9.

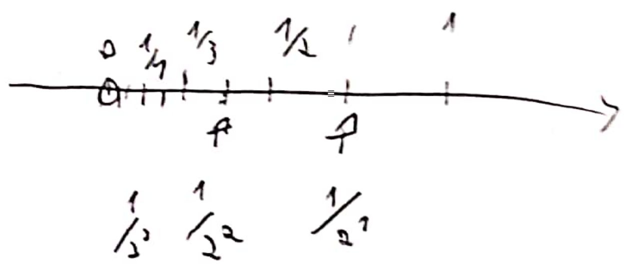
$s = \sup A$ se e só se:

i) s é uma cota superior de A

ii) s é a maior cota superior de A.

9c) $\forall n \in \mathbb{N}$, $\underline{n < 2^n}$ $\Rightarrow$ $\dfrac{1}{2^n} < \dfrac{1}{n}$ ⊛

$1 < 2$



$n+1 \leq n+n = 2n < 2 \cdot 2^n = 2^{n+1}$

$0 = \inf \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}$

i) $0 \leq \dfrac{1}{2^n} \quad \forall n \in \mathbb{N}$

$0 < m$ $\qquad 0 < 1$ $\qquad$ $\exists n \in \mathbb{N} \quad 1 < nm \Rightarrow \frac{1}{n} < m$

ii) $\underbrace{\dfrac{1}{2^n}}_{\in \{\dots\}} < \dfrac{1}{n} < m$ ✓

$0 = \inf \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right.$

$0 < m \quad \ldots \quad \frac{1}{\sqrt{2n}} < m \Rightarrow \frac{1}{2n} < m^2 \Rightarrow \frac{1}{2m^2} < n$ *

$m > 0 \Rightarrow \frac{1}{2m^2} > 0 \qquad 1 > 0 \qquad \frac{1}{2m^2}$

$\exists n \in \mathbb{N}$ t.q. $n \cdot 1 > \frac{1}{2m^2} \Rightarrow n > \frac{1}{2m^2}$

$\Rightarrow \frac{1}{2n} < m^2 \Rightarrow \frac{1}{\sqrt{2n}} < m$

$$\sum_{k=1}^{n} \frac{1}{3^k} = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^K}$$

$a < 1$

$\cancel{a = 1/3}$

$$S = a + a^2 + a^3 + \dots + a^n$$

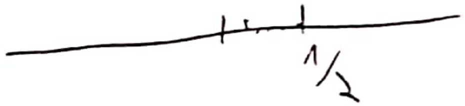
$$-aS = -a^2 - a^3 - a^4 - \dots - a^n - a^{n+1}$$

$$S - aS = (1-a)S = a - a^{n+1} \Rightarrow$$

$$\boxed{S = \frac{a - a^{n+1}}{1-a}}$$

$a = \frac{1}{3}$

$$\Rightarrow S = \frac{1/3 - 1/3^{n+1}}{1 - 1/3} = \frac{1}{3} \cdot \frac{1 - 1/3^n}{2/3}$$



$$\sum_{k=1}^{n} \frac{1}{3^k} = \frac{1 - 1/3^n}{2} = \frac{1}{2} - \frac{1}{2 \cdot 3^n}$$

$$\Rightarrow \sup\{\dots\} = \frac{1}{2}$$

$n < 3^n \Rightarrow n < 3^n$

A_n, $n \in \mathbb{N}$

enumerável

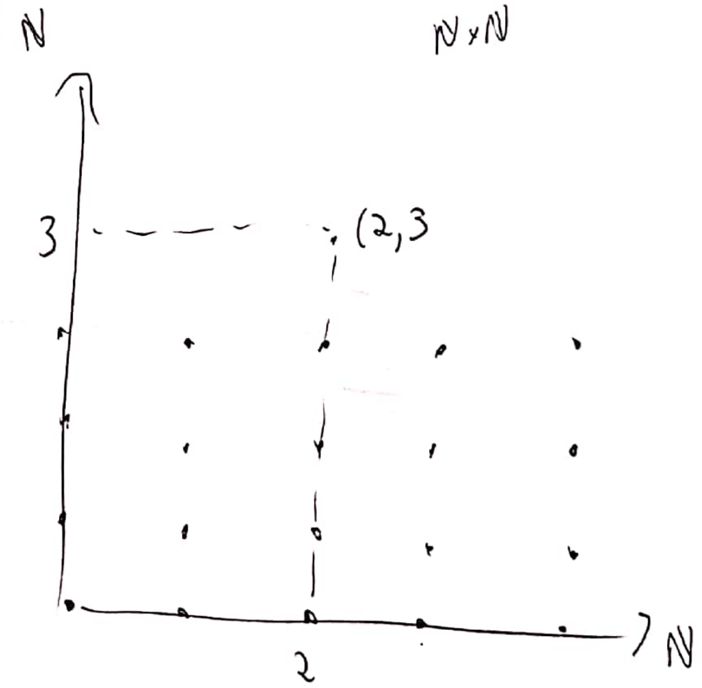
$A = \bigcup_{n \in \mathbb{N}} A_n$

A, B enumeráveis, $\Rightarrow$ $A \times B$ é enumerável

$\underbrace{\mathbb{N} \times \mathbb{N} \times \mathbb{N} \dots \mathbb{N}}_{n \text{ vezes}} = \mathbb{N}^n$ $\qquad \mathbb{N}^3$

A, B, $a \in A$, $b \in B$, $(a, b) \neq (b, a)$

$A \times B = \{(a, b) : a \in A,\ b \in B\}$



$A_1, \dots, A_n$, $(a_1, \dots, a_n)$
$\quad\quad\quad\quad\quad\quad\quad\; \cap \quad\quad\quad \cap$
$\quad\quad\quad\quad\quad\quad\quad\; A_1 \quad\quad\quad A_n$

$A_1 \times A_2 \times \dots \times A_n = \{(a_1, \dots, a_n) : a_1 \in A_1, \dots, a_n \in A_n\}.$

Suponha que A é enumerável e B é enumerável. $A \times B$ é enumerável?

