Herman's Theory Revisited

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Abstract

We prove that a $C^{2+\alpha}$-smooth orientation-preserving circle diffeomorphism with rotation number in Diophantine class D_δ, $0 < \delta < \alpha \le 1$, is $C^{1+\alpha-\delta}$-smoothly conjugate to a rigid rotation. We also derive the most precise version of Denjoy's inequality for such diffeomorphisms.

1   Introduction

An irrational number ρ is said to belong to Diophantine class D_δ if there exists a constant $C > 0$ such that $|\rho - p/q| \ge Cq^{-2-\delta}$ for any rational number p/q. The aim of this short note is to present the new and complete proof of the following

Theorem 1. *Let T be a $C^{2+\alpha}$-smooth orientation-preserving circle diffeomorphism with rotation number $\rho \in D_\delta$, $0 < \delta < \alpha \le 1$. Then T is $C^{1+\alpha-\delta}$-smoothly conjugate to the rigid rotation by angle ρ.*

(Note, that $C^{2+\alpha}$ with $\alpha = 1$ throughout this paper means $C^{2+\mathrm{Lip}}$ rather than C^3.)

This result was first announced in [1]. However, the complete proof was never published. Moreover, the argument in [1] contained a wrong intermediate estimate.

The first global results on smoothness of conjugation with rotations were obtained by M. Herman [2]. Later J.-C. Yoccoz extended the theory to the case of Diophantine rotation numbers [3]. The case of $C^{2+\alpha}$-smooth diffeomorphisms was considered by K. Khanin, Ya. Sinai [4, 1] and Y. Katznelson, D. Ornstein [5, 6].

In the present paper we use a conceptually new approach, which considerably simplifies the proof. We also believe that this approach will prove useful in other problems involving circle diffeomorphisms.

Let us remark that our result is stronger than the statement proven in [5], although their scope is wider (namely, we do not consider smoothness higher than C^3). It is also sharp, i.e. smoothness of conjugacy higher than $C^{1+\alpha-\delta}$ cannot be achieved in general settings, as it follows from the examples constructed in [5].

The paper is self-consistent although requires good understanding of combinatorics of circle homeomorphisms and Denjoy's theory, for which we refer a reader to the book [7].

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2 Cross-ratio tools

The *cross-ratio* of four pairwise distinct points x_1, x_2, x_3, x_4 is

$$\text{Cr}(x_1, x_2, x_3, x_4) = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_2 - x_3)(x_4 - x_1)}$$

Their *cross-ratio distortion* with respect to a strictly increasing function f is

$$\text{Dist}(x_1, x_2, x_3, x_4; f) = \frac{\text{Cr}(f(x_1), f(x_2), f(x_3), f(x_4))}{\text{Cr}(x_1, x_2, x_3, x_4)}$$

Clearly,

$$\text{Dist}(x_1, x_2, x_3, x_4; f) = \frac{\text{D}(x_1, x_2, x_3; f)}{\text{D}(x_1, x_4, x_3; f)}, \tag{1}$$

where

$$D(x_1, x_2, x_3; f) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} : \frac{f(x_2) - f(x_3)}{x_2 - x_3}$$

is the *ratio distortion* of three distinct points x_1, x_2, x_3 with respect to f.

In the case of smooth f such that f' does not vanish, both the ratio distortion and the cross-ratio distortion are defined for points, which are not necessarily pairwise distinct, as the appropriate limits (or, just by formally replacing ratios $(f(a) - f(a))/(a - a)$ with $f'(a)$ in the definitions above).

