REAL BOUNDS AND LYAPUNOV EXPONENTS

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ABSTRACT. We prove that a C^3 critical circle map without periodic points has zero Lyapunov exponent with respect to its unique invariant Borel probability measure. Moreover, no critical point of such a map satisfies the Collet-Eckmann condition. This result is proved directly from the well-known *real a-priori bounds*, without using Pesin's theory. We also show how our methods yield an analogous result for infinitely renormalizable unimodal maps of any combinatorial type. Finally we discuss an application of these facts to the study of neutral measures of certain rational maps of the Riemann sphere.

1. INTRODUCTION

This paper studies critical circle maps (as well as infinitely renormalizable unimodal maps) from the differentiable ergodic theory viewpoint. The ergodic aspects of one-dimensional dynamical systems have been the object of intense research for quite some time. In particular, the study of characteristic or Lyapunov exponents of invariant measures, or physical measures, was initiated in this context by Ledrappier, Bowen, Ruelle, and developed by Keller, Blokh and Lyubich among others. See [28, Chapter V] for a full account, and the references therein.

In this article we show that the Lyapunov exponent of a C^3 critical circle map (or of an infinitely renormalizable unimodal interval map) is always zero. The general approach leading to zero Lyapunov exponents is by arguing by contradiction and using Pesin's theory: non-zero Lyapunov exponent implies the existence of periodic orbits (see for instance [21, Supplement 4 and 5] and [34, Chapter 11]), and that would be a contradiction for critical circle maps with irrational rotation number. Our goal in this paper, however, is to prove that the exponent is zero directly from the real a-priori bounds (see Theorem 2.2), without using Pesin's theory. In fact, the only non-trivial result from ergodic theory we shall use here is Birkhoff's Ergodic Theorem.

By a *critical circle map* we mean an orientation preserving C^3 circle homeomorphism f with finitely many non-flat critical points $c_1, c_2, \ldots, c_N$ ($N \geq 1$). A critical point c is called *non-flat* if in a neighbourhood of c the map f can be written as $f(t) = \phi(t) |\phi(t)|^{d-1} + f(c)$, where ϕ is a C^3 local diffeomorphism with $\phi(c) = 0$, and where $d > 1$ is a real number known as the *criticality* (or *order*, or *type*, or *exponent*) of such critical point. Critical circle maps have been studied by several authors in the last three decades. From a strictly mathematical viewpoint, these studies started with basic topological aspects [18], [42], then evolved – in the special case of maps with a *single* critical point – to geometric bounds [20], [37], and

further to geometric rigidity and renormalization aspects, see [1], [9], [10], [11], [12], [14], [15], [16], [23], [25], [37], [38], [39], [40] and [41]. The geometric rigidity and renormalization aspects of the theory remain open for maps with more than one critical point, see Question 7.3. Such brief account bypasses important numerical studies by several physicists, as well as computer-assisted and conceptual work by Feigenbaum, Kadanoff, Lanford, Rand, Epstein and others; see [10] and references therein.

As we said before, this paper studies a critical circle map f from the differentiable ergodic theory viewpoint. We will focus on the case when the rotation number of f is irrational, in which case f is uniquely ergodic. Moreover, by a theorem of Yoccoz [42], f is minimal and therefore topologically conjugate to the corresponding rigid rotation. This implies that the support of its unique invariant Borel probability measure is the whole circle (see Section 3 for more details on the invariant measure). Our main result is the following.

Theorem A. *Let* $f : S^1 \to S^1$ *be a* C^3 *critical circle map with irrational rotation number, and let* μ *be its unique invariant Borel probability measure. Then* $\log Df$ *belongs to* $L^1(\mu)$ *and it has zero mean:*

$$\int_{S^1} \log Df \, d\mu = 0.$$

Moreover, no critical point of f *satisfies the Collet-Eckmann condition.*

Recall that f satisfies the *Collet-Eckmann* condition at a critical point $c \in \{c_1, c_2, \ldots, c_N\}$ if there exist $C > 0$ and $\lambda > 1$ such that $Df^n(f(c)) \geq C\lambda^n$ for all $n \in \mathbb{N}$ (see for instance [28, Chapter V]), or equivalently

$$\liminf_{n \to \infty} \frac{1}{n} \log Df^n(f(c)) \geq \log \lambda > 0 . \tag{1.1}$$

The integrability of $\log Df$ was obtained by Przytycki in [31, Theorem B], where he also proved that $\int \log Df \, d\mu \geq 0$ (see [35, Appendix A] for an easier proof). We will obtain the integrability again (see Proposition 3.1) on the way to proving that $\int \log Df \, d\mu = 0$. It is expected that $\log Df$ will not be integrable if we allow the presence of flat critical points, as in [18].

Theorem A applies to some classical examples of holomorphic dynamics in the Riemann sphere, see Theorem C in §6.

Remark 1.1. The analogue of Theorem A for diffeomorphisms is straightforward: if f is an orientation-preserving C^1 circle diffeomorphism, with irrational rotation number, the function $\psi : S^1 \to \mathbb{R}$ defined by $\psi = \log Df$ is a continuous function and therefore, by the unique ergodicity of f, the sequence of continuous functions:

$$\frac{1}{n} \sum_{j=0}^{n-1} \psi \circ f^j$$

converges uniformly to a constant, and this constant must be $\int_{S^1} \log Df \, d\mu$. By the chain rule $\sum_{j=0}^{n-1} \psi \circ f^j = \log Df^n$ and, therefore, the sequence of continuous functions $\log Df^n/n$ converges to the constant $\int_{S^1} \log Df \, d\mu$ uniformly in S^1. Since f^n is a diffeomorphism for all $n \in \mathbb{N}$, this constant must be zero. In our case, however, $\log Df$ is not a continuous function (it is defined only in $S^1 \setminus \{c_1, c_2, \ldots, c_N\}$, and it is unbounded, see Figure 1).
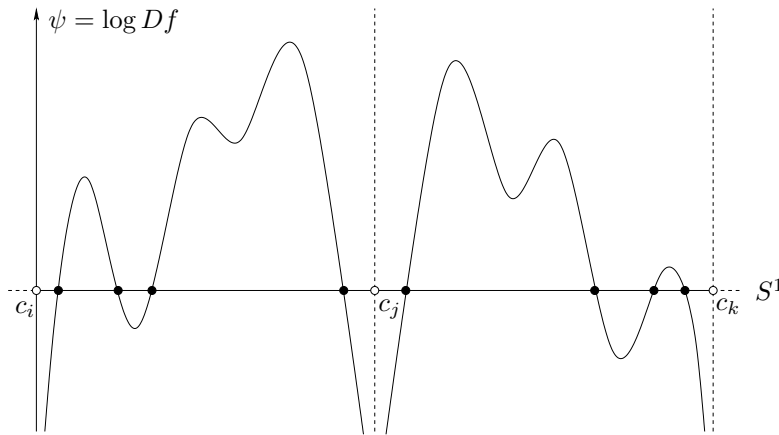
FIGURE 1. The cocycle $\psi = \log Df$ is unbounded.

Remark 1.2. The C^3-smoothness hypothesis of Theorem A could be relaxed to C^2 (or even $C^{1+Zygmund}$, see [28, Section IV.2.1]) – indeed whatever smoothness is necessary for the real bounds to hold.

1.1. **How the paper is organized.** In §2 we briefly recall some classical combinatorial facts about circle maps, and also the well-known *a-priori* bounds on the critical orbits of a critical circle map. We deduce from these facts two useful lemmas concerning dynamical partitions. In §3, we establish the integrability of $\log Df$ with respect to the unique f-invariant probability measure, for any critical circle map without periodic points for which the real bounds of §2 hold true. In §4, we use the results of §2 and §3 to prove our main result, namely Theorem A. In §5, we prove Theorem B, an analogous result to Theorem A for *infinitely renormalizable unimodal maps* with non-flat critical point. In §6, we discuss an application of Theorem A to the ergodic theory of certain Blaschke products as well as quadratic polynomials. Finally, in §7, we conclude by stating a few open questions concerning both critical circle maps and rational maps of the Riemann sphere.

2. THE REAL BOUNDS

Let f be a C^3 critical circle map as defined in the introduction, that is, f is an orientation preserving C^3 circle homeomorphism with finitely many non-flat critical points of odd type. As we have pointed out already, our standing assumption is that the rotation number $\rho(f) = \theta \in [0,1)$ is irrational. Therefore it has an infinite continued-fraction expansion, say

$$\theta = [a_0, a_1, ..., a_n, a_{n+1}, ...] = \lim_{n \to \infty} \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \cfrac{1}{a_n}}}}} .$$

We define recursively a sequence of *return times* of θ by:

$$q_0 = 1, \qquad q_1 = a_0 \quad \text{and} \quad q_{n+1} = a_n q_n + q_{n-1} \quad \text{for} \quad n \geq 1.$$

In particular the sequence $\{q_n\}_{n \geq 1}$ grows at least exponentially fast when n goes to infinity (we will use this fact in the proof of Proposition 3.1 and in the proof of Proposition 4.3 below). We recall that the numbers q_n are also obtained as the denominators of the truncated expansion of order n of θ:

$$\frac{p_n}{q_n} \;=\; [a_0, a_1, a_2, ..., a_{n-1}] \;=\; \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \, \cfrac{1}{a_{n-1}}}}}} \;.$$

We recall also the following well-known estimates.

Theorem 2.1. *For all $n \in \mathbb{N}$ we have:*

$$\frac{1}{q_n(q_n + q_{n+1})} < \left| \theta - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2} \;.$$

For a proof, see for instance Theorems 9 and 13 in [24, Ch. I], or Theorem 5 in [26, Ch. I].

2.1. **Dynamical partitions.** Denote by $I_n(c)$ the interval $[c, f^{q_n}(c)]$, where $c \in \{c_1, c_2, \ldots, c_N\}$ denotes a critical point of f, and define $\mathcal{P}_n(c)$ as:

$$\mathcal{P}_n(c) = \left\{ f^i(I_n(c)) : 0 \leq i \leq q_{n+1} - 1 \right\} \cup \left\{ f^j(I_{n+1}(c)) : 0 \leq j \leq q_n - 1 \right\} \;.$$

A crucial combinatorial fact is that $\mathcal{P}_n(c)$ is a partition (modulo boundary points) of the circle for every $n \in \mathbb{N}$. We call it the *n-th dynamical partition* of f associated with the point c. Note that the partition $\mathcal{P}_n(c)$ is determined by the finite piece of orbit

$$\left\{ f^j(c) : 0 \leq j \leq q_n + q_{n+1} - 1 \right\}.$$

As we are working with critical circle maps, our partitions in this article are always determined by a critical orbit. Our proof of Theorem A is based on the following result.

Theorem 2.2 (The real bounds). *There exists a constant $K_0 > 1$ with the following property. Given a C^3 critical circle map f with irrational rotation number there exists $n_0 = n_0(f)$ such that, for each critical point c of f, for all $n \geq n_0$, and for every pair I, J of adjacent atoms of $\mathcal{P}_n(c)$ we have:*

$$K_0^{-1}|I| \leq |J| \leq K_0|I|, \tag{2.1}$$

where $|I|$ denotes the Euclidean length of an interval I in the real line.

