PMR3523 Controle Moderno

Apoio à Aula

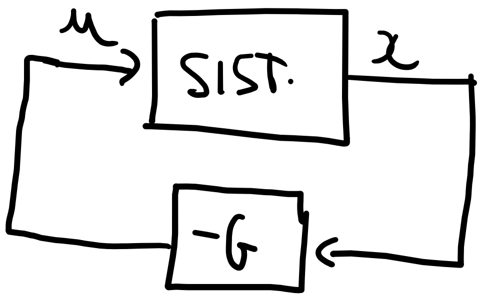
Projeto de controladores por realimentação dos estados
(alocação de polos)

ALOCAÇÃO DE POLOS

$$\dot{x} = Ax + Bu$$

CASO 1) REALIMENTAÇÃO DE ESTADOS → POSICIONAR POLOS ONDE DESEJAR

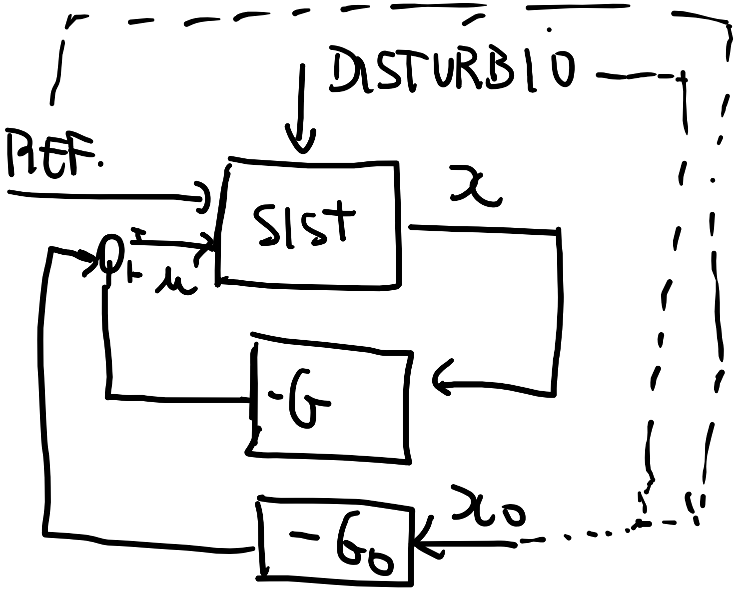
(*) $u = -G.x$ ⇒ MOLDA A DINÂMICA DA PLANTA (SEM DISTÚRBIOS E REFERÊNCIA)



CASO 2) REALIM. ESTADOS C/ DIST. E/OU REF.

$$u = -Gx - G_0 x_0$$

↳ VARIAVEIS EXTERNAS DISTURBIOS OU REFERENCIA



CASO 3) REALIMENTAÇÃO DA SAÍDA

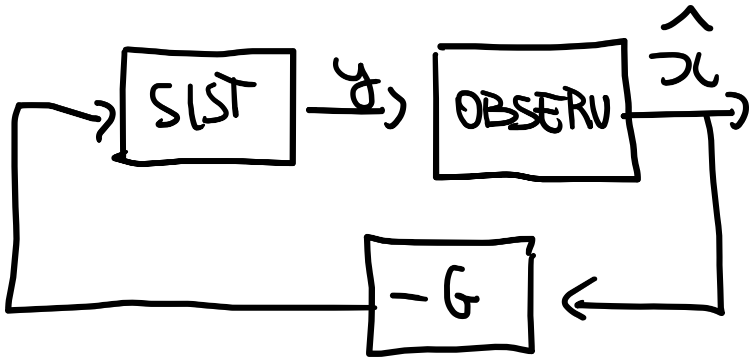
$\dot{x} = Ax + Bu$
$y = Cx + Du$ $\Rightarrow$ $u = -G_y \cdot y$

↳ NEM SEMPRE GARANTE A DINÂMICA DESEJADA

↳ PROJETAMOS UM OBSERVADOR ANTES

$$u = -G \cdot \hat{x}$$

↳ ESTADOS OBSERVADOS / ESTIMADOS



PROJETO DO CONTROLE (G) P/ SISO

$$\dot{x} = A x + B u$$

$\dot{x} \to K\times1$, $A \to K\times K$, $x \to K\times1$, $B \to K\times1$, $u \to 1\times1$

$$B = b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_K \end{bmatrix} \qquad G = g' = [g_1 \; g_2 \; \dots \; g_K] \quad (1\times K)$$

$$u = -G \cdot x$$

$G \to 1\times K$, $x \to K\times1$

FECHANDO A MALHA →

$$\dot{x} = Ax + B.(-G.x)$$

$$\boxed{\dot{x} = (A - b.g').x}$$

→ DINÂMICA EM MALHA FECHADA

OBJETIVO: ENCONTRAR $G = g'$ QUE POSICIONE POLOS DE $A_c = A - bg'$ NA POSIÇÃO DESEJADA

- **Key difference** in this case: since all poles are being placed, the assumption of dominant 2nd order behavior is pretty much guaranteed to be valid.

A_c = MATRIZ DA DINAMICA MALHA FECHADA

↳ COMO TEMOS K POLOS E K GANHOS $(g_1, g_2 \ldots g_K)$, É POSSIVEL, SE O SISTEMA FOR CONTROLÁVEL, POSICIONAR TODOS OS POLOS!

$$|sI - A_c| = |sI - A + b \cdot g'| = s^K + \hat{a}_1 s^{K-1} + \hat{a}_2 s^{K-2} + \ldots + \hat{a}_K$$

$\hat{a}_i$ = COEF. DO POLINÔMIO CARACTERÍSTICO QUE POSICIONA POLOS s_i NAS POSIÇÕES DESEJADAS.

EX) $P_{DESEJADOS} = [-1+2j, -1-2j, -5-20j, -5+20j]$

↳ $\hat{a}_i = ?$

$$(s+1-2j)(s+1+2j)(s+5+20j)(s+5-20j) = \ldots$$

$$s^4 + \underbrace{12}_{\hat{a}_1} s^3 + \underbrace{450}_{\hat{a}_2} s^2 + \underbrace{900}_{\hat{a}_3} s + \underbrace{2125}_{\hat{a}_4}$$

```
pdes = [-1+2*j, -1-2*j, -5-20*j,-5+20*j]
Gdes = zpk([],pdes,1)
[numdes,dendes]=tfdata(Gdes);
dendes{1}
```

Alocação de polos – forma canônica de controlabilidade

A. Abate, M. Forgione

Controllable Canonical Form from polynomials of Transfer Function

Consider the following transfer function:

$$\frac{Y(s)}{U(s)} = \frac{(s+10)(s^2+s+25)}{s^2(s+3)(s^2+s+36)}.$$

Give a state-space description of the system in its canonical controllable form.

Solution:
Let us recall some theory to begin with.
Consider a transfer function as a ratio between two polynomials, as follows:

$$\frac{Y(s)}{U(s)} = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n}.$$

There exists a state-space representation of this transfer function (a.k.a., a *realization*), which takes the form of the controllable

A. Abate, M. Forgione

canonical form, where:

$$A = \begin{bmatrix} -a_1 & & -a_3 & \ldots & -a_n \\ 1 & & 0 & \ldots & 0 \\ \vdots & & & & \\ 0 & & & & \vdots \\ 0 & \ldots & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} b_1 & b_2 & \ldots & \ldots & b_n \end{bmatrix}, \qquad D = [0].$$

In MATLAB, this realization is obtained with the command `tf2ss` . Notice that in general it is not always possible to obtain such a realization. (There are sufficient conditions on the rational functions for that, but we do not discuss them right now.) Also, notice that this realization is not unique. In particular, we may be interested to obtain the observable canonical form.