Notice that both ratio and cross-ratio distortions are multiplicative with respect to composition: for two functions f and g we have

$$D(x_1, x_2, x_3; f \circ g) = D(x_1, x_2, x_3; g) \cdot D(g(x_1), g(x_2), g(x_3); f) \tag{2}$$

$$\text{Dist}(x_1, x_2, x_3, x_4; f \circ g) = \text{Dist}(x_1, x_2, x_3, x_4; g) \cdot \text{Dist}(g(x_1), g(x_2), g(x_3), g(x_4); f) \tag{3}$$

Proposition 1. *Let $f \in C^{2+\alpha}$, $\alpha \in [0, 1]$, and $f' > 0$ on $[A, B]$. Then for any $x_1, x_2, x_3 \in [A, B]$ the following estimate holds:*

$$\text{D}(x_1, x_2, x_3; f) = 1 + (x_1 - x_3) \left(\frac{f''}{2f'} + \mathcal{O}(\Delta^\alpha) \right), \tag{4}$$

where $\Delta = \max\{x_1, x_2, x_3\} - \min\{x_1, x_2, x_3\}$, and the values of both f'' and f' can be taken at any points between $\min\{x_1, x_2, x_3\}$ and $\max\{x_1, x_2, x_3\}$.

Proof. First of all, it is easy to see why the arguments of f'' and f' in the estimate (4) be taken arbitrarily: $f''(\theta_1) - f''(\theta_2) = \mathcal{O}(\Delta^\alpha)$, $f'(\theta_1) - f'(\theta_2) = \mathcal{O}(\Delta)$, and $(f'(\theta))^{-1} = \mathcal{O}(1)$.

To prove (4), we need to consider three cases of relative locations of the points.
Case 1: x_2 lies between x_1 and x_3. It is easy to calculate that

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} - \frac{f(x_2) - f(x_3)}{x_2 - x_3} = (x_1 - x_3) \left(\frac{1}{2} f'' + \mathcal{O}(\Delta^\alpha) \right),$$

and (4) follows.

Case 2: x_1 lies between x_2 and x_3. One can check that

$$D(x_1, x_2, x_3; f) = 1 + \left[\frac{x_1 - x_3}{x_2 - x_3} \left(\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_3)}{x_1 - x_3} \right) \right] : \frac{f(x_2) - f(x_3)}{x_2 - x_3}.$$

The expression in the round brackets equals $(x_2 - x_3)(\frac{1}{2}f'' + \mathcal{O}(\Delta^\alpha))$, so in the square brackets we have $(x_1 - x_3)(\frac{1}{2}f'' + \mathcal{O}(\Delta^\alpha))$.
Case 3: x_3 lies between x_1 and x_2. Similar to Case 2. $\square$

Proposition 2. *Let $f \in C^{2+\alpha}$, $\alpha \in [0, 1]$, and $f' > 0$ on $[A, B]$. For any $x_1, x_2, x_3, x_4 \in [A, B]$ the following estimate holds:*

$$\mathrm{Dist}(x_1, x_2, x_3, x_4; f) = 1 + (x_1 - x_3)\mathcal{O}(\Delta^\alpha)$$

where $\Delta = \max\{x_1, x_2, x_3, x_4\} - \min\{x_1, x_2, x_3, x_4\}$.

Proof. Follows immediately from Proposition 1 due to (1). $\square$

Remark 1. While the ratio distortion satisfies an obvious estimate

$$\log D(x_1, x_2, x_3; f) = \mathcal{O}(x_1 - x_3), \tag{5}$$

Proposition 2 implies a stronger (for small Δ) estimate for cross-ratio distortion:

$$\log \mathrm{Dist}(x_1, x_2, x_3, x_4; f) = (x_1 - x_3)\mathcal{O}(\Delta^\alpha) \tag{6}$$

3 Circle diffeomorphisms

3.1 Settings and notations

For an orientation-preserving homeomorphism T of the unit circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, its *rotation number* $\rho = \rho(T)$ is the value of the limit $\lim_{i \to \infty} L_T^i(x)/i$ for a lift L_T of T from $\mathbb{T}^1$ onto $\mathbb{R}$. It is known since Poincare that rotation number is always defined (up to an additive integer) and does not depend on the starting point $x \in \mathbb{R}$. Rotation number ρ is irrational if and only if T has no periodic points. We restrict our attention in this paper to this case. The order of points on the circle for any trajectory $\xi_i = T^i \xi_0$, $i \in \mathbb{Z}$, coincides with the order of points for the *rigid rotation*

$$R_\rho : \quad \xi \mapsto \xi + \rho \mod 1$$

This fact is sometimes referred to as the *combinatorial equivalence* between T and R_ρ.