Of course for a particular f we can choose $K_0 > 1$ such that (2.1) holds for all $n \in \mathbb{N}$. Theorem 2.2 was proved by Świątek and Herman (see [20] and [37]) in the case when f has a single critical point. The original proof is based on the so-called *cross-ratio inequality* of Świątek. As it turns out, this inequality is valid also in the case when the map f has several critical points (all of non-flat type), see [30]. This fact combined with the method of proof presented in [11, Section 3] yields the above general result. A detailed proof will appear in [8].

Note that for a rigid rotation we have $|I_n| = a_{n+1}|I_{n+1}| + |I_{n+2}|$. If a_{n+1} is big, then I_n is much larger than I_{n+1}. Thus, even for rigid rotations, real bounds do not hold in general.

In the case of maps with a single critical point, one also has the following corollary, which suitably bounds the distortion of first return maps.

Corollary 2.3. *Given a C^3 critical circle map f with irrational rotation number and a unique critical point $c \in S^1$, there exists a constant $K_1 > 1$ such that the following facts hold true for each $n \geq n_0$:*

(i) *For all $x, y \in f(I_{n+1}(c))$, we have*

$$\frac{1}{K_1} \leq \frac{\left|Df^{q_n-1}(x)\right|}{\left|Df^{q_n-1}(y)\right|} \leq K_1 \ .$$

(ii) *For all $x, y \in f(I_n(c))$, we have*

$$\frac{1}{K_1} \leq \frac{\left|Df^{q_{n+1}-1}(x)\right|}{\left|Df^{q_{n+1}-1}(y)\right|} \leq K_1 \ .$$

The control on the distortion of the return maps in the above corollary follows from Koebe's distortion principle (see [11, Section 3]). When f has two or more critical points, the estimates given in the Corollary may fail, because the intervals $f(I_n(c))$ and $f(I_{n+1}(c))$ could in principle contain other critical points of $f^{q_{n+1}-1}$ and f^{q_n-1}, respectively.

Remark 2.4. We shall henceforth use the constant $K = \max\{K_0, K_1\} > 1$ whenever we invoke the real bounds.

For our purposes, an important consequence of Corollary 2.3 is the following auxiliary result.

Lemma 2.5. *Let f be as in Corollary 2.3. There exists $C > 1$ such that for all $n \in \mathbb{N}$ and for all $x \in I_n(c) \setminus I_{n+2}(c)$:*

$$\frac{1}{C} \leq Df^{q_{n+1}}(x) \leq C \ .$$

Proof of Lemma 2.5. For each $k \in \mathbb{N}$, let us write I_k instead of $I_k(c)$ in this proof. Fix $n \in \mathbb{N}$ and $x \in I_n \setminus I_{n+2}$. By Corollary 2.3 the map $f^{q_{n+1}-1} : f(I_n) \to f^{q_{n+1}}(I_n)$ has bounded distortion. In particular, there exists $C_0 > 1$ such that:

$$\frac{1}{C_0} \frac{\left|f^{q_{n+1}}(I_n)\right|}{\left|f(I_n)\right|} \leq Df^{q_{n+1}-1}(f(x)) \leq C_0 \frac{\left|f^{q_{n+1}}(I_n)\right|}{\left|f(I_n)\right|} \ .$$

Since $I_{n+1} \subset f^{q_{n+1}}(I_n) \subset I_n \cup I_{n+1}$ we obtain from the real bounds that $(1/K)|I_n| \leq \left|f^{q_{n+1}}(I_n)\right| \leq (1+K)|I_n|$. Therefore:

$$\frac{1}{C_0 K} \frac{|I_n|}{\left|f(I_n)\right|} \leq Df^{q_{n+1}-1}(f(x)) \leq C_0(1+K)\frac{|I_n|}{\left|f(I_n)\right|} \ .$$

Since c is a non-flat critical point of f of odd type d there exist $0 < C_1 < C_2$ such that $C_1|I_n|^d \leq |f(I_n)| \leq C_2|I_n|^d$ for all $n \in \mathbb{N}$, and then:

$$\frac{1}{C_0 C_2 K} \frac{1}{|I_n|^{d-1}} \leq Df^{q_{n+1}-1}(f(x)) \leq \frac{C_0(1+K)}{C_1} \frac{1}{|I_n|^{d-1}} \ .$$

Again using that c is non-flat there exist $0 < A < B$ such that $A|x-c|^{d-1} \leq Df(x) \leq B|x-c|^{d-1}$ for all x in a small but fixed neighbourhood around the critical point. In particular,

$$\left(A/K^{2(d-1)}\right)|I_n|^{d-1} \leq Df(x) \leq B|I_n|^{d-1}$$

for all $x \in I_n \setminus I_{n+2}$ and for all $n \in \mathbb{N}$, since $|x - c| \geq |I_{n+2}| \geq |I_n|/K^2$, again by the real bounds. With this at hand we deduce that

$$\frac{A}{C_0 C_2 K^{2d-1}} \leq Df^{q_{n+1}}(x) \leq \frac{BC_0(1+K)}{C_1},$$

for all $x \in I_n \setminus I_{n+2}$ and for all $n \in \mathbb{N}$. This finishes the proof of the lemma, provided we take

$$C \geq \max \left\{ C_0 C_2 K^{2d-1}/A, BC_0(1+K)/C_1 \right\}.$$

$\square$

The following consequence of the real bounds was inspired by [11, Lemma A.5, page 379]. It holds under the general assumptions of Theorem 2.2, for maps with an arbitrary number of critical points. For each $n \geq 1$ let:

$$S_n(c) = \sum_{I \in \mathcal{P}_n(c) \setminus \{I_n(c), I_{n+1}(c)\}} \frac{|I|}{d(c, I)},$$

where $d(c, I)$ denotes the Euclidean distance between an interval $I \subset S^1$ and the critical point c.

Lemma 2.6. *For each critical point c of f, the sequence $\left\{ \dfrac{S_n(c)}{n} \right\}_{n \geq 1}$ is bounded.*

Proof of Lemma 2.6. Given a critical point c, let us write in this proof, for simplicity of notation, $\mathcal{P}_k$, I_k instead of $\mathcal{P}_k(c)$, $I_k(c)$ respectively, for each $k \in \mathbb{N}$. Note that the transition from $\mathcal{P}_n$ to $\mathcal{P}_{n+1}$ can be described in the following way: the interval $I_n = [c, f^{q_n}(c)]$ is subdivided by the points $f^{jq_{n+1}+q_n}(c)$ with $1 \leq j \leq a_{n+1}$ into $a_{n+1} + 1$ subintervals. This sub-partition is spread by the iterates of f to all $f^j(I_n) = f^j([c, f^{q_n}(c)])$ with $0 \leq j < q_{n+1}$. The other elements of the partition $\mathcal{P}_n$, namely the intervals $f^j(I_{n+1})$ with $0 \leq j < q_n$, remain unchanged. On one hand, for any $I \in \mathcal{P}_n \setminus \{I_n, I_{n+1}\}$ we have:

$$\sum_{I \supset J \in \mathcal{P}_{n+1}} \frac{|J|}{d(c, J)} \leq \frac{1}{d(c, I)} \sum_{I \supset J \in \mathcal{P}_{n+1}} |J| = \frac{|I|}{d(c, I)}.$$

On the other hand:

$$\sum_{\mathcal{P}_{n+1} \ni J \subset I_n \setminus I_{n+2}} \frac{|J|}{d(c, J)} \leq \frac{1}{|I_{n+2}|} \sum_{\mathcal{P}_{n+1} \ni J \subset I_n \setminus I_{n+2}} |J| = \frac{|I_n \setminus I_{n+2}|}{|I_{n+2}|}.$$

This gives us:

$$0 \leq S_{n+1} - S_n \leq \frac{|I_n \setminus I_{n+2}|}{|I_{n+2}|} \quad \text{for all } n \geq 1.$$

By the real bounds $\frac{|I_n \setminus I_{n+2}|}{|I_{n+2}|} \leq K^2$ for all $n \geq 1$ and we are done. $\square$

3. The integrability of $\log Df$

As before let f be a C^3 critical circle map with finitely many non-flat critical points and with rotation number $\rho(f)$. Since we assume that $\rho(f)$ is irrational, f admits a unique invariant Borel probability measure μ. Moreover, by a theorem of Yoccoz [42], f has no wandering intervals and therefore there exists a circle homeomorphism $h : S^1 \to S^1$ which is a topological conjugacy between f and the

rigid rotation by angle $\rho(f)$, that we denote by $R_{\rho(f)}$. More precisely, the following diagram commutes:

$$
\begin{array}{ccc}
(S^1, \mu) & \xrightarrow{\ f\ } & (S^1, \mu) \\
{\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\
(S^1, \mathrm{Leb}) & \xrightarrow[\ R_{\rho(f)}\ ]{} & (S^1, \mathrm{Leb})
\end{array}
$$

where Leb denotes the normalized Lebesgue measure in the unit circle (the Haar measure for the multiplicative group of complex numbers of modulus 1). Therefore μ is just the push-forward of Lebesgue measure under h^{-1}, that is, $\mu(A) = \big(h_*^{-1}\,\mathrm{Leb}\big)(A) = \mathrm{Leb}\big(h(A)\big)$ for any Borel set A in the unit circle (recall that the conjugacy h is unique up to post-composition with rotations, so the measure μ is well-defined).

In this section we prove that $\log Df$ belongs to $L^1(\mu)$. As before, let us denote by $c_1, c_2, \ldots, c_N$ the critical points of f.

Let $\varphi : S^1 \to \mathbb{R}$ be given by $\varphi = |\log Df|$. For each $1 \le j \le N$ and each $n \ge 1$, let $J_n(c_j) = I_n(c_j) \cup I_{n+1}(c_j)$. We define $E_n = \bigcup_{j=1}^{N} J_n(c_j)$ and consider $\varphi_n : S^1 \to \mathbb{R}$ given by:

$$
\varphi_n = \chi_{S^1 \setminus E_n} \cdot \varphi \,,
$$

that is, $\varphi_n = 0$ on each $J_n(c_j)$ and $\varphi_n = \varphi$ on the complement of their union. We will use the following four facts:

Fact 1. From the real bounds (Theorem 2.2) there exists $0 < \lambda < 1$ such that $|I_k(c_j)| \ge \lambda^k$ for all $k \ge 1$ and each $1 \le j \le N$.

Fact 2. As explained above, the measure μ is the pullback of the Lebesgue measure under any topological conjugacy between f and the corresponding rigid rotation. In particular, for each $1 \le j \le N$ and for all $k \ge 1$, we have $\mu(I_k(c_j)) = |q_k\theta - p_k|$ and by Theorem 2.1:

$$
\frac{1}{q_k + q_{k+1}} < \mu(I_k(c_j)) \le \frac{1}{q_{k+1}} \quad \text{for all } k \ge 1 \text{ and each } 1 \le j \le N.
$$

Fact 3. By combinatorics, we have $\mu(I_k(c_j) \setminus I_{k+2}(c_j)) = a_{k+1}\mu(I_{k+1}(c_j))$, for all $k \ge 0$ and for each $1 \le j \le N$.