In our instance, we have:

$$\frac{Y(s)}{U(s)} = \frac{(s+10)(s^2+s+25)}{s^2(s+3)(s^2+s+36)}$$
$$= \frac{s^3+11s^2+35s+250}{s^5+4s^4+39s^3+108s^2}$$
$$= \frac{b(s)}{a(s)}.$$

From this expression, we have that $b_1 = 0, b_2 = 1, b_3 = 11, b_4 = 35, b_5 = 250; a_1 = 4, a_2 = 39, a_3 = 108, a_4 =$

Alocação de polos – forma canônica de controlabilidade

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$0, a_5 = 0$. The controllable canonical form is:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -4 & -39 & -108 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 11 & 35 & 250 \end{bmatrix} x + [0]u.$$

As a side note, let us remark that the controllable canonical form is not unique. This should be clear if, within the block diagram representation of a model, we re-order the state variable in the opposite sense. The corresponding canonical form would be:

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & & 0 & 0 & 0 \\ \vdots & & & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -a_n & \dots & -a_3 & -a_2 & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} b_n & b_{n-1} & \dots & b_2 & b_1 \end{bmatrix}, \qquad D = [0].$$

Both canonical forms display analogous features, and are equivalent in terms of their controllability properties. A similar thing happens for the case of the observable canonical form. □

– Ac.Yr. 2012/13, 1e Sem. Q1 – Exercise Session 7 – sc4026 10

Alocação de polos - introdução

Sistema na forma companheira de controlabilidade

$$\dot{x} = Ax + Bu$$

$$u = -Gx$$

$$H(s) = \frac{y(s)}{u(s)} = \frac{1}{s^k + a_1 s^{k-1} + \cdots + a_k}$$

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{k-1} & -a_k \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$C = [0 \quad 0 \quad \cdots \quad 0 \quad 1]$$

$$bg' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} [g_1, g_2, \ldots, g_k] = \begin{bmatrix} g_1 & g_2 & \cdots & g_k \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$A_c = A - bg' = \begin{bmatrix} -a_1 - g_1 & -a_2 - g_2 & \cdots & -a_k - g_k \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Alocação de polos - introdução

A

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{k-1} & -a_k \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$C = [0 \quad 0 \quad \cdots \quad 0 \quad 1]$$

SE A ESTÁ NA FORMA CANÔNICA DE CONTROLABILIDADE

- COMO PROJETAR $G = g' = [g_1, g_2 \ldots g_k]$?

Ac

$$A_c = A - bg' = \begin{bmatrix} -a_1 - g_1 & -a_2 - g_2 & \cdots & -a_k - g_k \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$-a_i - g_i = -\hat{a}_i$$

- a_i → COEF. POL. CARACT. SIST. ORIGINAL
- g_i → GANHOS CONTROLE
- $\hat{a}_i$ → COEF. POL. CARAC. DESEJADO

$$\boxed{g_i = \hat{a}_i - a_i}$$

Alocação de polos – posicionamento de polos
Onde colocar? Alternativa 1

- **Question:** where should we put the closed-loop poles?
- **Approach #1:** use time-domain specifications to locate dominant poles – roots of:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

  - Then place rest of the poles so they are "much faster" than the dominant 2nd order behavior.
  - Example: could keep the same damped frequency w_d and then move the real part to be 2–3 times faster than the real part of dominant poles $\zeta\omega_n$
    - Just be careful moving the poles too far to the left because it takes a lot of control effort

- Recall ROT for 2nd order response (4–??):

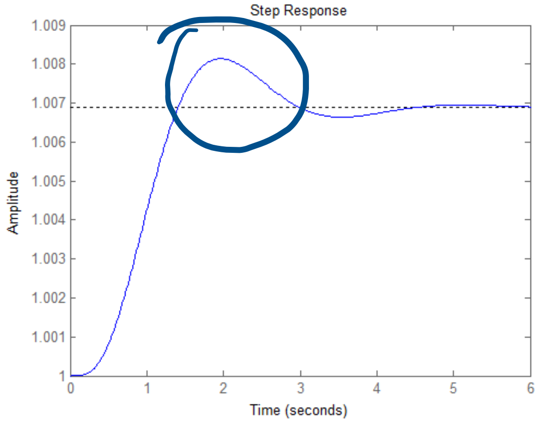
10-90% rise time	$t_r = \frac{1 + 1.1\zeta + 1.4\zeta^2}{\omega_n}$
Settling time (5%)	$t_s = \frac{3}{\zeta\omega_n}$
Time to peak amplitude	$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$
Peak overshoot	$M_p = e^{-\zeta\omega_n t_p}$

- **Key difference** in this case: since all poles are being placed, the assumption of dominant 2nd order behavior is pretty much guaranteed to be valid.

https://ocw.mit.edu/courses/aeronautics-and-astronautics/16-30-feedback-control-systems-fall-2010/lecture-notes/MIT16_30F10_lec12.pdf

Alocação de polos – Exemplo 1

```
>> tf(G)

ans =

                    145
  ------------------------------------
  s^4 + 12 s^3 + 54 s^2 + 108 s + 145
```




```
>> damp(G)

        Eigenvalue              Damping      Frequency

 -1.00e+00 + 2.00e+00i         4.47e-01      2.24e+00
 -1.00e+00 - 2.00e+00i         4.47e-01      2.24e+00
 -5.00e+00 + 2.00e+00i         9.28e-01      5.39e+00
 -5.00e+00 - 2.00e+00i         9.28e-01      5.39e+00

(Frequencies expressed in rad/seconds)
```

Requisito – ter uma resposta com tempo de assentamento 1s e zeta = 0.7

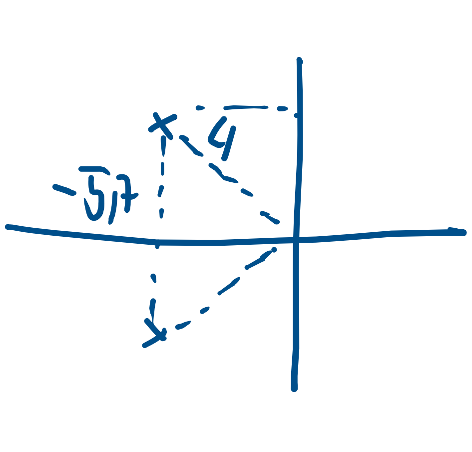
$$\Rightarrow \ \omega_n = 5{,}7 rad/s$$

Polos $4(-1 \pm j)$

Polos não dominantes, $20(-1 \pm j)$

$t_s = \frac{4}{\zeta \omega_n} \Rightarrow 1 = \frac{4}{0{,}7 \cdot \omega_n}$

$\Rightarrow \omega_n = 5{,}7 \, rad/s$



Alocação de polos – Exemplo 1

Passo 1 – obter a forma companheira de controlabilidade do sistema

```
[num,den]=tfdata(G);
[A,B,C,D] = tf2ss(num{1},den{1})
```

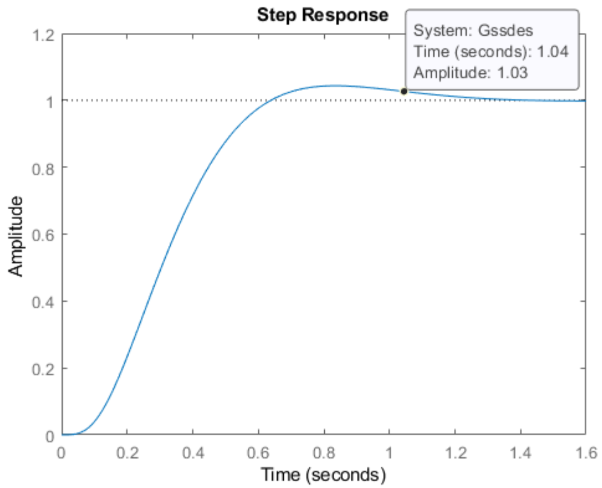
$-a_1 \quad -a_2 \quad -a_3 \quad -a_4$; a_i

```
A =

   -12    -54   -108   -145
     1      0      0      0
     0      1      0      0
     0      0      1      0


B =

     1
     0
     0
     0


C =

     0      0      0    145


D =

     0
```