We shall use the *continued fraction* expansion for the (irrational) rotation number:

$$\rho = [k_1, k_2, \ldots, k_n, \ldots] = \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{\cdots \cfrac{}{k_n + \cfrac{1}{\cdots}}}}} \in (0, 1) \tag{7}$$

which, as usual, is understood as a limit of the sequence of *rational convergents* $p_n/q_n = [k_1, k_2, \ldots, k_n]$. The positive integers k_n, $n \geq 1$, called *partial quotients*, are defined uniquely for irrational ρ. The mutually prime positive integers p_n and q_n satisfy the recurrent relation $p_n = k_n p_{n-1} + p_{n-2}$, $q_n = k_n q_{n-1} + q_{n-2}$ for $n \geq 1$, where it is convenient to define $p_0 = 0$, $q_0 = 1$ and $p_{-1} = 1$, $q_{-1} = 0$.

Given a circle homeomorphism T with irrational ρ, one may consider a *marked trajectory* (i.e. the trajectory of a marked point) $\xi_i = T^i \xi_0 \in \mathbb{T}^1$, $i \geq 0$, and pick out of it the sequence of the *dynamical convergents* ξ_{q_n}, $n \geq 0$, indexed by the denominators of the consecutive rational convergents to ρ. We will also conventionally use $\xi_{q_{-1}} = \xi_0 - 1$. The well-understood arithmetical properties of rational convergents and the combinatorial equivalence between T and R_ρ imply that the dynamical convergents approach the marked point, alternating their order in the following way:

$$\xi_{q_{-1}} < \xi_{q_1} < \xi_{q_3} < \cdots < \xi_{q_{2m+1}} < \cdots < \xi_0 < \cdots < \xi_{q_{2m}} < \cdots < \xi_{q_2} < \xi_{q_0} \tag{8}$$

We define the nth *fundamental segment* $\Delta^{(n)}(\xi)$ as the circle arc $[\xi, T^{q_n}\xi]$ if n is even and $[T^{q_n}\xi, \xi]$ if n is odd. If there is a marked trajectory, then we use the notations $\Delta_0^{(n)} = \Delta^{(n)}(\xi_0)$, $\Delta_i^{(n)} = \Delta^{(n)}(\xi_i) = T^i \Delta_0^{(n)}$. What is important for us about the combinatorics of trajectories can be formulated as the following simple

Lemma 1. *For any $\xi \in \mathbb{T}^1$ and $0 < i < q_{n+1}$ the segments $\Delta^{(n)}(\xi)$ and $\Delta^{(n)}(T^i\xi)$ are disjoint (except at the endpoints).*

Proof. Follows from the combinatorial equivalence of T to R_ρ and the following arithmetical fact: the distance from $i\rho$ to the closest integer is not less than Δ_n for $0 < i < q_{n+1}$ (and equals Δ_n only for $i = q_n$, in which case $\Delta^{(n)}(\xi)$ and $\Delta^{(n)}(T^i\xi)$ have a common endpoint $T^{q_n}\xi$). $\qquad\square$

In particular, for any ξ_0 all the segments $\Delta_i^{(n)}$, $0 \leq i < q_{n+1}$, are disjoint.

Let us denote $l_n = l_n(T) = \max_\xi |\Delta^{(n)}(\xi)| = \|T^{q_n} - \mathrm{Id}\|_0$ and $\Delta_n = l_n(R_\rho) = |q_n\rho - p_n|$. Obviously $l_n, \Delta_n \in (0, 1)$ for $n \geq 0$, while $l_{-1} = \Delta_{-1} = 1$.

Lemma 2. $l_n \geq \Delta_n$.