Fact 4. Since each c_j is a non-flat critical point, there exist $C_0 > 0$ and a neighbourhood V_j of c_j such that for all $x \in V_j$ we have:

$$
\varphi(x) \ \le \ C_0 \log \frac{1}{|x - c_j|} \ . \tag{3.1}
$$

We may assume, of course, that the V_j's are pairwise disjoint.

With all these facts at hand we are ready to prove the desired integrability result.

Proposition 3.1. *The function $\log Df$ is μ-integrable, i.e., $\log Df \in L^1(\mu)$.*

Proof of Proposition 3.1. Note that the sequence $\{\varphi_n\}$ converges monotonically to $\varphi = |\log Df|$. Let n_0 be the smallest positive integer such that $J_{n_0}(c_j) \subseteq V_j$ for all $1 \le j \le N$. We only look at values of n greater than n_0. Then, since φ_n is identically zero on E_n and agrees with φ everywhere else, we can write

$$
\int_{S^1} \varphi_n \, d\mu \ = \ \int_{S^1 \setminus E_{n_0}} \varphi \, d\mu + \sum_{j=1}^{N} \sum_{k=n_0}^{n-1} \int_{I_k(c_j) \setminus I_{k+2}(c_j)} \varphi \, d\mu \tag{3.2}
$$

The first integral on the right-hand side is a fixed number independent of n. Hence it suffices to bound the last double sum. Using (3.1) and the fact that in $I_k(c_j) \setminus I_{k+2}(c_j)$ the closest point to c_j is $f^{q_{k+2}}(c_j)$, we see that (see Figure 2)

$$\sum_{k=n_0}^{n-1} \int_{I_k(c_j) \setminus I_{k+2}(c_j)} \varphi \, d\mu \ \leq C_0 \sum_{k=n_0}^{n-1} \mu(I_k(c_j) \setminus I_{k+2}(c_j)) \log \frac{1}{|I_{k+2}(c_j)|} \qquad (3.3)$$

Applying facts 1, 2 and 3 to this last sum, we see that

$$\sum_{k=n_0}^{n-1} \mu(I_k(c_j) \setminus I_{k+2}(c_j)) \log \frac{1}{|I_{k+2}(c_j)|} \ \leq \ C_1 \sum_{k=n_0}^{n-1} (k+2)a_{k+1}|q_{k+1}\theta - p_{k+1}| \ (3.4)$$

However we know from Theorem 2.1 that

$$|q_{k+1}\theta - p_{k+1}| \ \leq \ \frac{1}{q_{k+2}} \ = \ \frac{1}{a_{k+1}q_{k+1} + q_k} \ < \ \frac{1}{a_{k+1}q_{k+1}} \qquad (3.5)$$

Putting (3.5) into (3.4) we get

$$\sum_{k=n_0}^{n-1} \mu(I_k(c_j) \setminus I_{k+2}(c_j)) \log \frac{1}{|I_{k+2}(c_j)|} \ \leq \ C_1 \sum_{k=n_0}^{n-1} \frac{(k+2)}{q_{k+1}} \, . \qquad (3.6)$$

Since the q_k's grow exponentially fast (at least as fast as the Fibonacci numbers), we have

$$\sum_{k=0}^{\infty} \frac{(k+2)}{q_{k+1}} \ < \ \infty \, .$$

Hence the left-hand side of (3.6) is uniformly bounded. Taking this information back to (3.3) and then to (3.2), we deduce that there exists a constant $C_2 > 0$ such that

$$\int_{S^1} \varphi_n \, d\mu \ \leq C_2 \quad \text{for all } n \geq 1.$$

But then, by the Monotone Convergence Theorem, φ is μ-integrable, as desired. $\quad\square$

Remark 3.2. The proof of Proposition 3.1 yields, *mutatis mutandis*, a slightly stronger result, namely that $\log Df \in L^p(\mu)$ for every finite $p \geq 1$. However, this more general fact will not be needed in this paper.

4. Proof of Theorem A

In this section we present two different proofs of Theorem A. The first proof works only when the map f has a single critical point, whereas the second works in the general multicritical case.

4.1. **First proof: the unicritical case.** Here we assume that f has a single critical point c. In particular, we are free to use Lemma 2.5. Once again we write $\mathcal{P}_k, I_k$, instead of $\mathcal{P}_k(c), I_k(c)$, etc., for simplicity of notation.

Consider the Borel set $A \subset S^1$ defined in the following way: $x \in A$ iff there exists an increasing sequence $\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$ such that for each $k \in \mathbb{N}$ there exists (a necessarily unique) $j_k \in \{0, 1, ..., q_{n_k+1} - 1\}$ such that $f^{j_k}(x) \in I_{n_k} \setminus I_{n_k+2}$.

Lemma 4.1. *The set A is f-invariant and $\mu(A) = 1$.*
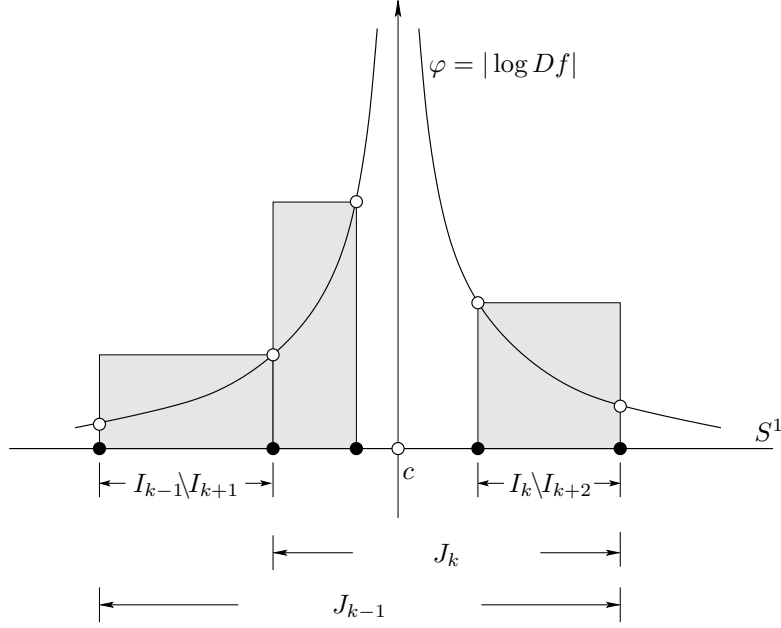
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FIGURE 2. Bounding the integral of $\varphi = |\log Df|$ near a critical point c.

Proof of Lemma 4.1. The first assertion follows immediately from the definition of A, hence we focus on proving that A has full μ-measure. For each $n \geq 1$ consider the disjoint union:

$$A_n = \bigcup_{i=0}^{q_{n+1}-1} f^{-i}(I_n \setminus I_{n+2}).$$

We claim that $\mu(A_n) > 1/3$ for all $n \geq 1$. Indeed, $\mu(A_n) = q_{n+1}\,\mu(I_n \setminus I_{n+2}) = a_{n+1}q_{n+1}\,\mu(I_{n+1})$, since $\mu(I_n \setminus I_{n+2}) = a_{n+1}\,\mu(I_{n+1})$. As explained at the beginning of Section 3, the measure μ is the pullback of the Lebesgue measure under any topological conjugacy between f and the corresponding rigid rotation. In particular:

$$\mu(I_{n+1}) = q_{n+1} \left| \theta - \frac{p_{n+1}}{q_{n+1}} \right|,$$

where $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is the rotation number of f. By Theorem 2.1:

$$\left| \theta - \frac{p_{n+1}}{q_{n+1}} \right| > \frac{1}{q_{n+1}(q_{n+2} + q_{n+1})}$$

and then:

$$\mu(A_n) > \frac{a_{n+1}q_{n+1}}{q_{n+2} + q_{n+1}}.$$

Since $q_{n+2} = a_{n+1}q_{n+1} + q_n < (a_{n+1} + 1)q_{n+1}$ we deduce that

$$\mu(A_n) > \frac{a_{n+1}}{a_{n+1} + 2}.$$

Since $a_{n+1} \geq 1$ for all $n \geq 0$ we obtain the claim, that is, $\mu(A_n) > 1/3$ for all $n \geq 1$. Moreover, since:

$$A = \limsup_{n \in \mathbb{N}} A_n = \bigcap_{k \geq 1} \bigcup_{n=k}^{+\infty} A_n \,,$$

we have $\mu(A) \geq 1/3$. The ergodicity of μ under f now implies that $\mu(A) = 1$, since A is f-invariant. $\qquad\square$

Now we consider the Borel set $B \subset S^1$ given by $B = A \setminus \mathcal{O}_f^-(c)$, where $\mathcal{O}_f^-(c)$ denotes the pre-orbit of the critical point, that is:

$$\mathcal{O}_f^-(c) = \left\{ c, f^{-1}(c), ..., f^{-n}(c), ... \right\} .$$

Proposition 4.2. *The set B has full μ-measure, i.e., $\mu(B) = 1$; moreover, the critical value $f(c)$ belongs to B.*

Proof of Proposition 4.2. Since μ has no atoms, $\mu\big(\mathcal{O}_f^-(c)\big) = 0$ (recall that μ is f-invariant and f has no periodic orbits). In particular $\mu(B) = 1$. The critical point of f belongs to A by definition, and by invariance, so does its critical value. Since there are no periodic orbits, $f(c) \notin \mathcal{O}_f^-(c)$ and then $f(c) \in B$. $\qquad\square$

The relation between B and the μ-integrability of $\log Df$ is given by the following:

Proposition 4.3. *Let $x \in B$ and let $\{n_k\}_{k \in \mathbb{N}}$ be its corresponding increasing sequence of natural numbers. Then:*

$$\lim_{k \to +\infty} \left\{ \frac{1}{q_{n_k+1}} \sum_{i=0}^{q_{n_k+1}-1} \log Df\big(f^i(x)\big) \right\} = 0 \,.$$

Proof of Proposition 4.3. Recall that, since c is a non-flat critical point of f, there exists $L > 0$ such that for any $x, y \in S^1 \setminus \{c\}$ we have:

$$\big| \log Df(x) - \log Df(y) \big| \leq \frac{L|x-y|}{\min\big\{|x-c|, |y-c|\big\}} \,.$$

Let $x \in B$ and let $\{n_k\}_{k \in \mathbb{N}}$ be its corresponding increasing sequence of natural numbers. Recall that for each $k \in \mathbb{N}$ there exists (a necessarily unique) $j_k \in \{0, 1, ..., q_{n_k+1} - 1\}$ such that $f^{j_k}(x) \in I_{n_k} \setminus I_{n_k+2}$. Then we have:

$$\frac{\big| \log Df^{q_{n_k+1}}(x) \big|}{q_{n_k+1}} \leq \frac{\big| \log Df^{q_{n_k+1}}(x) - \log Df^{q_{n_k+1}}\big(f^{j_k}(x)\big) \big|}{q_{n_k+1}}$$

$$+ \frac{\big| \log Df^{q_{n_k+1}}\big(f^{j_k}(x)\big) \big|}{q_{n_k+1}}$$

$$\leq \frac{1}{q_{n_k+1}} \sum_{i=0}^{j_k-1} \big| \log Df\big(f^i(x)\big) - \log Df\big(f^{i+q_{n_k+1}}(x)\big) \big| + \frac{\log C}{q_{n_k+1}}$$

$$\leq \frac{L}{q_{n_k+1}} \sum_{i=0}^{j_k-1} \frac{\big| f^i(x) - f^{i+q_{n_k+1}}(x) \big|}{\min\big\{|f^i(x) - c|, |f^{i+q_{n_k+1}}(x) - c|\big\}} + \frac{\log C}{q_{n_k+1}} \,,$$

where the second inequality is given by Lemma 2.5. By combinatorics we have the following facts:

(1) The points $f^i(x)$ and $f^{i+q_{n_k}+1}(x)$ do not belong to $I_{n_k} \cup I_{n_k+1}$ for any $i \in \{0, 1, ..., j_k - 1\}$.

