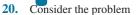
In each of Problems 1 through 8, find the steady-state solution of the heat conduction equation $\alpha^2 u_{xx} = u_t$ that satisfies the given set of boundary conditions.

- **1.** u(0, t) = 10, u(50, t) = 40
- **2.** u(0, t) = 30, u(40, t) = -20
- 3. $u_x(0,t) = 0$, u(L,t) = 0
- 4. $u_x(0,t) = 0, \ u(L,t) = T$
- 5. $u(0,t) = 0, \ u_x(L,t) = 0$
- **6.** u(0,t) = T, $u_x(L,t) = 0$
- 7. $u_x(0,t) u(0,t) = 0, \ u(L,t) = T$
- 8. u(0,t) = T, $u_x(L,t) + u(L,t) = 0$

9. Let an aluminum rod of length 20 cm be initially at the uniform temperature of 25°C. Suppose that at time t = 0, the end x = 0 is cooled to 0°C while the end x = 20 is heated to 60°C, and both are thereafter maintained at those temperatures.



a. Let u(x, t) = X(x)T(t), and show that

 $X'' + \lambda X = 0$, X(0) = 0, $X'(L) + \gamma X(L) = 0$, (48) and

$$T' + \lambda \alpha^2 T = 0,$$

where λ is the separation constant.

b. Assume that λ is real, and show that problem (48) has no nontrivial solutions if $\lambda \leq 0$.

c. If $\lambda > 0$, let $\lambda = \mu^2$ with $\mu > 0$. Show that problem (48) has nontrivial solutions only if μ is a solution of the equation

$$\mu \cos(\mu L) + \gamma \sin(\mu L) = 0. \tag{49}$$

d. Rewrite equation (49) as $\tan(\mu L) = -\mu/\gamma$. Then, by drawing the graphs of $y = \tan(\mu L)$ and $y = -\mu/\gamma$ for $\mu > 0$ on the same set of axes, show that equation (49) is satisfied by infinitely many positive values of μ ; denote these by $\mu_1, \mu_2, \ldots, \mu_n, \ldots$, ordered in increasing size.

e. Determine the set of fundamental solutions $u_n(x, t)$ corresponding to the values μ_n found in part d.

15. Consider a uniform bar of length *L* having an initial temperature distribution given by f(x), $0 \le x \le L$. Assume that the temperature at the end x = 0 is held at 0°C, while the end x = L is insulated so that no heat passes through it.

a. Show that the fundamental solutions of the partial differential emation and boundary conditions are

•
$$u_n(x,t) = e^{-(2n-1)^2 \pi^2 \alpha^2 t/4L^2} \sin\left(\frac{(2n-1)\pi x}{2L}\right),$$

 $n = 1, 2, 3, \dots$

b. Find a formal series expansion for the temperature u(x, t)

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t)$$

that also satisfies the initial condition u(x, 0) = f(x).

Thint: Even though the fundamental solutions involve only the odd sines, it is still possible to represent f by a Fourier series involving only these functions. See Problem 39 of Section 10.4.

An External Heat Source. Consider the heat conduction problem in a bar hat is in thermal contact with an external heat source or sink. Then the modified heat conduction equation is

$$u_t = \alpha^2 u_{xx} + s(x), \tag{50}$$

where the term s(x) describes the effect of the external agency; s(x) is positive for a source and negative for a sink. Suppose that the boundary conditions are

$$u(0,t) = T_1, \quad u(L,t) = T_2$$
 (51)

and the initial condition is

$$u(x,0) = f(x).$$
 (52)

Problems 21 through 23 deal with this kind of problem.

21. Write u(x,t) = v(x) + w(x,t), where v and w are the steady-state and transient parts of the solution, respectively. State the boundary value problems that v(x) and w(x, t), respectively, satisfy. Observe that the problem for w is the fundamental heat conduction problem discussed in Section 10.5, with a modified initial temperature distribution.

