D appendix

Wave Equations

The propagation of electromagnetic waves is governed by the following *Maxwell's equations:*

$$\nabla \cdot \mathbf{D} = \rho \tag{D.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{D.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{D.3}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{D.4}$$

Here, ρ is the charge density, and $\mathbf{J}$ is the current density. We assume that there are no free charges in the medium so that $\rho = 0$. For such a medium, $\mathbf{J} = \sigma\mathbf{E}$, where σ is the conductivity of the medium. Since the conductivity of silica is extremely low ($\sigma \approx 0$), we assume that $\mathbf{J} = 0$; this amounts to assuming a lossless medium.

In any medium, we also have, from (2.5) and (2.6),

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P},$$

where $\mathbf{P}$ is the electric polarization of the medium and

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}),$$

where $\mathbf{M}$ is the magnetic polarization of the medium. Since silica is a nonmagnetic material, we set $\mathbf{M} = 0$.

Using these relations, we can eliminate the flux densities from Maxwell's curl equations (D.3) and (D.4) and write them only in terms of the field vectors **E** and **H**, and the electric polarization **P**. For example,

$$\nabla \times \nabla \times \mathbf{E} = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}. \tag{D.5}$$

To solve this equation for **E**, we have to relate **P** to **E**. If we neglect nonlinear effects, we can assume the linear relation between **P** and **E** given by (2.7) and further, because of the homogeneity assumption, we can write $\chi(t)$ for $\chi(\mathbf{r}, t)$. We relax this assumption when we discuss nonlinear effects in Section 2.5.

We can solve (D.5) for **E** most conveniently by using Fourier transforms. The Fourier transform $\tilde{\mathbf{E}}$ of **E** is defined by (2.4); $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{H}}$ are defined similarly. It follows from the properties of Fourier transforms that

$$\mathrm{E}(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\mathbf{E}}(\mathbf{r}, \omega) \exp(-i\omega t)\, d\omega.$$

By differentiating this equation with respect to t, we obtain the Fourier transform of $\partial \mathbf{E}/\partial t$ as $-i\omega\tilde{\mathbf{E}}$.

Taking the Fourier transform of (D.5), we get

$$\nabla \times \nabla \times \tilde{\mathbf{E}} = \mu_0 \epsilon_0 \omega^2 \tilde{\mathbf{E}} + \mu_0 \omega^2 \tilde{\mathbf{P}}.$$

Using (2.8) to express $\tilde{\mathbf{P}}$ in terms of $\tilde{\mathbf{E}}$, this reduces to

$$\nabla \times \nabla \times \tilde{\mathbf{E}} = \mu_0 \epsilon_0 \omega^2 \tilde{\mathbf{E}} + \mu_0 \epsilon_0 \omega^2 \tilde{\chi} \tilde{\mathbf{E}}.$$

We denote $c = 1/\sqrt{\mu_0 \epsilon_0}$; c is the speed of light in a vacuum. When losses are neglected, as we have neglected them, $\tilde{\chi}$ is real, and we can write $n(\omega) = \sqrt{1 + \tilde{\chi}(\omega)}$, where n is the refractive index. Note that this is the same as (2.9), which we used as the definition for the refractive index. With this notation,

$$\nabla \times \nabla \times \tilde{\mathbf{E}} = \frac{\omega^2 n^2}{c^2} \tilde{\mathbf{E}}. \tag{D.6}$$

By using the identity,

$$\nabla \times \nabla \times \tilde{\mathbf{E}} = \nabla(\nabla \cdot \tilde{\mathbf{E}}) - \nabla^2 \tilde{\mathbf{E}},$$

(D.6) can be rewritten as

$$\nabla^2 \tilde{\mathbf{E}} + \frac{\omega^2 n^2}{c^2} \tilde{\mathbf{E}} = \nabla(\nabla \cdot \tilde{\mathbf{E}}). \tag{D.7}$$

Because of our assumption of a homogeneous medium (χ independent of $\mathbf{r}$) and using (D.1) and (2.9), we get

$$0 = \nabla \cdot \tilde{\mathbf{D}} = \epsilon_0 \nabla \cdot (1 + \tilde{\chi})\tilde{\mathbf{E}} = \epsilon_0 n^2 \nabla \cdot \tilde{\mathbf{E}}. \tag{D.8}$$

This enables us to simplify (D.7) and obtain the wave equation (2.10) for $\tilde{\mathbf{E}}$. Following similar steps, the wave equation (2.11) can be derived for $\tilde{\mathbf{H}}$.

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