(2) For each $i \in \{0, 1, ..., j_k - 1\}$ there exist Δ_i and Δ_i^*, adjacent elements of the partition $\mathcal{P}_{n_k}$, such that the points $f^i(x)$ and $f^{i+q_{n_k}+1}(x)$ belong to $\Delta_i \cup \Delta_i^*$ (the possibility that $\Delta_i = \Delta_i^*$ is not excluded; if they are different, we may suppose that $d(c, \Delta_i) \le d(c, \Delta_i^*)$). By the real bounds: $\left| f^i(x) - f^{i+q_{n_k}+1}(x) \right| \le |\Delta_i| + |\Delta_i^*| \le (1+K)|\Delta_i|$.

(3) If $i \ne l$ in $\{0, 1, ..., j_k - 1\}$ then $\Delta_i \ne \Delta_l$.

Therefore, with the notation of Lemma 2.6, we have that:

$$\sum_{i=0}^{j_k-1} \frac{\left| f^i(x) - f^{i+q_{n_k}+1}(x) \right|}{\min\left\{ |f^i(x) - c|, |f^{i+q_{n_k}+1}(x) - c| \right\}} \le (1+K) \sum_{I \in \mathcal{P}_{n_k} \setminus \{I_{n_k}, I_{n_k+1}\}} \frac{|I|}{d(c, I)},$$

and then:

$$\frac{\left| \log Df^{q_{n_k}+1}(x) \right|}{q_{n_k+1}} \le L(1+K)\left(\frac{S_{n_k}}{q_{n_k+1}} \right) + \frac{\log C}{q_{n_k+1}} \quad \text{for all } k \in \mathbb{N}.$$

At this point, recall that the sequence $\{q_n\}_{n \ge 1}$ grows at least exponentially fast with n, whereas S_n grows at most linearly with n, by Lemma 2.6. Hence both terms in the right-hand side of this last inequality go to zero as $k \to \infty$, and we are done. $\qquad\square$

With Proposition 3.1, Proposition 4.2 and Proposition 4.3 at hand our main result – in the *unicritical case* – follows in a straightforward manner.

Proof of Theorem A. By Proposition 3.1 we already know that $\log Df$ belongs to $L^1(\mu)$. Hence by Birkhoff's Ergodic Theorem we have

$$\lim_{n \to +\infty} \left\{ \frac{\log Df^n(x)}{n} \right\} = \int_{S^1} \log Df \, d\mu,$$

for μ-almost every $x \in S^1$. Combining this fact with Proposition 4.2 and Proposition 4.3 we obtain:

$$\int_{S^1} \log Df \, d\mu = 0.$$

Finally we have to prove that f does not satisfy the Collet-Eckmann condition. Indeed, if there were constants $C > 0$ and $\lambda > 1$ such that $Df^n(f(c)) \ge C\lambda^n$ for all $n \in \mathbb{N}$ we would have:

$$\liminf_{n \to +\infty} \left\{ \frac{\log Df^n(f(c))}{n} \right\} \ge \log \lambda > 0,$$

but this is impossible, since by Proposition 4.2 we know that $f(c)$ belongs to B, and this implies by Proposition 4.3 that there exists a subsequence of $\log Df^n(f(c))/n$ converging to zero. $\qquad\square$

Question 4.4. Theorem A suggests the question whether $\lim \left\{ \log Df^n(f(c))/n \right\} = 0$. For this purpose it would be enough to prove that the limit exists (since $f(c)$ belongs to B), for instance by proving that the critical value of f is a μ-typical point for the Birkhoff's averages of $\log Df$. Note, however, that this fact does not follow directly from the unique ergodicity of f since $\log Df$ is not a continuous function (it is defined only in $S^1 \setminus \{c\}$, and it is unbounded, see Figure 1 in the introduction).

4.2. **Second proof: the general multicritical case.** Let us now give a proof of Theorem A that works in general. Our proof relies on Proposition 4.8 below, which can be regarded as a suitable replacement for Lemma 2.5.

As before, let $\{q_n\}_{n\in\mathbb{N}}$ be the sequence of return times given by the irrational rotation number of f (see Section 2). Let us denote by $c_1, c_2, ..., c_N$ the critical points of f ($N \geq 1$) and let $d_i > 1$ denote the criticality of each c_i. Conjugating f by a suitable C^3-diffeomorphism (which does not affect its Lyapunov exponent) we may assume that each c_i has an open neighbourhood $V(c_i)$ where f is a *power-law* of the form:

$$f(x) = f(c_i) + (x - c_i)|x - c_i|^{d_i-1} \quad \text{for all } x \in V(c_i). \tag{4.1}$$

We also assume, of course, that $V(c_i) \cap V(c_j) = \varnothing$ whenever $i \neq j$.

Recall from the real bounds (Theorem 2.2) that, for each $c \in \{c_1, c_2, ..., c_N\}$, the dynamical partitions $\{\mathcal{P}_n(c)\}_{n\in\mathbb{N}}$ have the *comparability* property: any two consecutive atoms of $\mathcal{P}_n(c)$ have comparable lengths. We will also need the following three further consequences of the real bounds.

Lemma 4.5. *There exists $B_0 = B_0(f) > 1$ such that for each $c \in \{c_1, c_2, ..., c_N\}$, for each $n \in \mathbb{N}$ and for each atom $\Delta \in \mathcal{P}_n(c)$ we have:*

$$\frac{|\Delta|}{B_0} \leq \left|f^{q_n}(\Delta)\right| \leq B_0|\Delta| .$$

$\square$

Lemma 4.6. *There exists $B_1 = B_1(f) > 1$ with the following property: let $\Delta \in \mathcal{P}_n(c)$ and denote by Δ^* the union of Δ with its two immediate neighbours in $\mathcal{P}_n(c)$. If $0 \leq j < k \leq q_n$ are such that the intervals $f^j(\Delta^*)$, $f^{j+1}(\Delta^*)$,..., $f^{k-1}(\Delta^*)$ do not contain any critical point of f, then the map $f^{k-j} : f^j(\Delta) \to f^k(\Delta)$ has distortion bounded by B_1, that is:*

$$\frac{1}{B_1} \leq \frac{Df^{k-j}(x)}{Df^{k-j}(y)} \leq B_1 \quad \text{for all } x, y \in f^j(\Delta). \tag{4.2}$$

Proof of Lemma 4.6. The real bounds imply that $f^j(\Delta)$ has *space* inside $f^j(\Delta^*)$. Moreover, the map $f^{k-j} : f^j(\Delta^*) \to f^k(\Delta^*)$ is a diffeomorphism, and hence (4.2) follows from the standard Koebe distortion principle (see for instance [11, Lemma 2.4, page 348] and the references therein). $\square$

Lemma 4.7. *There exists $B_2 = B_2(f) > 1$ with the following property: if $c \neq c'$ are critical points of f and $\Delta \in \mathcal{P}_n(c)$, $\Delta' \in \mathcal{P}_n(c')$ for some $n \in \mathbb{N}$ are such that $\Delta \cap \Delta' \neq \varnothing$, then $B_2^{-1}|\Delta'| \leq |\Delta| \leq B_2|\Delta'|$.*

Proof of Lemma 4.7. This follows from the combinatorial fact that Δ is contained in the union of two adjacent atoms of $\mathcal{P}_n(c')$, one of which is Δ', and likewise for Δ'. $\square$

For each $k \geq 0$ and each critical point c we will use the notation $J_k(c) = I_k(c) \cup I_{k+1}(c) = \left[f^{q_{k+1}}(c), f^{q_k}(c)\right] \ni c$. The key to prove Theorem A is the following fact.

Proposition 4.8. *There exists $C = C(f) > 0$ with the following properties:*

(1) *For each $x \in S^1$ and all $n \geq 0$ we have $\log Df^{q_n}(x) \leq C$.*

(2) *For all $n \geq 0$, if $x \in S^1$ is such that $f^i(x) \notin \bigcup_{j=1}^{j=N} J_{2n}(c_j)$ for all $0 \leq i \leq q_n$, then $\log Df^{q_n}(x) \geq -Cn$.*

In what follows we denote by C_0, C_1, C_2, C_3,... positive constants (greater than 1, in fact) depending only on f. Moreover, for any two positive numbers a and b we use the notation $a \asymp b$ to mean that $C^{-1}a \leq b \leq Ca$ for some constant $C > 1$ depending only on f.

Proof of Proposition 4.8. Let us fix once and for all a critical point $c \in \{c_1, c_2, ..., c_N\}$. We assume that $n \geq 0$ is large enough so that each atom of $\mathcal{P}_n(c)$ contains at most one critical point of f. Let $x \in S^1$ and let $\Delta \in \mathcal{P}_n(c)$ be such that $x \in \Delta$. Let $\Delta^* \supseteq \Delta$ be as in Lemma 4.6. Just by taking n larger still, we may assume that, for $0 \leq k < q_n$, each $f^k(\Delta^*)$ contains at most one critical point of f. We say that $0 \leq k < q_n$ is a *critical time* for x if $f^k(\Delta^*)$ contains a critical point of f. Let us write $0 \leq k_1 < k_2 < ... < k_m < q_n$ for the sequence of all critical times for x. Note that $m \leq 3N$ since the family $\{f^k(\Delta^*)\}_{0 \leq k < q_n}$ has intersection multiplicity equal to 3. Using these critical times and the chain rule we can write:

$$Df^{q_n}(x) = Df^{k_1}(x) \left[\prod_{j=1}^{m-1} Df^{k_{j+1}-k_j-1}\left(f^{k_j+1}(x)\right)Df\left(f^{k_j}(x)\right) \right] \qquad (4.3)$$
$$\times Df^{q_n-k_m-1}\left(f^{k_m+1}(x)\right)Df\left(f^{k_m}(x)\right).$$

We proceed to estimate each term in the product (4.3) above. From Lemma 4.6 (with $j = 0$ and $k = k_1$) we have:

$$Df^{k_1}(x) \asymp \frac{\left|f^{k_1}(\Delta)\right|}{|\Delta|}. \qquad (4.4)$$

Again from Lemma 4.6 (with $k_j + 1$ and k_{j+1} replacing j and k respectively) we have for all $j \in \{1, ..., m-1\}$:

$$Df^{k_{j+1}-k_j-1}\left(f^{k_j+1}(x)\right) \asymp \frac{\left|f^{k_{j+1}}(\Delta)\right|}{\left|f^{k_j+1}(\Delta)\right|}. \qquad (4.5)$$