Proof. Denote by μ the unique probability invariant measure for T. It follows from the ergodicity of T^{q_n} with respect to μ that

$$\int_{\mathbb{T}^1} (T^{q_n}(\xi) - \xi) d\mu(\xi) = \rho(T^{q_n}) \mod 1$$

Since $\rho(T^{q_n}) = \rho(R_\rho^{q_n}) = (-1)^n \Delta_n \mod 1$, we have

$$\int_{\mathbb{T}^1} |\Delta^{(n)}(\xi)| d\mu(\xi) = \Delta_n,$$

which implies the statement of the lemma. $\qquad\square$

It is well known that $\Delta_n \sim \frac{1}{q_{n+1}}$, thus the Diophantine properties of $\rho \in D_\delta$ can be equivalently expressed in the form:

$$\Delta_{n-1}^{1+\delta} = \mathcal{O}(\Delta_n) \tag{9}$$

3.2 Denjoy's theory

The following set of statements essentially summarizes the classical Denjoy theory (see [7]), which holds for any orientation-preserving circle diffeomorphism $T \in C^{1+BV}(\mathbb{T}^1)$ with irrational rotation number ρ.

A. $\log(T^{q_n})'(\xi_0) = \mathcal{O}(1)$.

B. There exists $\lambda \in (0,1)$ such that $\frac{|\Delta_0^{(n+m)}|}{|\Delta_0^{(n)}|} = \mathcal{O}(\lambda^m)$.

C. There exists a homeomorphism ϕ that conjugates T to R_ρ:

$$\phi \circ T \circ \phi^{-1} = R_\rho \tag{10}$$

In order to prove Theorem 1 one has to show that $\phi \in C^{1+\alpha-\delta}(\mathbb{T}^1)$ and $\phi' > 0$.

3.3 Denjoy-type inequality

The aim of this subsection is to prove the following result that does not require any restrictions on the rotation number of T.

Proposition 3 (Denjoy-type inequality). *Let T be a $C^{2+\alpha}$-smooth, $\alpha \in [0,1]$, orientation-preserving circle diffeomorphism with irrational rotation number. Then*

$$(T^{q_n})'(\xi) = 1 + \mathcal{O}(\varepsilon_n), \quad \text{where} \quad \varepsilon_n = l_{n-1}^\alpha + \frac{l_n}{l_{n-1}}l_{n-2}^\alpha + \frac{l_n}{l_{n-2}}l_{n-3}^\alpha + \cdots + \frac{l_n}{l_0} \tag{11}$$

Remark 2. The inequality (11) can be re-written as

$$\log(T^{q_n})'(\xi) = \mathcal{O}(\varepsilon_n)$$

Remark 3. In the paper [1] there was a wrong claim (Lemma 12) that one can simply put $\varepsilon_n = l_{n-1}^\alpha$ in (11). This is not true in the case when l_{n-1} is too small in comparison with l_{n-2}, though comparable with l_n.

In order to prove Proposition 3, we introduce the functions

$$M_n(\xi) = \mathrm{D}(\xi_0, \xi, \xi_{q_{n-1}}; T^{q_n}), \quad \xi \in \Delta_0^{(n-1)};$$

$$K_n(\xi) = \mathrm{D}(\xi_0, \xi, \xi_{q_n}; T^{q_{n-1}}), \quad \xi \in \Delta_0^{(n-2)},$$

where ξ_0 is arbitrary fixed. The following three exact relations (all of them are easy to check) are crucial for our proof:

$$M_n(\xi_0) \cdot M_n(\xi_{q_{n-1}}) = K_n(\xi_0) \cdot K_n(\xi_{q_n}); \tag{12}$$

$$K_{n+1}(\xi_{q_{n-1}}) - 1 = \frac{|\Delta_0^{(n+1)}|}{|\Delta_0^{(n-1)}|}\left(M_n(\xi_{q_{n+1}}) - 1\right); \tag{13}$$

$$\frac{(T^{q_{n+1}})'(\xi_0)}{M_{n+1}(\xi_0)} - 1 = \frac{|\Delta_0^{(n+1)}|}{|\Delta_0^{(n)}|}\left(1 - \frac{(T^{q_n})'(\xi_0)}{K_{n+1}(\xi_0)}\right) \tag{14}$$

We also need the following lemmas.

Lemma 3. $\frac{|\Delta_i^{(n+m)}|}{|\Delta_i^{(n)}|} \sim \frac{|\Delta_j^{(n+m)}|}{|\Delta_j^{(n)}|}$, $0 \leq j - i < q_{n+1}$.