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{k-1} & -a_k \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$C = [0 \quad 0 \quad \cdots \quad 0 \quad 1]$$

Alocação de polos – Exemplo 1



Passo 2 – obter a matriz A desejada na mesma forma

Requisito – ter uma resposta com tempo de assentamento 1s e zeta = 0.7

$$\Rightarrow \ \omega_n = 5{,}7 rad/s$$

Polos requeridos $4(-1 \pm j)$

Polos não dominantes, $20(-1 \pm j)$

```
pdes = [-4+4*j, -4-4*j, 20*(-1-1*j), 20*(-1+1*j)]
Gdes = zpk([],pdes,1)
[numdes,dendes]=tfdata(Gdes);

[Ades,Bdes,Cdes,Ddes] = tf2ss(numdes{1},dendes{1})
Gssdes=ss(Ades,Bdes,Cdes,Ddes)
step(Gssdes)
```

```
Gssdes =

  A =
              x1          x2          x3          x4
   x1        -48       -1152       -7680   -2.56e+04
   x2          1           0           0           0
   x3          0           1           0           0
   x4          0           0           1           0

  B =
         u1
   x1     1
   x2     0
   x3     0
   x4     0

  C =
             x1         x2         x3         x4
   y1         0          0          0   2.56e+04

  D =
         u1
   y1     0
```

Alocação de polos – Exemplo 1

Passo 3 – obter a matriz G desejada de realimentação

Lembrar que:

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{k-1} & -a_k \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$g_i = \hat{a}_i - a_i$$

a_i: Coeficiente do polinômio característico atual

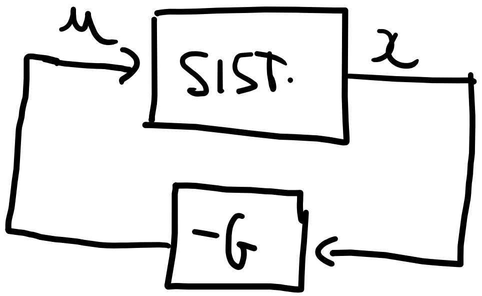
$\hat{a}_i$: Coeficiente do polinômio característico desejado

```
g = (-Ades(1,:) + A(1,:))'
Aclosed = A - B*g';
damp(Aclosed)

sys_closed = ss(Aclosed,zeros(4,1),eye(4),0);
t = [0:0.01:3];
u = zeros(length(t),1);
lsim(sys_closed,u,t,[0 0 0 1])
```

Pole	Damping	Frequency (rad/TimeUnit)	Time Constant (TimeUnit)
-2.00e+01 + 2.00e+01i	7.07e-01	2.83e+01	5.00e-02
-2.00e+01 - 2.00e+01i	7.07e-01	2.83e+01	5.00e-02
-4.00e+00 + 4.00e+00i	7.07e-01	5.66e+00	2.50e-01
-4.00e+00 - 4.00e+00i	7.07e-01	5.66e+00	2.50e-01

Alocação de polos – Exemplo 1

Passo 4 – simulação de verificação



SIST) $\dot{x} = Ax + Bu$

SIST. CONT) $\dot{x} = Ax + B.(-g'.x)$

$$\dot{x} = (A - Bg')\cdot x$$

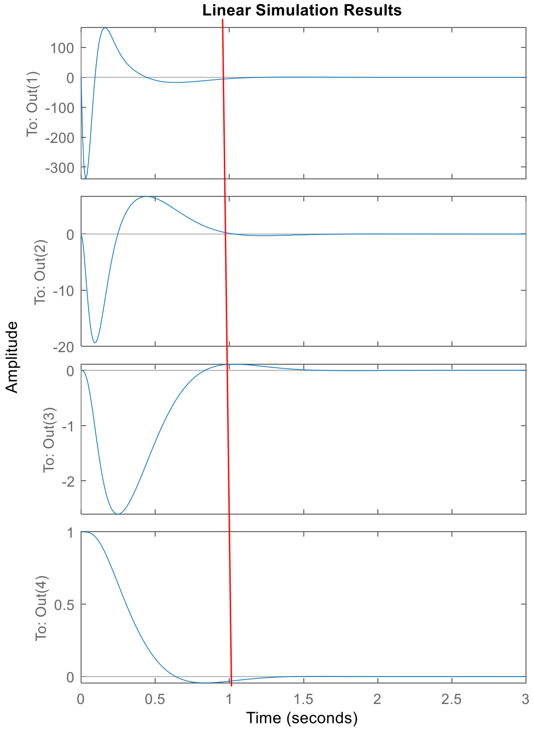
$A_c = A - Bg'$

$B_c = 0$

$C_c = I \Rightarrow$ MEDINDO TODOS ESTADOS

$D_c = 0$

```
sys_closed = ss(Aclosed,zeros(4,1),eye(4),0);
t = [0:0.01:3];
u = zeros(length(t),1);
lsim(sys_closed,u,t,[0 0 0 1])
```



CASO GERAL

$\dot{x} = Ax + Bu$ → A não está na forma canônica controlabilidade

↳ OBTEMOS A TRANSFORMAÇÃO DE ESTADOS QUE LEVA O SISTEMA À FORMA CANÔNICA DE CONTROLAB.

$\bar{x} = T.x \Rightarrow x = T^{-1}.\bar{x} \Rightarrow \dot{\bar{x}} = \bar{A}\bar{x} + \bar{b}u$

- $\bar{x}$ ↳ ESTADOS P/ FORMA C.C.
- T ↳ MATRIZ TRANSF. ESTADOS
- x ↳ ESTADOS ORIGINAIS

COM: $\bar{A} = T.A.T^{-1}$ $\bar{A}, \bar{b}$ NA FORMA C.C.

$\bar{b} = T.b$

$\bar{g} = \hat{a} - \bar{a}$ → POL. CARAC. DO SIST NÃO CONTROLADO

- $\bar{g}$ ↳ MATRIZ REAL. ESTADOS
- $\hat{a}$ ↳ POL. CARACT. DESEJADO

$= \hat{a} - a$ POIS $a = \bar{a}$

CASO GERAL

$\bar{g} = \hat{a} - \bar{a}$ → POL. CARAC. DO SIST. NÃO CONTROLADO $= \hat{a} - a$

$\bar{g}$ ↳ MATRIZ REAL. ESTADOS

$\hat{a}$ ↳ POL. CARACT. DESEJADO

POIS $a = \bar{a}$

(EQ. CARACTERÍSTICA É INVARIANTE COM MUDANÇA DE ESTADOS)

$$\boxed{\bar{g} = \hat{a} - a}$$

MAS)

$$u = -g' \cdot x = -g' \cdot T^{-1} \cdot \bar{x} = -\bar{g}' \bar{x}$$

$$\bar{g}' = g' \cdot T^{-1} \Rightarrow \bar{g} = (T^{-1})' \cdot g \Rightarrow g = T' \cdot \bar{g}$$

$$\boxed{g = T' \cdot (\hat{a} - a)}$$

CASO GERAL

$$g = T'.(\hat{a} - a)$$

MATRIZ GANHO g É A DIF. ENTRE OS COEF. DO POL. CARACT. DESEJADO E ATUAL, PRÉ MULTIPLICADO PELA MATRIZ TRANSF. T (QUE LEVA À FORMA C.C.)