For each $j \in \{1, ..., m\}$ let $\beta_j \in \{c_1, c_2, ..., c_N\}$ be the (unique) critical point of f in $f^{k_j}(\Delta^*)$, and let d_j be its criticality. Since we are assuming that n is sufficiently large, we may suppose that $f^{k_j}(\Delta^*) \subseteq V(\beta_j)$ for all $j \in \{1, ..., m\}$. Then, from the power-law expression (4.1) we have:

$$Df\left(f^{k_j}(x)\right) \asymp \left|f^{k_j}(x) - \beta_j\right|^{d_j-1}, \qquad (4.6)$$

and recall that $d_j - 1 > 1$ for all $j \in \{1, ..., m\}$. Still using the power-law expression we see that:

$$\left|f^{k_j+1}(\Delta)\right| \asymp \left|f^{k_j}(\Delta)\right|^{d_j} \quad \text{for all } j \in \{1, ..., m\}. \qquad (4.7)$$

Using Lemma 4.6 yet again, we also see that:

$$Df^{q_n-k_m-1}\left(f^{k_m+1}(x)\right) \asymp \frac{\left|f^{q_n}(\Delta)\right|}{\left|f^{k_m+1}(\Delta)\right|}. \qquad (4.8)$$

Let us now prove item (1) of the conclusion of Proposition 4.8. Note that (4.6) yields:

$$Df\left(f^{k_j}(x)\right) \leq C_0\left|f^{k_j}(\Delta)\right|^{d_j-1} \quad \text{for all } j \in \{1, ..., m\}, \qquad (4.9)$$

where $C_0 = C_0(f) > 0$. Combining all these facts, namely (4.4)-(4.9), we deduce the following (upper) telescoping estimate:

$$Df^{q_n}(x) \le C_1 \frac{\left|f^{k_1}(\Delta)\right|}{|\Delta|} \left[\prod_{j=1}^{m-1} \frac{\left|f^{k_{j+1}}(\Delta)\right|}{\left|f^{k_j+1}(\Delta)\right|}\left|f^{k_j}(\Delta)\right|^{d_j-1}\right] \left|f^{k_m}(\Delta)\right|^{d_m-1} \frac{\left|f^{q_n}(\Delta)\right|}{\left|f^{k_m+1}(\Delta)\right|}$$

$$\asymp \frac{\left|f^{k_1}(\Delta)\right|}{|\Delta|}\left[\prod_{j=1}^{m-1}\frac{\left|f^{k_{j+1}}(\Delta)\right|}{\left|f^{k_j}(\Delta)\right|}\right]\frac{\left|f^{q_n}(\Delta)\right|}{\left|f^{k_m}(\Delta)\right|} = \frac{\left|f^{q_n}(\Delta)\right|}{|\Delta|} \le C_2\,, \qquad (4.10)$$

where in the last line we have used (4.7) and finally Lemma 4.5. This proves item (1). In order to prove item (2) note first that all estimates provided above are two-sided, except (4.9). In order to get a lower bound for the left side of (4.9) we use the hypothesis in (2). Since $f^{k_j}(x) \notin J_{2n}(\beta_j)$ we have:

$$\left|f^{k_j}(x) - \beta_j\right| \ge C_3\left|I_{2n}(\beta_j)\right|. \qquad (4.11)$$

From the real bounds we know that there exists $\lambda \in (0,1)$ depending only on f such that $C_4^{-1}\lambda^n\left|I_n(\beta_j)\right| \le \left|I_{2n}(\beta_j)\right| \le C_4\lambda^n\left|I_n(\beta_j)\right|$. Moreover, we claim that $\left|I_n(\beta_j)\right|$ is comparable to $\left|f^{k_j}(\Delta)\right|$. Indeed, this follows from Lemma 4.7 because $I_n(\beta_j) \in \mathcal{P}_n(\beta_j)$ intersects an atom of $\mathcal{P}_n(c)$ in $f^{k_j}(\Delta^*)$, and this atom has length comparable to $\left|f^{k_j}(\Delta)\right|$ (such atom is either $f^{k_j}(\Delta)$ itself, or one of its neighbours). Using these facts in (4.11) we deduce that:

$$Df\left(f^{k_j}(x)\right) \ge C_5\,\lambda^{n(d_j-1)}\left|f^{k_j}(\Delta)\right|^{d_j-1}. \qquad (4.12)$$

Using this lower estimate in place of the upper estimate (4.9) and proceeding as in (4.10) we arrive at the estimate

$$Df^{q_n}(x) \ge C_6\,\lambda^{n(d_1+d_2+...+d_m-m)}\,, \qquad (4.13)$$

where again $C_6 = C_6(f) > 1$. Note that $0 < d_1 + d_2 + ... + d_m - m < 3(d_1 + d_2 + ... + d_N)$, and since $\alpha = 3(d_1 + d_2 + ... + d_N)$ is a positive constant depending only on f we get:

$$Df^{q_n}(x) \ge C_6\,\lambda^{n\alpha}\,,$$

and then:

$$\log Df^{q_n}(x) \ge -n\alpha\log\frac{1}{\lambda} + \log C_6 \ge -C_7\,n\,.$$

$\square$

With Proposition 4.8 at hand we are ready to prove Theorem A in the general multicritical case.

Proof of Theorem A. The fact that no critical point of f satisfies the Collet-Eckmann condition follows at once from item (1) of Proposition 4.8. By Proposition 3.1 we know that $\log Df \in L^1(\mu)$, and then we know from Birkhoff's ergodic theorem that:

$$\lim_{n\to+\infty}\left\{\frac{\log Df^n(x)}{n}\right\} = \int_{S^1}\log Df\,d\mu\,,$$

for μ-almost every $x \in S^1$. For each $n \geq 0$ let:

$$A_n = S^1 \setminus \bigcup_{j=1}^{j=N} \bigcup_{i=0}^{q_n-1} f^{-i}\big(J_{2n}(c_j)\big)$$

$$= \left\{ x \in S^1 : \forall \ 0 \leq i \leq q_n - 1 : f^i(x) \in S^1 \setminus \bigcup_{j=1}^{j=N} J_{2n}(c_j) \right\},$$

and consider

$$A = \limsup_{n \in \mathbb{N}} A_n = \bigcap_{k=1}^{+\infty} \bigcup_{n=k}^{+\infty} A_n \,.$$

We claim that A has full μ-measure. Indeed, since:

$$\mu\big(J_{2n}(c_j)\big) = \mu\big(I_{2n}(c_j)\big) + \mu\big(I_{2n+1}(c_j)\big) \leq \frac{1}{q_{2n+1}} + \frac{1}{q_{2n+2}} \,,$$

we deduce that $q_n\mu\big(J_{2n}(c_j)\big) \to 0$ (exponentially fast in n, in fact) and since $\mu(A_n) \geq 1 - Nq_n\mu\big(J_{2n}(c_j)\big)$ we see that $\mu(A_n) \to 1$ as $n \to +\infty$. This implies the claim that $\mu(A) = 1$. Now for each $x \in A$ we have from Proposition 4.8 that there exists a sequence $n_k \to +\infty$ such that:

$$\frac{-Cn_k}{q_{n_k}} \leq \frac{\log Df^{q_{n_k}}(x)}{q_{n_k}} \leq \frac{C}{q_{n_k}} \,,$$

and letting $k \to +\infty$ we get that:

$$\lim_{k \to +\infty} \frac{\log Df^{q_{n_k}}(x)}{q_{n_k}} = 0 \,.$$

Therefore:

$$\lim_{n \to +\infty} \left\{ \frac{\log Df^n(x)}{n} \right\} = 0$$

for μ-almost every $x \in A$, and then we are done since A has full μ-measure. $\qquad\square$

Remark 4.9. Note that for maps with two or more critical points the analogue of Question 4.4 has, in general, a negative answer. Indeed, it may well happen that one of the critical points, say c_1, lies in the pre-orbit of one of the other critical points. In that case, the Lyapunov exponent of $f(c_1)$ will be equal to $-\infty$.

5. Analogous results for unimodal maps

The main result of this paper, Theorem A, has an analogue in the context of *infinitely renormalizable unimodal maps*. We wish to state the result (see Theorem B below) and this will involve a slight digression into the renormalization theory of unimodal maps. We refer the reader to [28, Chapter VI] for general background on the subject.

Let $I_0 = [-1, 1] \subset \mathbb{R}$, and consider C^3 maps $f : I_0 \to I_0$ which are *unimodal* with $f'(0) = 0$, $f(0) = 1$, *i.e.*, with a unique critical point at 0 and critical value at 1. Without loss of generality assume that f is *even*, in the sense that $f(-x) = f(x)$ for all $x \in I_0$. We assume throughout that the critical point is *non-flat*, as defined in the introduction. Such an f is said to be *renormalizable* if there exist $p = p(f) > 1$ and $\lambda = \lambda(f) = f^p(0)$ such that $f^p|[-|\lambda|, |\lambda|]$ is unimodal and maps $[-|\lambda|, |\lambda|]$

into itself. With p smallest possible, the *first renormalization* of f is the map $Rf : I_0 \to I_0$ given by

$$Rf(x) \;=\; \frac{1}{\lambda} \, f^p(\lambda x) \tag{5.1}$$

The intervals $\Delta_j \;=\; f^j([-|\lambda|, |\lambda|])$, for $0 \le j \le p - 1$, have pairwise disjoint interiors, and their relative order inside I_0 determines a *unimodal* permutation θ of $\{0, 1, \ldots, p - 1\}$. Thus, renormalization consists of a first return map to a small neighbourhood of the critical point rescaled to unit size via a linear rescale (see Figure 3).
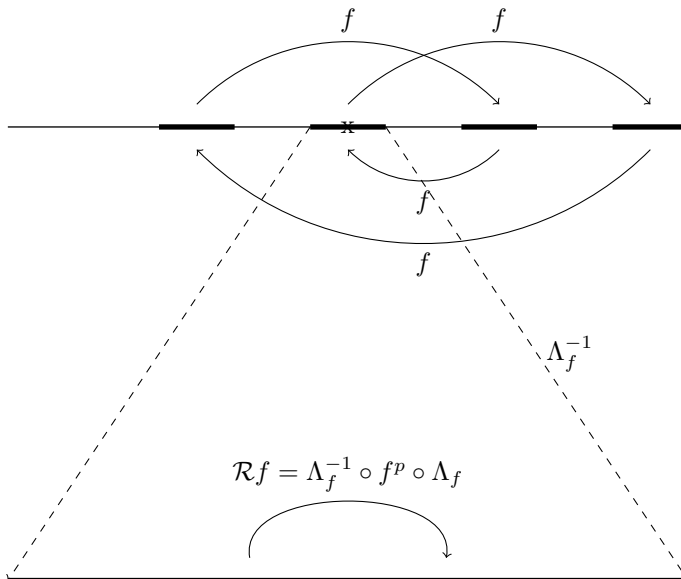

FIGURE 3.   Renormalizing a unimodal map.

Since Rf is again a normalized unimodal map, one can ask whether Rf is also renormalizable, and if the answer is yes then one can define $R^2 f = R(Rf)$, and so on. Thus, a unimodal map f is said to be *infinitely renormalizable* if the entire sequence $f, Rf, R^2 f, \ldots, R^n f, \ldots$ is well-defined.