Proof. Due to (2) and (5)

$$\left| \log \frac{|\Delta_j^{(n+m)}| \cdot |\Delta_i^{(n)}|}{|\Delta_i^{(n+m)}| \cdot |\Delta_j^{(n)}|} \right| = |\log D(\xi_{i+q_{n+m}}, \xi_i, \xi_{i+q_n}; T^{j-i})|$$

$$\leq \sum_{k=i}^{j-1} |\log D(\xi_{k+q_{n+m}}, \xi_k, \xi_{k+q_n}; T)| = \mathcal{O}\left(\sum_{k=i}^{j-1} (|\Delta_k^{(n+m)}| + |\Delta_k^{(n)}|) \right) = \mathcal{O}(1)$$

since the circle arcs $\Delta_k^{(n)}$, $i \leq k < j$, are disjoint due to Lemma 1; the same is true for the arcs $\Delta_k^{(n+m)}$, $i \leq k < j$. $\qquad \square$

Lemma 4. $\frac{|\Delta_0^{(n+m)}|}{|\Delta_0^{(n)}|} = \mathcal{O}\left(\frac{l_{n+m}}{l_n} \right)$.

Proof. Pick out the point $\xi^* \in \mathbb{T}^1$ such that $|\Delta^{(n)}(\xi^*)| = l_n$. Due to combinatorics of trajectories, there exists $0 \leq i < q_{n+1} + q_n$ such that $\xi_{i+q_n} \in \Delta^{(n)}(\xi^*)$, and so $\Delta^{(n)}(\xi^*) \subset \Delta_i^{(n)} \cup \Delta_{i+q_n}^{(n)}$. It follows that there exists $0 \leq i^* < q_{n+1} + 2q_n$ (either $i^* = i$ or $i^* = i + q_n$) such that $\Delta_{i^*}^{(n)} \geq l_n/2$, and so $\frac{|\Delta_{i^*}^{(n+m)}|}{|\Delta_{i^*}^{(n)}|} \leq \frac{2l_{n+m}}{l_n}$. The statement now follows from Lemma 3 (since $q_{n+1} + 2q_n < 3q_{n+1}$, we need to apply Lemma 3 at most three times). $\qquad \square$

Lemma 5.

$$\log \text{Dist}(\xi_0, \xi, \xi_{q_{n-1}}, \eta; T^{q_n}) = \mathcal{O}(l_{n-1}^\alpha), \quad \xi, \eta \in \Delta_0^{(n-1)};$$
$$\log \text{Dist}(\xi_0, \xi, \xi_{q_n}, \eta; T^{q_{n-1}}) = \mathcal{O}(l_n^\alpha), \quad \xi, \eta \in \Delta_0^{(n-2)}$$

Proof. Follows from (3), (6) and Lemma 1 similar to the proof of Lemma 3. $\qquad \square$

Lemma 6. $\frac{l_{n+m}}{l_n} = \mathcal{O}(\lambda^m)$.

Proof. Pick out the point $\xi^* \in \mathbb{T}^1$ such that $|\Delta^{(n+m)}(\xi^*)| = l_{n+m}$. It is easy to see that $\frac{l_{n+m}}{l_n} \leq \frac{|\Delta^{(n+m)}(\xi^*)|}{|\Delta^{(n)}(\xi^*)|} = \mathcal{O}(\lambda^m)$ due to the statement **A** above. $\qquad \square$

Proof of Proposition 3. Since $M_n(\xi)/M_n(\eta) = \text{Dist}(\xi_0, \xi, \xi_{q_{n-1}}, \eta; T^{q_n})$ and $K_n(\xi)/K_n(\eta) = \text{Dist}(\xi_0, \xi, \xi_{q_n}, \eta; T^{q_{n-1}})$, Lemma 5 implies that $M_n(\xi)/M_n(\eta) = 1 + \mathcal{O}(l_{n-1}^\alpha)$ and $K_n(\xi)/K_n(\eta) = 1 + \mathcal{O}(l_n^\alpha)$. Due to the statement **B** above, the functions M_n and K_n are bounded from above and from below uniformly in n. This gives us

$$M_n(\xi) = m_n + \mathcal{O}(l_{n-1}^\alpha), \qquad K_n(\xi) = m_n + \mathcal{O}(l_n^\alpha) \tag{15}$$

where m_n^2 denotes the products in (12). Due to (13) and (15) we have

$$m_{n+1} - 1 = \frac{|\Delta_0^{(n+1)}|}{|\Delta_0^{(n-1)}|}(m_n - 1) + \mathcal{O}(l_{n+1}^\alpha), \tag{16}$$