↳ COMO OBTER T? LIVRO TEXTO

$$T = (Q.W)^{-1}$$

Q = MATRIZ CONTROLABILIDADE = $[b \;\; Ab \;\; A^2b \; \ldots \; A^{k-1}b]$

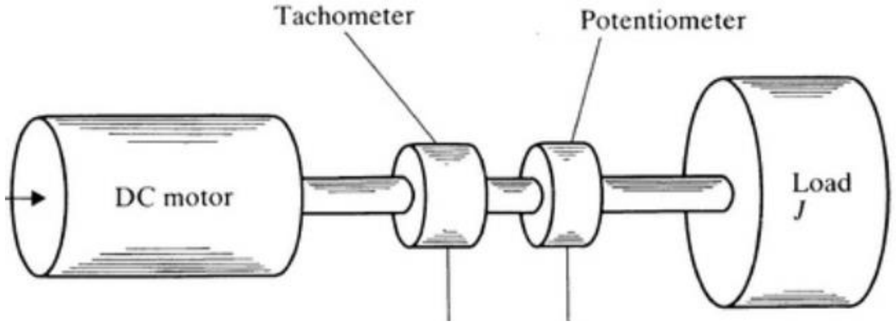
$$W = \begin{bmatrix} 1 & a_1 & \cdots & a_{k-1} \\ 0 & 1 & a_1 \cdots & a_{k-2} \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

TOEPLITZ DIAGONAL

Alocação de polos – Exemplo 2

$$\dot{\theta} = \omega$$
$$\dot{\omega} = -\alpha\omega + \beta u$$

$\alpha = -K^2/JR$ $\beta = K/JR$



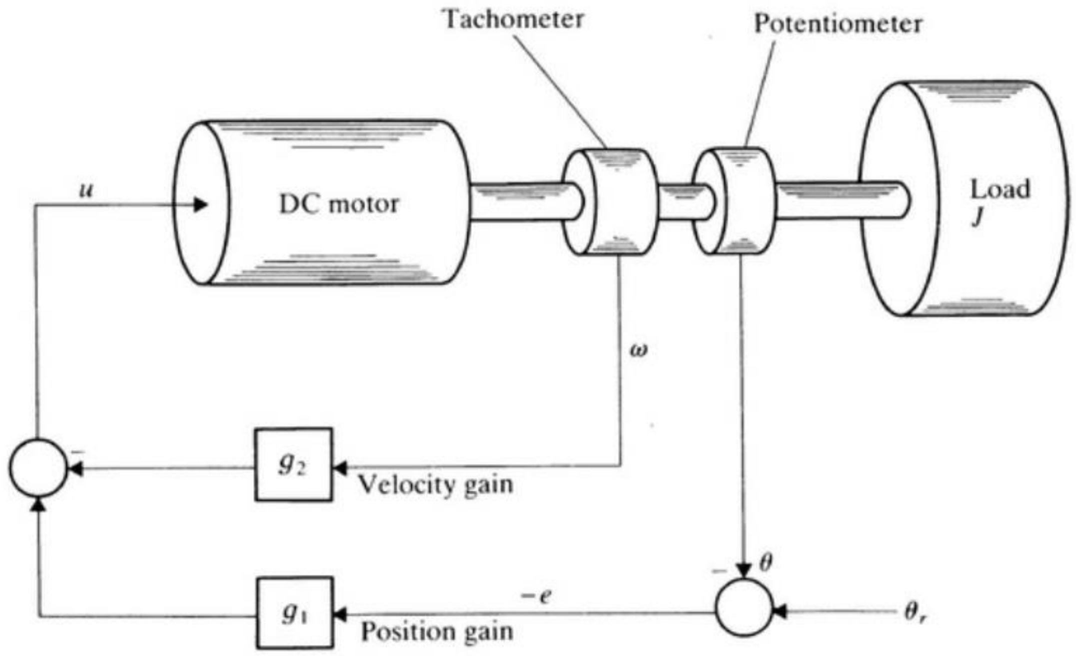
- θ e ω posição e velocidade angular
- α e β dependem dos parâmetros físicos do motor e carga

$$e = \theta - \theta_r$$

$$\dot{e} = \dot{\theta} - \dot{\theta}_r = \omega \qquad (\theta_r = \text{const})$$

$$\begin{bmatrix} \dot{e} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix} \begin{bmatrix} e \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u$$

Alocação de polos – exemplo 2



$$u = -g_2\omega - g_1 e$$

$$\dot{e} = \omega$$

$$\dot{\omega} = -g_1\beta e - (\alpha + \beta g_2)\omega$$

Polinômio característico do sistema

$$|sI - A| = \begin{vmatrix} s & -1 \\ 0 & s+\alpha \end{vmatrix} = s^2 + \alpha s$$

$$a = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$$

Matrizes Q e W

$$Q = [b, Ab] = \begin{bmatrix} 0 & \beta \\ \beta & -\alpha\beta \end{bmatrix} \qquad W = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$$

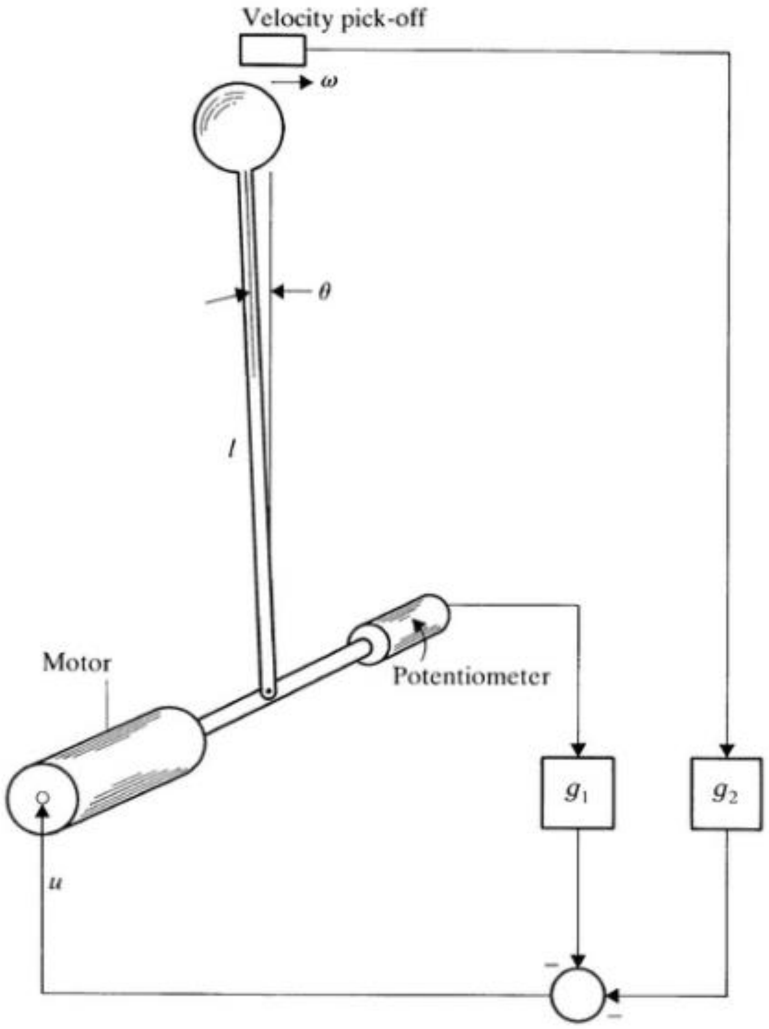
$$[(QW)']^{-1} = \begin{bmatrix} 0 & 1/\beta \\ 1/\beta & 0 \end{bmatrix}$$

Controlador

$$g = [(QW)']^{-1}(\hat{a} - a)$$

$$g = \begin{bmatrix} 0 & 1/\beta \\ 1/\beta & 0 \end{bmatrix} \begin{bmatrix} \bar{a}_1 - \alpha \\ \bar{a}_2 \end{bmatrix} = \begin{bmatrix} \bar{a}_2/\beta \\ (\bar{a}_1 - \alpha)/\beta \end{bmatrix}$$