We assume henceforth that f is infinitely renormalizable. Let us denote by $\mathcal{I}_f \subseteq I_0$ the closure of the orbit of the critical point of f. The set $\mathcal{I}_f$ is a Cantor set with zero Lebesgue measure, and is the *global attractor* of f both from the *topological* and *metric* points of view: the set $\mathcal{B}(\mathcal{I}_f) = \left\{ x \in I_0 : \omega(x) = \mathcal{I}_f \right\}$ is a residual set in I_0, and it has full Lebesgue measure (see [28, Section V.1] and the references therein). Let us point out, however, that $\mathcal{B}(\mathcal{I}_f)$ has empty interior.

For each $n \ge 0$, we can write

$$R^n f(x) = \frac{1}{\lambda_n} \cdot f^{q_n}(\lambda_n x)$$

where $q_0 = 1$, $\lambda_0 = 1$, $q_n = \prod_{i=0}^{n-1} p(R^i f)$ and $\lambda_n = \prod_{i=0}^{n-1} \lambda(R^i f) = f^{q_n}(0)$. The positive integers $a_i = p(R^i f) \ge 2$ are called the *renormalization periods* of f, and the q_n's are the *closest return times* of the orbit of the critical point (in perfect

analogy with the case of critical circle maps). Note that $q_{n+1} = a_n q_n = \prod_{i=0}^{i=n} a_i \geq 2^{n+1}$, in particular the sequence q_n goes to infinity at least exponentially fast.

Next, consider the *renormalization intervals* $\Delta_{0,n} = [-|\lambda_n|, |\lambda_n|] \subset I_0$, and $\Delta_{i,n} = f^i(\Delta_{0,n})$ for $i = 0, 1, \ldots, q_n - 1$. The collection $\mathcal{C}_n = \{\Delta_{0,n}, \ldots, \Delta_{q_n-1,n}\}$ consists of pairwise disjoint intervals. Moreover, $\bigcup\{\Delta : \Delta \in \mathcal{C}_{n+1}\} \subseteq \bigcup\{\Delta : \Delta \in \mathcal{C}_n\}$ for all $n \geq 0$ and we have

$$\mathcal{I}_f = \bigcap_{n=0}^{\infty} \bigcup_{i=0}^{q_n-1} \Delta_{i,n} \ .$$

It is also well-known that $f|_{\mathcal{I}_f}$ is a homeomorphism, and that the dynamics of f restricted to $\mathcal{I}_f$ is conjugate to that of an *adding machine* (see [28, Section III.4, Prop. 4.5]). More precisely, consider the following inverse system of cyclic groups (each endowed with the discrete topology):

$$\cdots \to \mathbb{Z}_{a_0 a_1 \cdots a_n a_{n+1}} \to \mathbb{Z}_{a_0 a_1 \cdots a_n} \to \cdots \to \mathbb{Z}_{a_0 a_1} \to \mathbb{Z}_{a_0} \ .$$

Here, the morphisms obviously correspond to multiplication by the successive periods a_n. The inverse limit $\mathbb{A}$ of this system together with the translation τ induced by $x \mapsto x+1$ on each factor (with carryover to the left) is a compact abelian group, called the *adding machine with periods* $(a_0, a_1, \ldots)$ (*cf.* [34, p. 212]). The system $(\mathbb{A}, \tau)$ is a minimal and uniquely ergodic dynamical system (and so is $f|_{\mathcal{I}_f}$). The assertion is that there exists a homeomorphism $H : \mathbb{A} \to \mathcal{I}_f$ such that the diagram

$$
\begin{array}{ccc}
\mathbb{A} & \xrightarrow{\ \tau\ } & \mathbb{A} \\
{\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H} \\
\mathcal{I}_f & \xrightarrow[\ f\ ]{} & \mathcal{I}_f
\end{array}
$$

commutes. In particular, if ν denotes the unique invariant probability measure under τ, then the pushforward $\mu = H_*\nu$ is the unique invariant probability measure for $f|_{\mathcal{I}_f}$. It is not difficult to check that this measure satisfies

$$\mu(\Delta_{i,n} \cap \mathcal{I}_f) = \frac{1}{q_n} = \frac{1}{a_0 a_1 \cdots a_{n-1}}$$

for each $n \geq 0$ and each $0 \leq i \leq q_n - 1$. This occurs simply because, at each level n, the intervals $\Delta_{i,n}$ with $0 \leq i \leq q_n - 1$ are permuted by f.

Since $\mathcal{B}(\mathcal{I}_f)$ has full Lebesgue measure in I_0 and $f|_{\mathcal{I}_f}$ is uniquely ergodic, it is easy to check that the measure μ is the unique *physical* measure for f in I_0, see [28, Section V.1, Theorem 1.6]. Now, the exact analogue of Theorem A can be stated as follows.

Theorem B. *Let $f : I_0 \to I_0$ be a C^3 infinitely renormalizable unimodal map with non-flat critical point, and let μ be the unique f-invariant Borel probability measure supported in the closure of its post-critical set. Then $\log|Df|$ belongs to $L^1(\mu)$ and it has zero mean:*

$$\int_{I_0} \log|Df| \, d\mu = 0.$$

Moreover, f does not satisfy the Collet-Eckmann condition.

Note that Theorem B is stated for C^3 infinitely renormalizable unimodal maps of any combinatorial type, with non-flat critical point of any criticality and without any assumption about the Schwarzian derivative.

If f has negative Schwarzian derivative and non-degenerate critical point (that is, $D^2 f(c) \neq 0$), Theorem B goes back to Keller [22, Theorem 3, page 722]. It is also well-known that an infinitely renormalizable unimodal map does not satisfy the Collet-Eckmann condition (otherwise it would admit an absolutely continuous invariant probability measure, see [3] and the references therein, which is impossible since its non-wandering set has zero Lebesgue measure [28, Section VI.2, Theorem 2.1], see also [27]). The lack of the Collet-Eckmann condition will be re-obtained here, just as in Theorem A, by showing that there exists a subsequence of $\log \left|Df^n\big(f(c)\big)\right|/n$ converging to zero.

Remark 5.1. Just as in the case of critical circle maps, Theorem B suggests the question whether $\lim \left\{ \log \left|Df^n\big(f(c)\big)\right|/n \right\} = 0$, see Question 4.4. Again, for this purpose it would be enough to prove that the limit exists, for instance by proving that the critical value of f is a μ-typical point for the Birkhoff's averages of $\log|Df|$. We remark that Nowicki and Sands have proven in [29] that

$$\liminf_{n \to \infty} \left\{ \frac{1}{n} \log |Df^n\left(f(c)\right)| \right\} \geq 0 \ .$$

Combined with what we said above and will prove below, this fact implies that, if the limit exists, it must be equal to zero.

5.1. **Proof of Theorem B.** Sullivan has shown in [36] that, at every level of renormalization, each renormalization interval $\Delta_{j,n}$ has a definite space around itself inside I_0. We can state this particular result by Sullivan as follows (*cf.* [28, Section VI.2, Lemma 2.1]). Given a closed interval Δ and $\delta > 0$, we define the δ-*scaled neighborhood* of Δ to be the open interval $V \supset \Delta$ such that each component of $V \setminus \Delta$ has length equal to $\delta|\Delta|$.

Lemma 5.2. *There exists $\tau = \tau(f) > 0$ such that, for each $n \geq 1$, the τ-scaled neighborhoods $V_{j,n} \supset \Delta_{j,n}$, for $0 \leq j \leq q_n - 1$, are pairwise disjoint.*

This fact combined with Koebe distortion principle implies that $f^{k-j} : \Delta_{j,n} \to \Delta_{k,n}$ is a diffeomorphism with *bounded distortion* for all $1 \leq j \leq k \leq q_n$. This is the contents of the following lemma, whose proof can be found in [28, Section VI.2, Theorem 2.1, item 1].

Lemma 5.3. *There exists $K_1 = K_1(f) > 1$ such that:*

$$\frac{1}{K_1} \frac{|\Delta_{k,n}|}{|\Delta_{j,n}|} \leq \left|Df^{k-j}(x)\right| \leq K_1 \frac{|\Delta_{k,n}|}{|\Delta_{j,n}|} \tag{5.2}$$

for all $x \in \Delta_{j,n}$ and all $n \geq 0$.

The proposition below follows from the fact that the successive renormalizations of f form a bounded sequence in the C^1-topology, which in turn is a consequence of Lemma 5.2 and Lemma 5.3. Nevertheless, we provide a proof as a courtesy to the reader.

Proposition 5.4. *There exists a constant $C_0 = C_0(f) > 1$ such that for all $n \geq 0$ and each $x \in \bigcup_{j=0}^{q_n - 1} \Delta_{j,n}$ we have $\left|Df^{q_n}(x)\right| \leq C_0$. In particular f does not satisfy the Collet-Eckmann condition.*

Proof of Proposition 5.4. Since $0 \in I_0$ is a non-flat critical point with exponent $d > 1$ there exist $0 < a < b$ such that $a|x|^{d-1} \le |Df(x)| \le b|x|^{d-1}$ for all $x \in \Delta_{0,n}$ and all $n \in \mathbb{N}$. In particular

$$|Df(x)| \le b|\Delta_{0,n}|^{d-1} \tag{5.3}$$

for all $x \in \Delta_{0,n}$ and all $n \in \mathbb{N}$.

Now, given $n \in \mathbb{N}$, $j \in \{0, 1, ..., q_n - 1\}$ and $x \in \Delta_{j,n}$ note that $f^{q_n-j}(x) \in \Delta_{0,n}$ and then $f^{q_n-j+1}(x) \in \Delta_{1,n}$. From (5.2) we deduce that:

$$\frac{1}{K_1}\frac{|\Delta_{0,n}|}{|\Delta_{j,n}|} \le |Df^{q_n-j}(x)| \le K_1 \frac{|\Delta_{0,n}|}{|\Delta_{j,n}|} \tag{5.4}$$

and also that

$$\frac{1}{K_1}\frac{|\Delta_{j,n}|}{|\Delta_{1,n}|} \le |Df^{j-1}(f^{q_n-j+1}(x))| \le K_1 \frac{|\Delta_{j,n}|}{|\Delta_{1,n}|}, \tag{5.5}$$

while from (5.3) we have that

$$|Df(f^{q_n-j}(x))| \le b|\Delta_{0,n}|^{d-1}. \tag{5.6}$$

From the chain rule and (5.4), (5.5) and (5.6) we get

$$|Df^{q_n}(x)| \le b\,K_1^2\,\frac{|\Delta_{0,n}|^d}{|\Delta_{1,n}|}. \tag{5.7}$$

Since $|Df(y)| \ge a|y|^{d-1}$ for all $y \in \Delta_{0,n}$ and since $d > 1$ we get:

$$|\Delta_{1,n}| = \int_0^{|\lambda_n|} |Df(y)|\,dy \ge a \int_0^{|\lambda_n|} |y|^{d-1}dy = a \int_0^{|\lambda_n|} y^{d-1}dy = \frac{a}{d}\,|\lambda_n|^d,$$

that is, $|\Delta_{1,n}| \ge \frac{a}{d}\frac{1}{2^d}|\Delta_{0,n}|^d$. With this at hand and (5.7) we finally obtain[1]

$$|Df^{q_n}(x)| \le \frac{b\,K_1^2\,d\,2^d}{a} =: C_0 \quad \text{for all } x \in \Delta_{j,n}.$$

$\square$

Proposition 5.4 gives us large iterates where $|Df|$ is far away from infinity. With this at hand, we just need to find large iterates where $|Df|$ is far away from zero. For this purpose we give the following definition:

Definition 5.5. For each $x \in \mathcal{I}_f$ and each $n \ge 0$ we define the *n-th entrance time* of x, denoted by $v_n(x)$, to be:

$$v_n(x) = \min\{j \ge 0 : f^j(x) \in \Delta_{0,n}\}.$$

Note that $v_n(x) \in \{0, 1, ..., q_n - 1\}$ and that $v_{n+1}(x) \ge v_n(x)$ for all $n \ge 0$ and all $x \in \mathcal{I}_f$. Note also that $v_n(f(c)) = q_n - 1$ for all $n \ge 0$.