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which is iterated into

$$m_n - 1 = \mathcal{O}\left(\sum_{k=0}^{n} l_{n-k}^{\alpha} \frac{|\Delta_0^{(n)}|}{|\Delta_0^{(n-k)}|} \frac{|\Delta_0^{(n-1)}|}{|\Delta_0^{(n-k-1)}|}\right)$$

$$= \mathcal{O}\left(l_n^{\alpha} \sum_{k=0}^{n} \left(\frac{l_n}{l_{n-k}}\right)^{1-\alpha} \frac{l_{n-1}}{l_{n-k-1}}\right) = \mathcal{O}\left(l_n^{\alpha} \sum_{k=0}^{n} (\lambda^{2-\alpha})^k\right) = \mathcal{O}(l_n^{\alpha})$$

due to Lemmas 4 and 6. Hence,

$$M_n(\xi) = 1 + \mathcal{O}(l_{n-1}^{\alpha}), \qquad K_n(\xi) = 1 + \mathcal{O}(l_n^{\alpha}) \tag{17}$$

Due to (14) and (17) we have

$$(T^{q_{n+1}})'(\xi_0) - 1 = \frac{|\Delta_0^{(n+1)}|}{|\Delta_0^{(n)}|}(1 - (T^{q_n})'(\xi_0)) + \mathcal{O}(l_n^{\alpha}) \tag{18}$$

which is iterated into

$$(T^{q_n})'(\xi_0) - 1 = \mathcal{O}\left(\sum_{k=0}^{n} l_{n-k-1}^{\alpha} \frac{|\Delta_0^{(n)}|}{|\Delta_0^{(n-k)}|}\right)$$

The statement of the proposition now follows from Lemma 4. $\qquad \square$

Remark 4. Due to Lemma 6, $\varepsilon_n = \mathcal{O}(\lambda^{\alpha n})$ for $0 \le \alpha < 1$ and $\varepsilon_n = \mathcal{O}(n\lambda^n)$ for $\alpha = 1$, so ε_n decays exponentially for $\alpha > 0$.

3.4 Exponential bound on $k_{n+1}\varepsilon_n$

Let $r(n+m, n)$, $m \ge 0$, be the number of indices $0 \le i < q_{n+m+1}$ such that $\Delta_i^{(n+m)} \subset \Delta_0^{(n)}$. It is easy to see that $r(n, n) = 1$, $r(n+1, n) = k_{n+2}$, $r(n+m, n) = r(n+m-1, n)k_{n+m+1} + r(n+m-2, n)$ for $m \ge 2$.

Lemma 7. *There exists a constant $C > 0$ such that*

$$\frac{l_n}{l_{n+m}} \ge r(n+m, n)\left(1 - C\sum_{s=n+1}^{n+m} k_{s+1}\varepsilon_s\right) \tag{19}$$

Proof. If $\Delta_i^{(n+m)}$ and $\Delta_j^{(n+m)}$, $0 \le i, j < q_{n+m+1}$, are contained in $\Delta_0^{(n)}$, then

$$\log \frac{|\Delta_i^{(n+m)}|}{|\Delta_j^{(n+m)}|} = \mathcal{O}\left(\sum_{s=n+1}^{n+m} k_{s+1}\varepsilon_s\right)$$

due to the combinatorics of dynamical partitions and Proposition 3. (One of the segments $\Delta_i^{(n+m)}$ and $\Delta_j^{(n+m)}$ is mapped onto another by a composition of no more than k_{n+2} maps $T^{q_{n+1}}$, no more than k_{n+3} maps $T^{q_{n+2}}$, ..., and no more than k_{n+m+1} maps $T^{q_{n+m}}$.) It follows that there exists $C > 0$ such that

$$|\Delta_0^{(n)}| \ge r(n+m, n)|\Delta_i^{(n+m)}|\left(1 - C\left(\sum_{s=n+1}^{n+m} k_{s+1}\varepsilon_s\right)\right)$$