Alocação de polos – exemplo 3



$$\dot{\theta} = \omega$$

$$\dot{\omega} = \Omega^2\theta - \alpha\omega + \beta u$$

$\alpha = -K^2/JR$ $\beta = K/JR$ $\quad J = J_m + ml^2 \quad \Omega^2 = \dfrac{mgl}{J + ml^2} = \dfrac{g}{l + J/ml}$

$$A = \begin{bmatrix} 0 & 1 \\ \Omega^2 & -\alpha \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ \beta \end{bmatrix}$$

$$|sI - A| = \begin{vmatrix} s & -1 \\ -\Omega^2 & s + \alpha \end{vmatrix} = s^2 + \alpha s - \Omega^2 \Rightarrow \begin{matrix} a_1 = \alpha \\ a_2 = -\Omega^2 \end{matrix}$$

$$Q = \begin{bmatrix} 0 & \beta \\ \beta & -\alpha\beta \end{bmatrix} \quad W = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \Rightarrow [(QW)']^{-1} = \begin{bmatrix} 0 & 1/\beta \\ 1/\beta & 0 \end{bmatrix}$$

Alocação de polos – exemplo 3

$$\dot{\theta} = \omega$$

$$\dot{\omega} = \Omega^2\theta - \alpha\omega + \beta u$$

$$Q = \begin{bmatrix} 0 & \beta \\ \beta & -\alpha\beta \end{bmatrix} \qquad W = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \quad \Rightarrow \quad [(QW)']^{-1} = \begin{bmatrix} 0 & 1/\beta \\ 1/\beta & 0 \end{bmatrix}$$

Controlador

$$g = [(QW)']^{-1}(\hat{a} - a)$$

$$g = \begin{bmatrix} 0 & 1/\beta \\ 1/\beta & 0 \end{bmatrix} \begin{bmatrix} (\bar{a}_1 - \alpha) \\ \bar{a}_2 + \Omega^2 \end{bmatrix} = \begin{bmatrix} (\bar{a}_2 + \Omega^2)/\beta \\ (\bar{a}_1 - \alpha)/\beta \end{bmatrix}$$

Alocação de polos – exemplo 4

$s^4 - 5s^2$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} u,$$

If legitimate, design a state-feedback controller so that the closed-loop dynamics have eigenvalues in $-1, -2, -1 \pm j$.

$$T = (Q \cdot W)^{-1}$$

$$= \begin{pmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & 1 & a_1 & a_2 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

```
%% Exemplo 4
A = [0 1 0 0
    0 0 -1 0
    0 0 0 1
    0 0 5 0];
B = [0 ; 1 ; 0 ; -2];

Q = ctrb(A,B);
rank(Q)
% polinomio caracteristico de A
pc = charpoly(A);
W = triu(toeplitz(pc(1:end-1)));
% vetor a com coeficientes a1, a2, a3... do polinomio caracteristico de A
a = pc(2:end)

% polos desejados
p = [-1, -2, -1-j,-1+j]
G = zpk([],p,1);
[num,den]=tfdata(G);
% vetor a_til com coeficientes a1, a2, a3... do pol. caract. de Adesejado
a_til =den{1}(2:end);

% matriz de realimentação de ganhos
g = inv((Q*W))' * (a_til-a)'
```

$$g = T' \cdot (\hat{a} - a)$$

```
% verificação
damp(A-B*g');

sys_closed = ss(A - B*g',zeros(4,1),eye(4),0);
t = [0:0.01:10];
u = zeros(length(t),1);
lsim(sys_closed,u,t,[0 0 0 1])
```

Alocação de polos – exemplo 4 - Alternativa

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} u,$$

If legitimate, design a state-feedback controller so that the closed-loop dynamics have eigenvalues in $-1, -2, -1 \pm j$.

$$u = [k_1 \; k_2 \; k_3 \; k_4] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Consider the feedback $u = -Kx$, where

$$K = [\; k_1 \quad k_2 \quad k_3 \quad k_4 \;].$$

The characteristic polynomial of

$$A - BK = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_1 & -k_2 & -k_3 - 1 & -k_4 \\ 0 & 0 & 0 & 1 \\ 2k_1 & 2k_2 & 5 + 2k_3 & 2k_4 \end{bmatrix},$$

can be computed as:

$$\det(sI - (A - BK)) =$$
$$s^4 - (2k_4 + k_2)s^3 - (2k_3 + k_1 + 5)s^2 - 3k_2 s - 3k_1$$

Let the desired eigenvalues be $-1, -2, -1 \pm j$, which

– Ac.Yr. 2012/13, 1e Sem. Q1 – Exercise Session 5 – sc4026 7

A. Abate, J. Antonello

corresponds to the following characteristic polynomial:

$$(s+1)(s+2)(s+1+j)(s+1-j) =$$
$$s^4 + 5s^3 + 10s^2 + 10s + 4.$$

By directly equating the coefficients of the two characteristic polynomials, we obtain:

$$k_1 = -4/3, \quad k_2 = -10/3, \quad k_3 = -41/6, \quad k_4 = -5/6.$$

The feedback input can thus be synthesized as

$$u = -[\; 4/3 \quad 10/3 \quad 41/6 \quad 5/6 \;]\, x.$$

Alocação de polos – exemplo 4 - PLACE

COMANDO PLACE RETORNA $K = g'$

↓ POLOS DESEJADOS

```
g = place(A,B,p)

g =

   -1.3333   -3.3333   -8.1667   -4.1667
```

Alocação de polos – Sistema MIMO

$$\dot{x} = Ax + Bu$$

- $u \rightarrow m \times 1$
- $B \rightarrow k \times m$

$$u = -G \cdot x$$

- u: $m \times 1$
- $x \rightarrow k \times 1$
- $G \rightarrow m \times k$

Ex) 3ª ordem, 2 entradas

$$u = -\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

⇒ m linhas e k colunas

⇒ $m*k$ ganhos a ajustar

⇒ k polos a ajustar

⇒ mais incógnitas ($m*k$) do que equações (k)

Alocação de polos – Sistema MIMO

OPÇÕES ⇒ AJUSTAR 1 COLUNA PARA ∅

$$u = -\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

↳ ∅ ⇒ ESTADO x_2 NÃO PRECISARIA SER MEDIDO!

→ USAR COMANDO PLACE

Algorithms

`place` uses the algorithm of [1] which, for multi-input systems, optimizes the choice of eigenvectors for a robust solution.

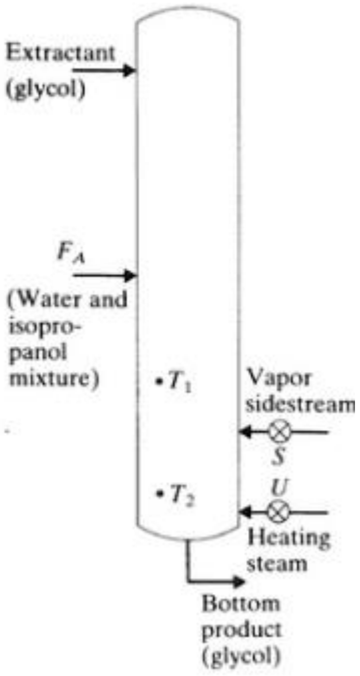
In high-order problems, some choices of pole locations result in very large gains. The sensitivity problems attached with large gains suggest caution in the use of pole placement techniques. See [2] for results from numerical testing.

References

[1] Kautsky, J., N.K. Nichols, and P. Van Dooren, "Robust Pole Assignment in Linear State Feedback," *International Journal of Control*, 41 (1985), pp. 1129-1155.

[2] Laub, A.J. and M. Wette, *Algorithms and Software for Pole Assignment and Observers*, UCRL-15646 Rev. 1, EE Dept., Univ. of Calif., Santa Barbara, CA, Sept. 1984.

