

a. Find the displacement $u(x, t)$ for the given initial position $f(x)$.

G **b.** Plot $u(x, t)$ versus x for $0 \leq x \leq 10$ and for several values of t between $t = 0$ and $t = 20$.

G **c.** Plot $u(x, t)$ versus t for $0 \leq t \leq 20$ and for several values of x.

G **d.** Construct an animation of the solution in time for at least one period.

e. Describe the motion of the string in a few sentences.

1. $f(x) = \begin{cases} 2x/L, & 0 \leq x \leq L/2, \\ 2(L - x)/L, & L/2 < x \leq L \end{cases}$

2. $f(x) = \begin{cases} 4x/L, & 0 \leq x \leq L/4, \\ 1, & L/4 < x < 3L/4, \\ 4(L - x)/L, & 3L/4 \leq x \leq L \end{cases}$

3. $f(x) = 8x(L - x)^2/L^3$

4. $f(x) = \begin{cases} 0 & 0 \leq x \leq L/2 - 1 \\ 1, & L/2 - 1 < x < L/2 + 1 \quad (\text{assume } L > 2), \\ 0, & L/2 + 1 \leq x \leq 1 \end{cases}$

Consider an elastic string of length L whose ends are held fixed. The string is set in motion from its equilibrium position with an initial velocity $u_t(x, 0) = g(x)$. In each of Problems 5 through 8, carry out the following steps. Let $L = 10$ and $a = 1$ in parts (b) through (d).

a. Find the displacement $u(x, t)$ for the given $g(x)$.

G **b.** Plot $u(x, t)$ versus x for $0 \leq x \leq 10$ and for several values of t between $t = 0$ and $t = 20$.

G **c.** Plot $u(x, t)$ versus t for $0 \leq t \leq 20$ and for several values of x.

G **d.** Construct an animation of the solution in time for at least one period.

e. Describe the motion of the string in a few sentences.

5. $g(x) = \begin{cases} 2x/L, & 0 \leq x \leq L/2, \\ 2(L - x)/L, & L/2 < x \leq L \end{cases}$

6. $g(x) = \begin{cases} 4x/L, & 0 \leq x \leq L/4, \\ 1, & L/4 < x < 3L/4, \\ 4(L - x)/L, & 3L/4 \leq x \leq L \end{cases}$

7. $g(x) = 8x(L - x)^2/L^3$

8. $g(x) = \begin{cases} 0 & 0 \leq x \leq L/2 - 1 \\ 1, & L/2 - 1 < x < L/2 + 1 \quad (\text{assume } L > 2), \\ 0, & L/2 + 1 \leq x \leq 1 \end{cases}$

9. If an elastic string is free at one end, the boundary condition to be satisfied there is that $u_x = 0$. Find the displacement $u(x, t)$ in an elastic string of length L, fixed at $x = 0$ and free at $x = L$, set in motion with no initial velocity from the initial position $u(x, 0) = f(x)$, where f is a given function. *Hint:* Show that the fundamental solutions for this problem, satisfying all conditions except the nonhomogeneous initial condition, are

$$u_n(x, t) = \sin(\lambda_n x) \cos(\lambda_n a t),$$

where $\lambda_n = (2n - 1)\pi/(2L)$, $n = 1, 2, \ldots$. Compare this problem with Problem 15 of Section 10.6; pay particular attention to the extension of the initial data out of the original interval $[0, L]$.

10. Consider an elastic string of length L. The end $x = 0$ is held fixed, while the end $x = L$ is free; thus the boundary conditions are $u(0, t) = 0$ and $u_x(L, t) = 0$. The string is set in motion with no initial velocity from the initial position $u(x, 0) = f(x)$, where

$$f(x) = \begin{cases} 0 & 0 \leq x \leq L/2 - 1 \\ 1, & L/2 - 1 < x < L/2 + 1 \quad (\text{assume } L > 2), \\ 0, & L/2 + 1 \leq x \leq 1 \end{cases}$$

a. Find the displacement $u(x, t)$.

G **b.** With $L = 10$ and $a = 1$, plot u versus x for $0 \leq x \leq 10$ and for several values of t. Pay particular attention to values of t between 3 and 7. Observe how the initial disturbance is reflected at each end of the string.

G **c.** With $L = 10$ and $a = 1$, plot u versus t for several values of x.

G **d.** Construct an animation of the solution in time for at least one period.

e. Describe the motion of the string in a few sentences.

G **11.** Suppose that the string in Problem 10 is started instead from the initial position $f(x) = 8x(L - x)^2/L^3$. Follow the instructions in Problem 10 for this new problem.

12. Dimensionless variables can be introduced into the wave equation $a^2 u_{xx} = u_{tt}$ in the following manner:

a. Let $s = x/L$ and show that the wave equation becomes

$$a^2 u_{ss} = L^2 u_{tt}.$$

b. Show that L/a has the dimensions of time and therefore can be used as the unit on the time scale. Let $\tau = at/L$ and show that the wave equation then reduces to

$$u_{ss} = u_{\tau\tau}.$$

Problems 13 and 14 indicate the form of the general solution of the wave equation and the physical significance of the constant a.

13. **a.** Show that the wave equation

$$a^2 u_{xx} = u_{tt}$$

can be reduced to the form $u_{\xi\eta} = 0$ by the change of variables $\xi = x - at$, $\eta = x + at$.

b. Show that $u(x, t)$ can be written as

$$u(x, t) = \phi(x - at) + \psi(x + at),$$

where ϕ and ψ are arbitrary functions.

14. G **a.** Plot the value of $\phi(x - at)$ for $t = 0$, $1/a$, $2/a$, and t_0/a if $\phi(s) = \sin s$. Note that for any $t \neq 0$, the graph of $y = \phi(x - at)$ is the same as that of $y = \phi(x)$ when $t = 0$, but displaced a distance at in the positive x direction. Thus a represents the velocity at which a disturbance moves along the string.

b. What is the interpretation of $\phi(x + at)$?

15. A steel wire 5 ft in length is stretched by a tensile force of 50 lb. The wire has a weight per unit length of 0.026 lb/ft.

a. Find the velocity of propagation of transverse waves in the wire.

b. Find the natural frequencies of vibration.

c. If the tension in the wire is increased, how are the natural frequencies changed? Are the natural modes also changed?