Fixa $a_0 \in A$. $\{a_0\} \times B = \{(a_0, b) : b \in B\}$

$\{a_0\} \times B \cong B$

$\varphi : B \to \{a_0\} \times B$, $\varphi(b) = (a_0, b)$

i) É sobrejetora: $\forall \bar{x} \in \{a_0\} \times B$, $\exists\ b \in B$ t.q. $\varphi(b) = \bar{x}$

$\bar{x} \in \{a_0\} \times B \Rightarrow \bar{x} = (a_0, b)$, para algum $b \in B \Rightarrow \bar{x} = (a_0, b) = \varphi(b)$

Sobrejetora ✓

ii) <u>Injetora</u>: $b_1, b_2 \in B$, $b_1 \neq b_2 \Rightarrow \varphi(b_1) \neq \varphi(b_2)$

Sejam $b_1, b_2 \in B$, $b_1 \neq b_2$

$\varphi(b_1) = (a_0, b_1)$ $\qquad b_1 \neq b_2 \Rightarrow (a_0, b_1) \neq (a_0, b_2)$

$\varphi(b_2) = (a_0, b_2)$

$\Rightarrow \varphi(b_1) \neq \varphi(b_2)$ ✓ é Injetora.

φ sobre. e inj. $\Rightarrow \varphi$ é bijeção. $\Rightarrow B \equiv \underbrace{\{a_0\} \times B}$

$$\begin{tikzcd} \mathbb{N} \arrow[r, leftrightarrow, "\psi \text{ bij}"] \arrow[rr, leftrightarrow, bend right, "\varphi \circ \psi"'] & B \arrow[r, leftrightarrow, "\varphi \text{ bij.}"] & \{a_0\} \times B \end{tikzcd}$$

$\Rightarrow \varphi \circ \psi$ bij

$\Rightarrow \exists$ bijeção $\varphi \circ \psi : \mathbb{N} \to \{a_0\} \times B$ $\Rightarrow \{a_0\} \times B$ é enumerável.

$$A \times B = \boxed{\bigcup_{a \in A} \underbrace{\{a\} \times B}_{\text{enumerável}}} =: \{x : x \in \{a\} \times B \text{, para algum } a \in A\}$$

$A \times B \subseteq \bigcup \{a\} \times B$ $\qquad \bigcup \{a\} \times B \subseteq A \times B$

$(a,b) \in A \times B \Rightarrow$ ~~...~~

$(a,b)\ ,\ b \in B \Rightarrow (a,b) \in \{a\} \times B \Rightarrow (a,b) \in \bigcup \{a\} \times B$

$\bar{x} \in \bigcup \{a\} \times B \Rightarrow \exists a \in A$ t.q. $\bar{x} \in \{a\} \times B \Rightarrow \bar{x} = (a,b),\ b \in B \Rightarrow \bar{x} = (a,b) \in A \times B$

$A \times B = \bigcup \{a\} \times B$

$\Rightarrow A \times B$ é enumerável.

// //

$A \times B$ enumerável

? $n \in \mathbb{N}$!

Tese: $\underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{n \text{ vezes}}$ é enumerável.

<u>caso base</u>: $n=2$

$\mathbb{N} \times \mathbb{N}$ ✓

<u>passo indutivo</u>: $\underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{n \text{ vezes}}$ é enumerável

$\underbrace{\mathbb{N} \times \cdots \times \mathbb{N} \times \mathbb{N}}_{n+1 \text{ vezes}}$ $\quad \underbrace{(\underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{n \text{ vezes}})}_{\text{enumerável}} \times \underbrace{\mathbb{N}}_{\text{enumerável}} \Rightarrow$ enumerável.

$\mathbb{Q}$ $\quad$ ~~$f: \mathbb{Q} \to$~~

$$f: \underbrace{\cancel{\mathbb{Q}, \mathbb{Z}, \mathbb{Z}^*}}^{\mathbb{Z} \times \mathbb{Z}^*} \to \mathbb{Q}$$

$$(a, b) \mapsto \frac{a}{b}$$

$f: E \to A$

sobrejetora

$f: \cancel{\mathbb{Q}}$

$$K = \{(a,b) \in \underbrace{\mathbb{Z} \times \mathbb{Z}^*}_{\Rightarrow \text{ enumerável}} : b > 0, \ mdc(a,b) = 1\} \subseteq \underbrace{\mathbb{Z} \times \mathbb{Z}^*}_{\text{enumerável}}$$

$$(a, b) \mapsto \frac{a}{b}$$

$f: K \to \mathbb{Q}$