Alocação de polos – exemplo MIMO (1)



$$x = \begin{bmatrix} \Delta Q_l \\ \Delta V_l \\ \Delta z_1 \\ \Delta z_2 \end{bmatrix} \quad u = \begin{bmatrix} \Delta u_1 \\ \Delta S \end{bmatrix} \quad x_0 = \begin{bmatrix} \Delta x_{FA1} \\ \Delta F_A \end{bmatrix} \quad y = \begin{bmatrix} \Delta T_1 \\ \Delta T_2 \end{bmatrix}$$

ΔQ_l = heat flow to reboiler "holdup"
ΔV_l = vapor flow rate
Δu_1 = steam flow rate

where ΔS = flow rate of vapor side stream
Δx_{FA1} = feed composition
ΔF_A = feed flow rate

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & a_{32} & 0 & 0 \\ 0 & a_{42} & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & 0 \\ 0 & 0 \\ 0 & b_{32} \\ 0 & b_{42} \end{bmatrix} \quad E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ f_{31} & f_{32} \\ 0 & f_{42} \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & c_{13} & 0 \\ 0 & 0 & 0 & c_{24} \end{bmatrix}$$

Alocação de polos – exemplo MIMO (1)

Δu_1 depends on x_1, x_2, and x_3;

Δu_2 depends on x_4.

ΔQ_l = heat flow to reboiler "holdup"
ΔV_l = vapor flow rate
Δu_1 = steam flow rate

where ΔS = flow rate of vapor side stream
Δx_{FA1} = feed composition
ΔF_A = feed flow rate

$$G = \begin{bmatrix} g_1 & g_2 & g_3 & 0 \\ 0 & 0 & 0 & g_4 \end{bmatrix}$$

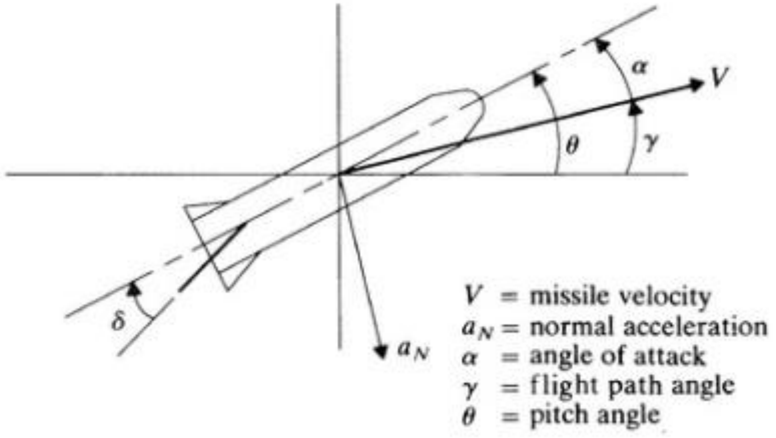
$$A_c = A - BG = \begin{bmatrix} a_{11} - b_{11}g_1 & -b_{11}g_2 & -b_{11}g_3 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & a_{32} & 0 & -b_{32}g_4 \\ 0 & 0 & 0 & -b_{42}g_4 \end{bmatrix}$$

$$|sI - A_c| = \left| \begin{array}{ccc|c} s - a_{11} + b_{11}g_1 & b_{11}g_2 & b_{11}g_3 & 0 \\ -a_{21} & s - a_{22} & 0 & 0 \\ 0 & -a_{32} & s & 0 \\ \hline 0 & 0 & 0 & s + b_{42}g_4 \end{array} \right|$$

$$= (s + b_{42}g_4) \begin{vmatrix} s - a_{11} + b_{11}g_1 & b_{11}g_2 & b_{11}g_3 \\ -a_{21} & s - a_{22} & 0 \\ 0 & -a_{32} & s \end{vmatrix}$$

$$= (s + b_{42}g_4)(s^3 + \bar{a}_1 s^2 + \bar{a}_2 s + \bar{a}_3)$$

Alocação de polos – exemplo MIMO



$$\dot{\alpha} = q + \frac{Z_\alpha}{V}\alpha + \frac{Z_\delta}{V}\delta$$

$$\dot{q} = M_\alpha \alpha + M_\delta \delta \qquad \text{(assuming } M_q \approx 0\text{)}$$

$$\dot{\delta} = \frac{1}{\tau}(u - \delta)$$

$$e = a_{NC} - a_N$$

$$A = \begin{bmatrix} Z_\alpha/V & -Z_\alpha & Z_\delta/\tau \\ -M_\alpha/Z_\alpha & M_q & \bar{M}_\delta \\ 0 & 0 & -1/\tau \end{bmatrix} \quad B = \begin{bmatrix} -Z_\delta/\tau \\ 0 \\ 1/\tau \end{bmatrix} \quad E = \begin{bmatrix} -Z_\alpha/V \\ M_\alpha/Z_\alpha \\ 0 \end{bmatrix}$$

Alocação de polos – exemplo MIMO (2)

- Consider the lateral dynamics of a B747 at cruise (40,000ft, M=0.8)
  - Variables of interest now include lateral velocity (side slip, β), yaw ψ and yaw rate r, roll ϕ and roll rate p.
  - Actuators are aileron δ_a and rudder δ_r (Figure 10.30 from FPE)

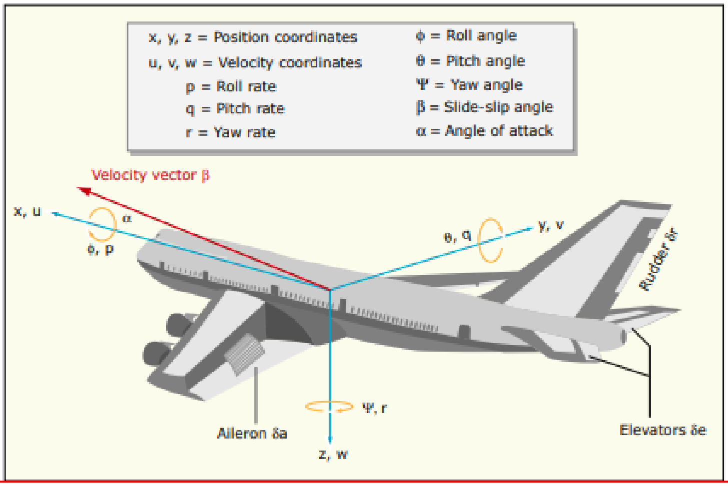


Image by MIT OpenCourseWare.

- Form nonlinear dynamics as before and linearize about forward cruise flight condition to get equations of motion for **lateral dynamics**

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}\ ,\ \mathbf{x} = \begin{bmatrix} \beta \\ r \\ p \\ \phi \end{bmatrix}\ ,\ \mathbf{u} = \begin{bmatrix} \delta_r \\ \delta_a \end{bmatrix} \qquad \dot{\psi} = r$$