Lemma 5.6. *For μ-almost every $x \in \mathcal{I}_f$ we have $v_n(x) \to +\infty$ as $n \to +\infty$.*

---

[1]Another way to obtain a positive lower bound for the sequence of ratios $|\Delta_{1,n}|/|\Delta_{0,n}|^d$ is by noting that $|\Delta_{1,n}| = |\phi(\Delta_{0,n})|^d = |D\phi(x_n)|^d|\Delta_{0,n}|^d$, where the C^3 local diffeomorphism ϕ, which fixes the origin, is given by the definition of non-flat critical point (see the introduction) and x_n is some point in $\Delta_{0,n}$.

Proof of Lemma 5.6. For each $n \geq 0$ we define:

$$A_n = \big\{ x \in \mathcal{I}_f : v_{n+1}(x) > v_n(x) \big\}.$$

We also define:

$$A = \limsup_{n \in \mathbb{N}} A_n = \bigcap_{k \geq 0} \bigcup_{n=k}^{+\infty} A_n \, .$$

The set A is f-invariant, and $x \in A$ if and only if $v_n(x) \to +\infty$ as $n \to +\infty$. We claim that $\mu(A_n) = 1 - 1/a_n \geq 1/2$ for all $n \geq 0$.

Indeed, fix $n \in \mathbb{N}$ and $j \in \{0, 1, ..., q_n - 1\}$, and note that $v_n(x) = q_n - j$ for all $x \in \Delta_{j,n} \cap \mathcal{I}_f$. The intervals of the $(n+1)$-th level of renormalization contained in $\Delta_{j,n}$ are easily seen to be $\Delta_{j+kq_n, n+1}$ where $k \in \{0, 1, ..., a_n - 1\}$.

Note that if $x \in \Delta_{j+kq_n, n+1}$ then $v_{n+1}(x) = q_{n+1} - (j + kq_n) = a_n q_n - (q_n - v_n(x) + kq_n) = v_n(x) + (a_n - 1 - k)q_n$. Thus, if $k = a_n - 1$ then $v_{n+1}(x) = v_n(x)$, but if $k \in \{0, 1, ..., a_n - 2\}$ then $v_{n+1}(x) \geq v_n(x) + q_n > v_n(x)$. Hence we have $A_n \cap \Delta_{j,n} = \bigcup_{k=0}^{a_n - 2} \Delta_{j+kq_n, n+1}$ and then:

$$\mu(A_n \cap \Delta_{j,n}) = \sum_{k=0}^{a_n - 2} \mu(\Delta_{j+kq_n, n+1}) = (a_n - 1)\mu(\Delta_{0,n+1}) = \frac{a_n - 1}{q_{n+1}} = \frac{1}{q_n}\left(1 - \frac{1}{a_n}\right).$$

Thus $\mu(A_n) = \sum_{j=0}^{q_n - 1} \mu(A_n \cap \Delta_{j,n}) = 1 - \frac{1}{a_n} \geq \frac{1}{2}$ as was claimed. The claim implies that $\mu(A) \geq \frac{1}{2} > 0$, and since f is ergodic with respect to μ and the set A is f-invariant, it follows that $\mu(A) = 1$. $\qquad\square$

We shall use the fact, due to Guckenheimer in the late seventies [17], that at each renormalization level the interval containing the critical point is the largest (up to multiplication by a constant). More precisely:

Lemma 5.7. *There exists a constant $\varepsilon = \varepsilon(f) > 0$ such that $|\Delta_{0,n}| \geq \varepsilon |\Delta_{j,n}|$ for all $j \in \{0, 1, ..., q_n - 1\}$ and all $n \geq 0$.*

Proof of Lemma 5.7. This is proven in [13, page 760]; see also [17, page 144]. $\quad\square$

The argument used by Guckenheimer relies on the Minimum Principle for maps with negative Schwarzian derivative (see [28, Section II.6, Lemma 6.1] for its statement) so at this point we need the fact that f is of class C^3.

With Proposition 5.4, Lemma 5.6 and Lemma 5.7 at hand we are ready to prove Theorem B:

Proof of Theorem B. The integrability of $\log |Df|$ was obtained by Przytycki [31, Theorem B] so let us prove that $\int_{I_0} \log |Df| \, d\mu$ is zero. For this purpose define $A \subset \mathcal{I}_f$ to be the set of points $x \in \mathcal{I}_f$ such that $v_n(x) \to +\infty$ as $n \to +\infty$, where $\{v_n(x)\}$ is the sequence of entrance times of $x \in \mathcal{I}_f$, see Definition 5.5. We also define $B \subset \mathcal{I}_f$ to be the set of points $x \in \mathcal{I}_f$ such that:

$$\lim_{n \to +\infty} \left\{ \frac{\log |Df^n(x)|}{n} \right\} = \int_{I_0} \log |Df| \, d\mu \, ,$$

and note that $\mu(A \cap B) = 1$ by Lemma 5.6 and Birkhoff's ergodic theorem. By Proposition 5.4 we get

$$\limsup_{n \in \mathbb{N}} \left\{ \frac{\log |Df^{q_n}(x)|}{q_n} \right\} \leq 0 \quad \text{for all } x \in \mathcal{I}_f,$$

and recall that from this it follows at once that f does not satisfy the Collet-Eckmann condition. Moreover we already conclude that:

$$\lim_{n \to +\infty} \left\{ \frac{\log \left| Df^n(x) \right|}{n} \right\} \leq 0 \quad \text{for all } x \in B. \tag{5.8}$$

Now let $x \in A$. For each $n \in \mathbb{N}$, $f^{v_n(x)}$ maps $\Delta_{q_n - v_n(x),n} \ni x$ into $\Delta_{0,n}$. By Lemma 5.3 there exists $K_1 = K_1(f) > 1$ such that:

$$\frac{1}{K_1} \frac{|\Delta_{0,n}|}{|\Delta_{q_n - v_n(x),n}|} \leq \left| Df^{v_n(x)}(x) \right| \leq K_1 \frac{|\Delta_{0,n}|}{|\Delta_{q_n - v_n(x),n}|} \quad \text{for all } n \geq 1.$$

Applying Lemma 5.7 we deduce in particular that for all $x \in A$ and for all $n \geq 1$:

$$\left| Df^{v_n(x)}(x) \right| \geq \frac{\varepsilon}{K_1} > 0$$

and since $v_n(x) \to +\infty$ for all $x \in A$ it follows that:

$$\liminf_{n \in \mathbb{N}} \left\{ \frac{\log \left| Df^{v_n(x)}(x) \right|}{v_n(x)} \right\} \geq 0 \quad \text{for all } x \in A.$$

Therefore:

$$\lim_{n \to +\infty} \left\{ \frac{\log \left| Df^n(x) \right|}{n} \right\} \geq 0 \quad \text{for all } x \in A \cap B. \tag{5.9}$$

Combining (5.8) with (5.9) we obtain

$$\lim_{n \to +\infty} \left\{ \frac{\log \left| Df^n(x) \right|}{n} \right\} = 0 \quad \text{for all } x \in A \cap B$$

and since $\mu(A \cap B) = 1$ this implies $\int_{I_0} \log |Df| \, d\mu = 0$, as we wanted to prove. $\quad\square$

Let us finish Section 5 with the following remark about the μ-integrability of $\log |Df|$.

Lemma 5.8. *There exists $C = C(f) > 1$ such that:*

$$\frac{1}{C} \sum_{n \in \mathbb{N}} \frac{1}{q_{n+1}} \log \frac{|\Delta_{0,n}|}{|\Delta_{0,n+1}|} \leq \int_{I_0} \left| \log |Df| \right| d\mu \leq C \sum_{n \in \mathbb{N}} \frac{1}{q_n} \log \frac{|\Delta_{0,n}|}{|\Delta_{0,n+1}|} \tag{5.10}$$

Proof of Lemma 5.8. We just need to imitate the procedure of Section 3, during the proof of Proposition 3.1. Indeed, let $\psi = \left| \log |Df| \right|$ and for each $n \in \mathbb{N}$ write $\psi_n = \psi \cdot \chi_{I_0 \setminus \Delta_{0,n}}$. Using the power-law at the critical point $c = 0$ we see that:

$$\frac{1}{C_0} \log \frac{1}{|x|} \leq \psi(x) \leq C_0 \log \frac{1}{|x|} \tag{5.11}$$

for all $x \in \Delta_{0,n} \setminus \{0\}$ and $n \in \mathbb{N}$ big enough. Now observe that:

$$\int_{I_0} \psi_n \, d\mu = \int_{I_0 \setminus \Delta_{0,n}} \psi_n \, d\mu = \sum_{j=0}^{n-1} \int_{\Delta_{0,j} \setminus \Delta_{0,j+1}} \psi_n \, d\mu. \tag{5.12}$$

Since $|\Delta_{0,n+1}| \leq |x| \leq |\Delta_{0,n}|$ for all $x \in \Delta_{0,n} \setminus \Delta_{0,n+1}$ we see from (5.11) and (5.12) that:

$$\frac{1}{C_1} \sum_{j=0}^{n-1} \mu(\Delta_{0,j} \setminus \Delta_{0,j+1}) \log \frac{1}{|\Delta_{0,j}|} \leq \int_{I_0} \psi_n \, d\mu \leq C_1 \sum_{j=0}^{n-1} \mu(\Delta_{0,j} \setminus \Delta_{0,j+1}) \log \frac{1}{|\Delta_{0,j+1}|}.$$
(5.13)

Since $\mu(\Delta_{0,j} \setminus \Delta_{0,j+1}) = \dfrac{1}{q_j} - \dfrac{1}{q_{j+1}}$ for all $j \in \mathbb{N}$ we obtain from (5.13) and a simple telescoping trick that:

$$\frac{1}{C_2} \sum_{j=0}^{n-1} \frac{1}{q_{j+1}} \log \frac{|\Delta_{0,j}|}{|\Delta_{0,j+1}|} \leq \int_{I_0} \psi_n \, d\mu \leq C_2 \sum_{j=0}^{n-1} \frac{1}{q_j} \log \frac{|\Delta_{0,j}|}{|\Delta_{0,j+1}|}$$

and since $\psi_n \nearrow \psi$ as n goes to infinity we are done. $\qquad\square$

Note in particular that if f is infinitely renormalizable of bounded type, then the series on the right-hand side of (5.10) converges, since by the real bounds we know that $|\Delta_{0,n}| \asymp |\Delta_{0,n+1}|$ (the so-called *bounded geometry*, see [28, Section VI.2, Theorem 2.1, item 2]), and the q_n's grow exponentially fast. Note also that the integrability of $\log |Df|$ proved by Przytycki in [31, Theorem B] already mentioned shows that, in the general case, the series on the left-hand side of (5.10) always converges.