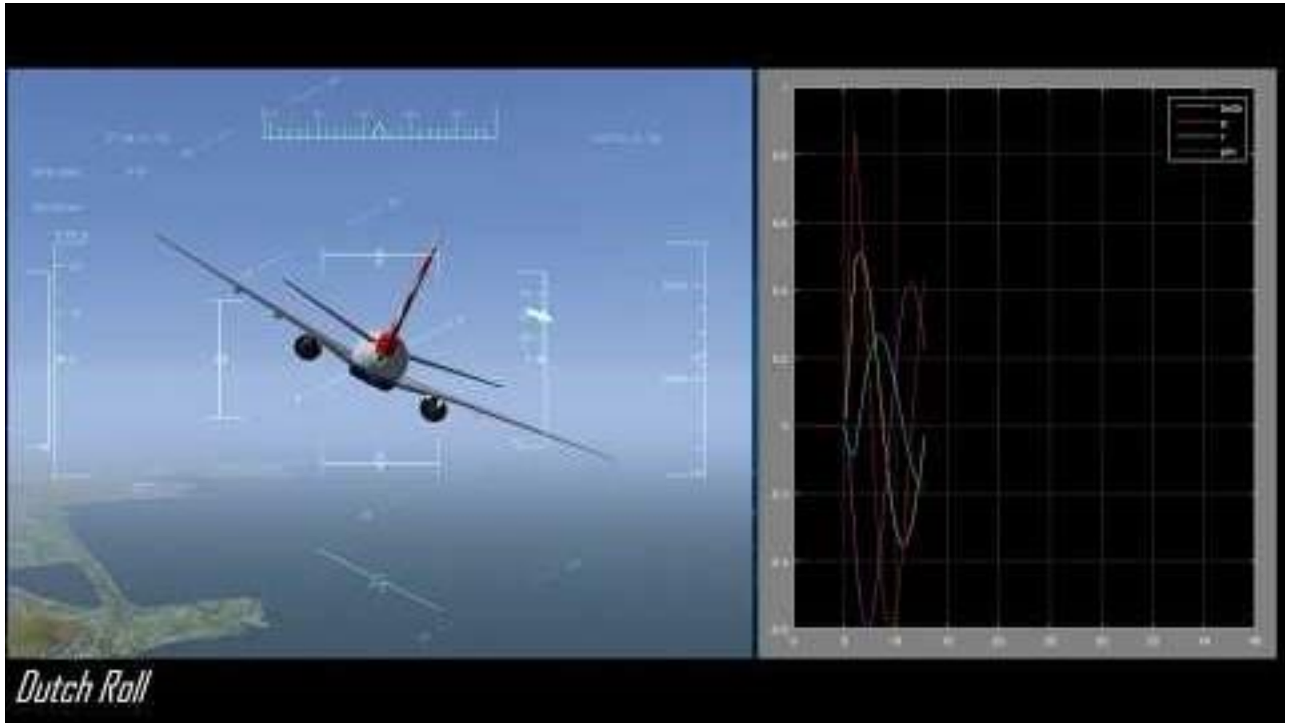
$$A = \begin{bmatrix} -0.0558 & -0.9968 & 0.0802 & 0.0415 \\ 0.598 & -0.115 & -0.0318 & 0 \\ -3.05 & 0.388 & -0.4650 & 0 \\ 0 & 0.0805 & 1 & 0 \end{bmatrix} \qquad (1)$$

$$B = \begin{bmatrix} 0.00729 & 0 \\ -0.475 & 0.00775 \\ 0.153 & 0.143 \\ 0 & 0 \end{bmatrix} \qquad (2)$$

and we can sense the yaw rate r and bank angle ϕ.

https://ocw.mit.edu/courses/aeronautics-and-astronautics/16-30-feedback-control-systems-fall-2010/lecture-notes/MIT16_30F10_lec12.pdf

Alocação de polos – exemplo MIMO (2)



- Code gives the numerical values for all of the stability derivatives.
  - Solve for the eigenvalues of the matrix A to find the **lateral modes**.

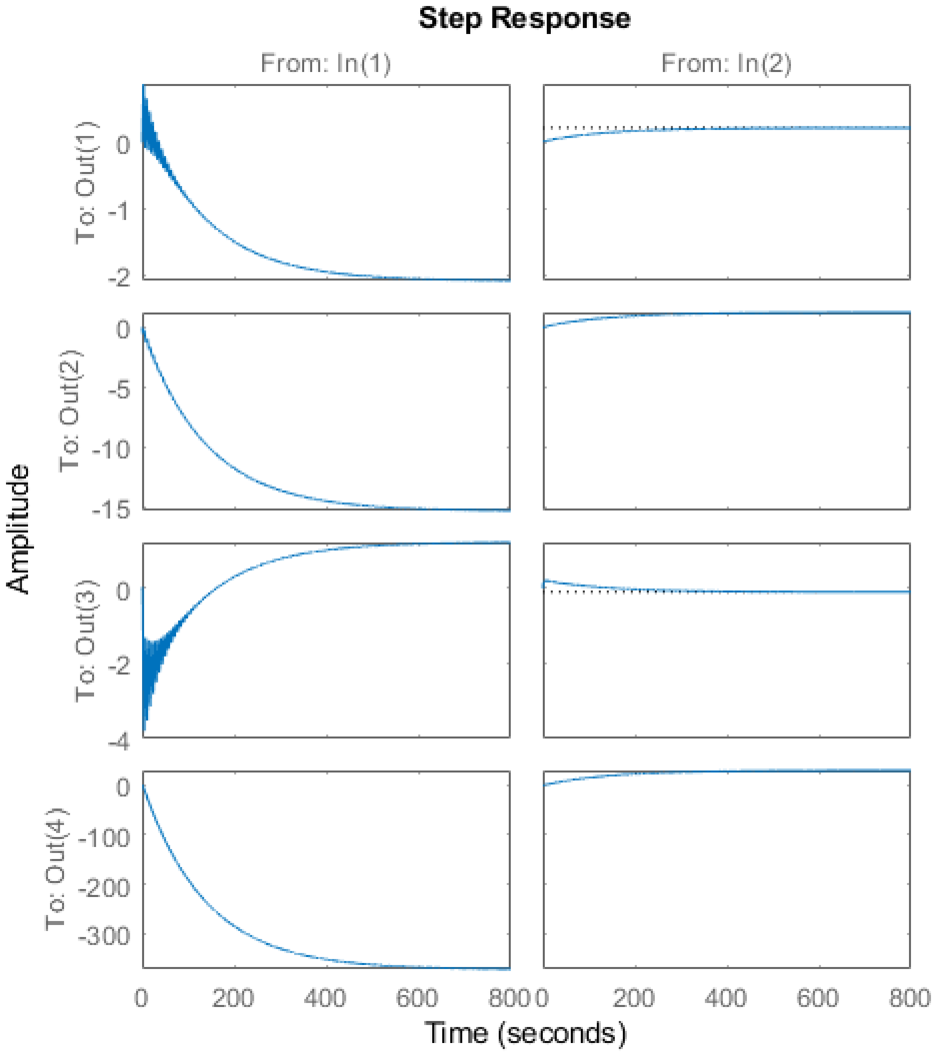
$$-0.0329 \pm 0.9467\mathbf{i}, -0.5627, -0.0073$$

  - Stable, but there is one very slow pole.
- There are 3 modes, but they are a **lot more complicated** than the longitudinal case.

Slow mode	-0.0073	$\Rightarrow$ **Spiral Mode**
Fast real	-0.5627	$\Rightarrow$ **Roll Damping**
Oscillatory	$-0.0329 \pm 0.9467\mathbf{i}$	$\Rightarrow$ **Dutch Roll**

https://ocw.mit.edu/courses/aeronautics-and-astronautics/16-30-feedback-control-systems-fall-2010/lecture-notes/MIT16_30F10_lec12.pdf

Alocação de polos – exemplo MIMO (2)



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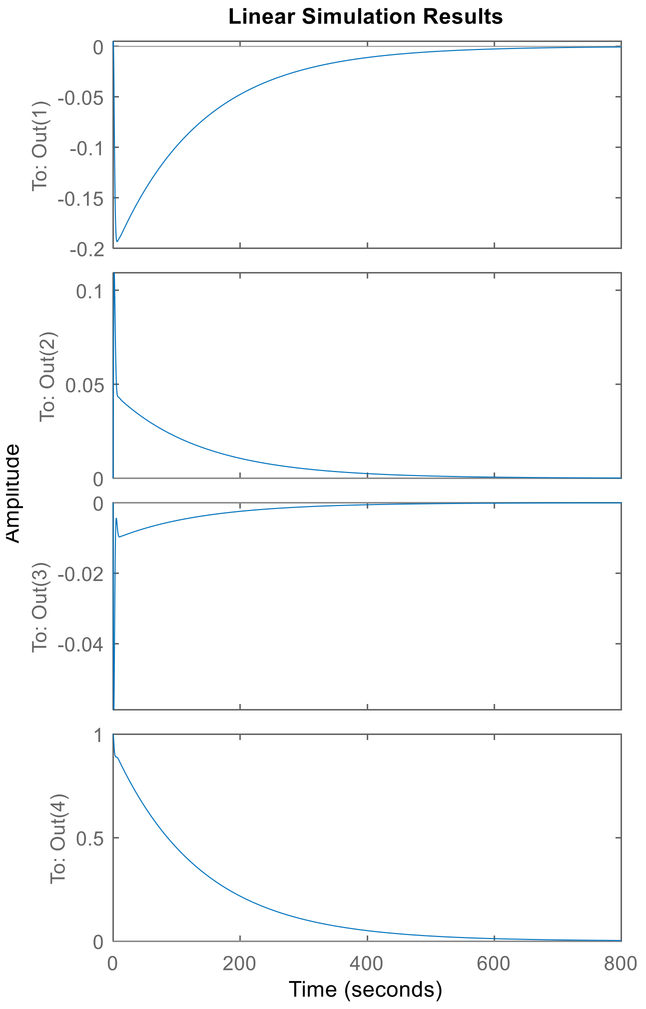
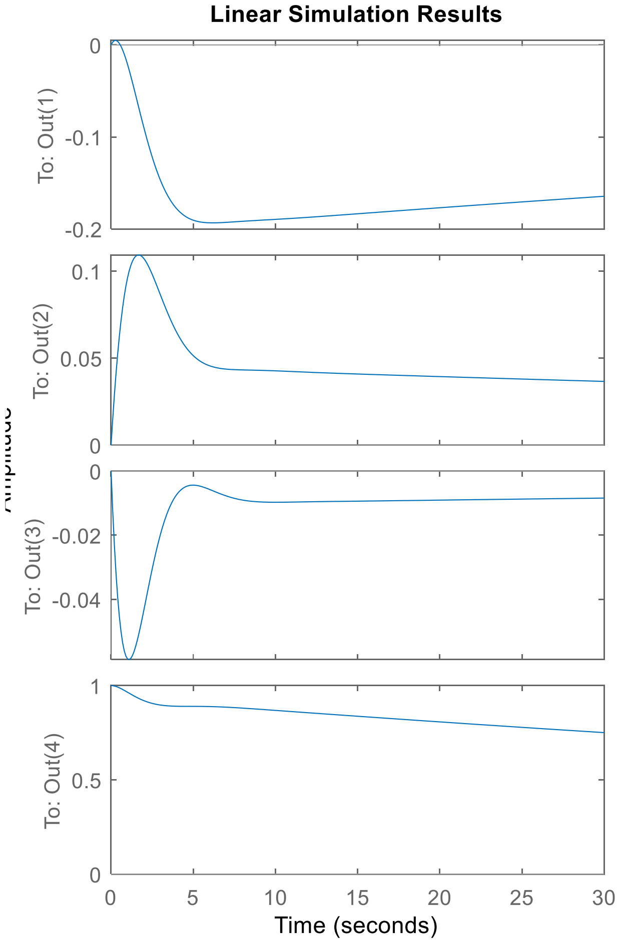
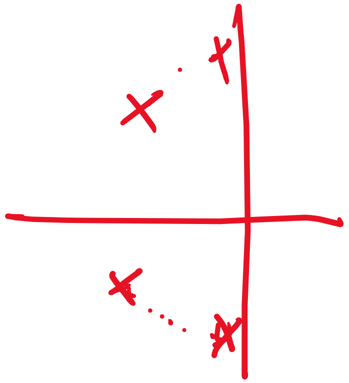
Alocação de polos – exemplo MIMO (2)







```
%novos polos
p = [-0.0073 -0.5627 -0.7+0.7*j -0.7-0.7*j]

% forma usando place Matlab
G = place(A,B,p)

% verificação
damp(A-B*G);

sys_closed = ss(A - B*G,zeros(4,1),eye(4),0);
t = [0:0.1:30];
u = zeros(length(t),1);
figure
lsim(sys_closed,u,t,[0 0 0 1])
```

https://ocw.mit.edu/courses/aeronautics-and-astronautics/16-30-feedback-control-systems-fall-2010/lecture-notes/MIT16_30F10_lec12.pdf

Alocação de polos – exemplo MIMO (3)

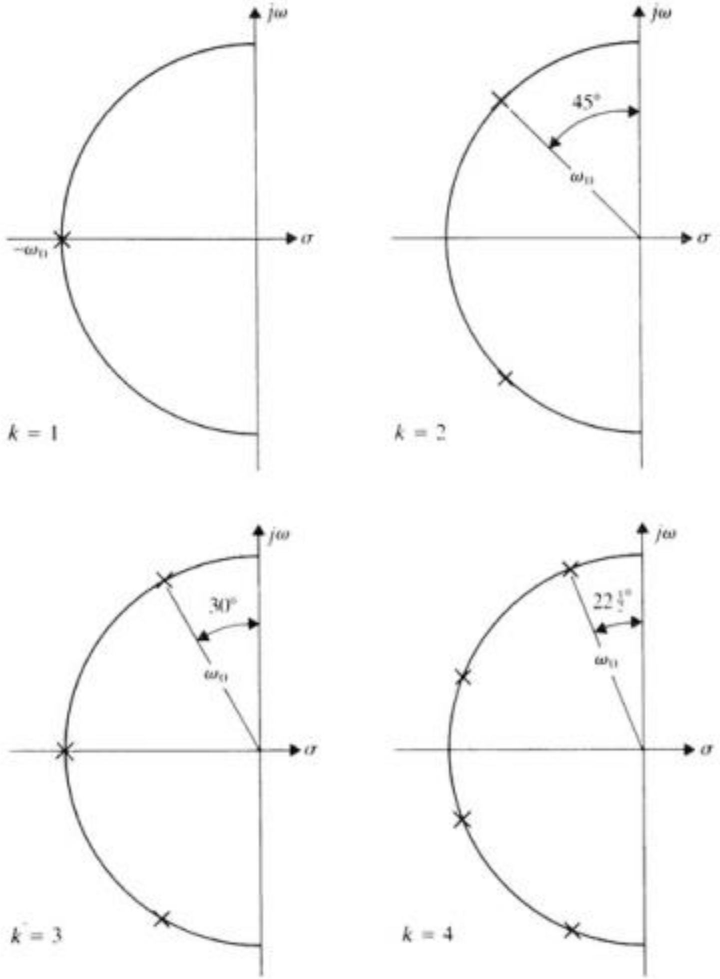
5.4. A *distillation column* in a chemical plant has the following state-coefficient matrices:

$$\mathbf{A} = \begin{bmatrix} -21 & 0 & 0 & 0 \\ 0.1 & -5 & 0 & 0 \\ 0 & -1.5 & 0 & 0 \\ 0 & -4 & 0 & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 6000 & 0 \\ 0 & 0 \\ 0 & 2.3 \\ 0 & 0.1 \end{bmatrix} \quad (5.142)$$

(a) Is the plant controllable?

(b) Suppose we would like to control the plant using *only one* input at a time. Is the plant controllable with only the *first* input, i.e. with $\mathbf{B} = [6000; 0; 0; 0]^T$? Is the plant controllable with only the *second* input, i.e. with $\mathbf{B} = [0; 0; 2.3; 0.1]^T$?

Alocação de polos – posicionamento de polos Onde colocar? Aletrnativa 2



$$\left(\frac{s}{\omega_0}\right)^{2k} = (-1)^{k+1}$$

[z,p,k]=buttap(4)