6. Neutral measures on Julia sets

In this section we show some applications of Theorems A and B to holomorphic dynamics. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of the Riemann sphere with degree greater than or equal to two, and let $\mathcal{M}_f$ be the set of all f-invariant ergodic Borel probability measures in the Riemann sphere, whose support is contained in the Julia set of f. By [31, Theorem A] we know that $\log |Df|$ is μ-integrable for any $\mu \in \mathcal{M}_f$ (here Df denotes the derivative considered with respect to the spherical metric). The *Lyapunov exponent* of f with respect to μ is the real number $\chi(\mu) = \int_{\widehat{\mathbb{C}}} \log |Df| \, d\mu$. By Birkhoff's Ergodic Theorem,

$$\chi(\mu) = \lim_{n \to +\infty} \left\{ \frac{\log |Df^n(z)|}{n} \right\}$$

for μ-almost every $z \in \widehat{\mathbb{C}}$. If f is *hyperbolic*, *i.e.* if each critical point is either periodic or contained in the basin of an attracting periodic orbit, then there exists $\chi > 0$ such that $\chi(\mu) > \chi$ for any $\mu \in \mathcal{M}_f$. If f is not hyperbolic, then either f has a parabolic periodic orbit or one of its critical points lies in its Julia set (both phenomena can occur simultaneously). In the first case, the average of the Dirac measures supported along some parabolic periodic orbit gives an element $\mu \in \mathcal{M}_f$ such that $\chi(\mu) = 0$ (take for example $f(z) = z^2 + 1/4$ and $\mu = \delta_{1/2} \in \mathcal{M}_f$). Being purely atomic, these measures are not so interesting. In the second case (when the Julia set contains a critical point) one may have a non-atomic measure $\mu \in \mathcal{M}_f$ such that $\chi(\mu) = 0$. Following [31] we call such a measure *neutral*.

As an example, consider the one-parameter family $f_\omega : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of Blaschke products in the Riemann sphere given by:

$$f_\omega(z) = e^{2\pi i \omega} z^2 \left(\frac{z-3}{1-3z} \right) \quad \text{for } \omega \in [0,1). \tag{6.1}$$

Just as any Blaschke product, every map in this family commutes with the geometric involution around the unit circle $\Phi(z) = 1/\bar{z}$ (note that Φ is the identity in the unit circle). In particular every map in this family leaves invariant the unit circle (Blaschke products *are* the rational maps leaving invariant the unit circle), and its restriction to S^1 is a real-analytic critical circle map with a unique critical point at 1, which is of cubic type, and with critical value $e^{2\pi i\omega}$ (the fact that f_ω has topological degree one, when restricted to the unit circle, follows from the Argument Principle, since it has two zeros and one pole in the unit disk). By monotonicity of the rotation number (see for instance [19] and [28]) we know that for each irrational number θ in $[0,1)$ there exists a unique ω in $[0,1)$ such that the rotation number of $f_\omega|_{S^1}$ is θ. As an application of Theorem A we have the following result.

Theorem C. *Let ω in $[0,1)$ such that the rotation number of $f_\omega|_{S^1}$ is irrational. Then the rational function f_ω admits a non-atomic invariant ergodic Borel probability measure μ whose support is contained in the Julia set J_ω of f_ω and such that $\chi(\mu) = 0$. Moreover, f_ω does not satisfy the Collet-Eckmann condition.*

Proof of Theorem C. The measure μ is precisely the unique invariant measure supported on the unit circle. By Theorem A, we only need to prove that the unit circle is contained in the Julia set. Since S^1 is f_ω-invariant and $f_\omega|_{S^1}$ is minimal (by Yoccoz's result [42]), either we have $S^1 \subset J_\omega$, either $S^1 \cap J_\omega = \emptyset$. Suppose, by contradiction, that $S^1 \cap J_\omega = \emptyset$, and let U be the Fatou component of f_ω containing S^1. By the invariance of the unit circle, U is mapped into itself by f_ω, and therefore it must be a Siegel disk or a Herman ring (precisely because it has an invariant simple closed curve on its interior). But in that case $f_\omega : U \to U$ would be a biholomorphism, which is impossible since it has a critical point in the unit circle. Therefore $S^1 \subset J_\omega$, and Theorem C follows directly from Theorem A. $\square$

By the main result of [32], and since there is only one critical point in the Julia set J_ω, we deduce that f_ω does not satisfy any of the standard definitions of non-uniform hyperbolicity for rational maps in the Riemann sphere (topological Collet-Eckmann condition, uniform hyperbolicity of periodic orbits in the Julia set, etc.).

We remark that Theorem B in §5 yields analogous examples to those given in Theorem C above, in the context of polynomials. For instance, one can take $f(z) = z^2 + c_\infty$, where $c_\infty \in [-2, 1/4]$ denotes the infinitely renormalizable parameter for period doubling (the so-called *Feigenbaum* parameter). In that case, the neutral measure is the unique invariant measure supported in the closure of the post-critical set of f, which is non-atomic as explained in Section 5.

6.1. **Other examples.** Let $\theta \in [0,1]$ be an irrational number, and consider the quadratic polynomial $P_\theta : \mathbb{C} \to \mathbb{C}$ given by:

$$P_\theta(z) = e^{2\pi i\theta} z + z^2 . \tag{6.2}$$

The origin is a fixed point of P_θ, and $DP_\theta(0) = e^{2\pi i\theta}$ has modulus one. Let us assume that θ is of *bounded type*, that is, there exists $\varepsilon > 0$ such that:

$$\left| \theta - \frac{p}{q} \right| \geq \frac{\varepsilon}{q^2} ,$$

for any integers p and $q \neq 0$. By a classical result of Siegel P_θ is *linearizable* around the origin: there exists a simply-connected component Ω_θ of the Fatou set

of P_θ, a *Siegel disk*, that contains the origin and where P_θ is conformally conjugate to its linear part $z \mapsto e^{2\pi i\theta}z$, an irrational rotation acting on the unit disk. A famous theorem of Douady [7] (see also the recent paper [43]) asserts that $\partial\Omega_\theta$ is a quasi-circle that contains the (unique) critical point of P_θ. Moreover, there exist $\omega = \omega(\theta) \in (0,1)$ and a quasiconformal homeomorphism $\phi_\theta : \mathbb{C} \to \mathbb{C}$ with $\phi_\theta(\mathbb{D}) = \Omega_\theta$ and such that $\phi_\theta \circ f_\omega = P_\theta \circ \phi_\theta$ on $\mathbb{C}\backslash\mathbb{D}$, where f_ω is the Blaschke product given by (6.1). In particular $\phi_\theta : S^1 \to \partial\Omega_\theta$ is a quasisymmetric homeomorphism which conjugates f_ω with P_θ. Combining this with Theorem C we get the following result.

Corollary 6.1. *Let $\theta \in [0,1]$ be an irrational number of bounded type, and let P_θ be the quadratic polynomial given by* (6.2). *Then the harmonic measure on the boundary of the Siegel disk Ω_θ, viewed from the origin, is a neutral measure for P_θ.*

Corollary 6.1 follows from Theorem C and the general fact that a quasiconformal conjugacy between rational maps preserves zero Lyapunov exponents (one way to see this is to combine Koebe's distortion lemma with the fact that a quasiconformal homeomorphism is bi-Hölder and the argument in [33, Lemma 8.3]).

More generally, the harmonic measure at the boundary of any Siegel disk or at the boundary of any Herman ring is a neutral measure, as stated in the introduction of [31]. In particular, by taking some polynomial or rational function f with a Siegel disk whose boundary does not contain critical points (for such an example see [2] and the references therein) we obtain $\mu \in \mathcal{M}_f$ such that $\chi(\mu) = 0$, μ has no atoms and $\mathrm{supp}(\mu)$ contains no critical point of f. Note, however, that for any rational function f and any neutral measure μ we have that $\mathrm{supp}(\mu)$ must be contained in the closure of the forward orbit of some critical point of f, otherwise f would be expanding on $\mathrm{supp}(\mu)$ and then μ would not be neutral.

Finally, let us mention that Cortez and Rivera-Letelier have constructed in [5] and [6] real quadratic polynomials for which the ω-limit set of the critical point is a minimal Cantor set (necessarily contained in the Julia set) supporting *any prescribed* number of neutral measures (finite, countable infinite or uncountable). This is in sharp contrast with the examples discussed above, where $f|_{\mathrm{supp}(\mu)}$ is uniquely ergodic.

7. Further questions

We conclude with some questions (in addition to Question 4.4 posed at the end of Section 4).

Let f be a rational map of the Riemann sphere with degree greater than or equal to two, and let μ be a neutral measure for f, that is, a non-atomic f-invariant ergodic Borel probability measure, whose support is contained in the Julia set of f, and with Lyapunov exponent equal to zero (just as in the situation of Theorem C).

Question 7.1. Is it true that $h_{\mathrm{top}}(f|_{\mathrm{supp}(\mu)}) = 0$? Is $f|_{\mathrm{supp}(\mu)}$ a minimal dynamical system? Is it true, at least, that $\mathrm{supp}(\mu)$ has no periodic orbits?

Question 7.2. What are all the examples of rational maps having some non-atomic invariant ergodic Borel probability measure, supported inside its Julia set, and with zero Lyapunov exponent?

A difficult problem in the context of critical circle maps with finitely many critical points is the one of *geometric rigidity*. More precisely, let f and g be two orientation preserving C^3 circle homeomorphisms with the same irrational rotation number, and with $N \geq 1$ non-flat critical points of odd type. Denote by $S_f = \{c_1, ..., c_N\}$ the ordered critical set of f, by $S_g = \{c'_1, ..., c'_N\}$ the ordered critical set of g, and suppose that the criticalities of c_i and c'_i are the same for all $i \in \{1, ..., N\}$ (the cubic case is the generic one). Finally, denote by μ_f and μ_g the corresponding unique invariant measures of f and g.

By Yoccoz's result [42] we know that f and g are topologically conjugate to each other. By elementary reasons, the condition $\mu_f\big([c_i, c_{i+1}]\big) = \mu_g\big([c'_i, c'_{i+1}]\big)$ for all $i \in \{1, ..., N-1\}$ is necessary (and sufficient) in order to have a topological conjugacy between f and g that sends the critical points of f to the critical points of g. Under this assumption, it turns out that this conjugacy is in fact a *quasisymmetric* homeomorphism. This follows from a recent general result of Clark and van Strien [4]. In this context it also follows from the real bounds (see [11, Corollary 4.6] for the case of a single critical point, and the forthcoming article [8] for the multicritical case).

Question 7.3. Is this conjugacy a smooth diffeomorphism?

In the case of exactly one critical point this question has been answered in the affirmative, in the real-analytic category. This is due to the efforts of several mathematicians during the last twenty years (see [1], [9], [10], [11], [12], [23], [25], [38], [39], [40] and [41]). A positive answer was recently obtained also in the C^3 category (see [14], [15] and [16]). To the best of our knowledge, the case of more than one critical point remains completely open.

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