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for any fixed $0 \le i < q_{n+m+1}$ such that $\Delta_i^{(n+m)} \subset \Delta_0^{(n)}$. Now we choose ξ_0 in such a way that $|\Delta_i^{(n+m)}| = l_{n+m}$ and obtain (19). $\qquad\square$

Proposition 4. *For any chosen $\lambda_0 \in (\lambda^{\alpha-\delta}, 1)$, the following asymptotics hold:*

$$k_{n+1}\varepsilon_n = \mathcal{O}(\lambda_0^n)$$

Proof. Consider the sequence n_i, $i \ge 0$, of all indices n such that $k_{n+1}\varepsilon_n > \lambda_0^n$, and assume it to be infinite. Similarly to proof of Lemma 7, we have $|\Delta_{q_{n-1}+kq_n}^{(n)}| \ge |\Delta_{q_{n-1}}^{(n)}|(1 - Ck\varepsilon_n)$. Choosing $k_{n_i+1}^* \le k_{n_i+1}$ in such a way that $k_{n_i+1}^*\varepsilon_{n_i} > \lambda_0^{n_i}$ but $Ck_{n_i+1}^*\varepsilon_{n_i} \le \frac{1}{2}$ (it is possible for large enough i since both λ_0^n and ε_n decay exponentially), we achieve $|\Delta_0^{(n_i-1)}| \ge \sum_{k=0}^{k_{n_i+1}^*-1} |\Delta_{q_{n_i-1}+kq_{n_i}}^{(n_i)}| \ge \frac{1}{2}k_{n_i+1}^*|\Delta_{q_{n_i-1}}^{(n_i)}|$. With ξ_0 such that $|\Delta_{q_{n_i-1}}^{(n_i)}| = l_{n_i}$, this implies

$$\frac{l_{n_i-1}}{l_{n_i}} > \frac{\lambda_0^{n_i}}{2\varepsilon_{n_i}} \tag{20}$$

From the equality $\varepsilon_{n_i} = l_{n_i-1}^\alpha + \frac{l_{n_i}}{l_{n_i-1}}\varepsilon_{n_i-1}$, in view of (20) we get $\varepsilon_{n_i}(1 - 2\varepsilon_{n_i-1}\lambda_0^{-n_i}) < l_{n_i-1}^\alpha$. Since $\varepsilon_n\lambda_0^{-n}$ decays exponentially (see Remark 4), this proves that $\varepsilon_{n_i} = \mathcal{O}(l_{n_i-1}^\alpha)$. Hence, (20) implies

$$l_{n_i} = \mathcal{O}(l_{n_i-1}^{1+\alpha}\lambda_0^{-n_i}) \tag{21}$$

Due to Lemma 7, $\frac{l_{n_i-1}}{l_{n_i-1}} \ge r(n_i-1, n_{i-1})\left(1 - C\sum_{s=n_{i-1}+1}^{n_i-1} \lambda_0^s\right) \ge \frac{1}{2}r(n_i-1, n_{i-1})$ for large enough i, so

$$l_{n_i-1} = \mathcal{O}\left(\frac{l_{n_i-1}}{r(n_i-1, n_{i-1})}\right) \tag{22}$$

The estimate (21) and Lemma 6 imply $l_{n_i} = \mathcal{O}(l_{n_i-1}^{1+\delta+\kappa}\lambda^{(\alpha-\delta-\kappa)n_i}\lambda_0^{-n_i})$ for any $\kappa \in (0, \alpha - \delta)$. Having taken κ so small that $\lambda^{\alpha-\delta-\kappa} < \lambda_0$ and using (22), we achieve

$$l_{n_i} \le \left(\frac{l_{n_i-1}}{r(n_i-1, n_{i-1})}\right)^{1+\delta+\kappa} \tag{23}$$

for large enough i.

Now we start to use the Diophantine properties of rotation number ρ. We have $\Delta_n = r(n+m, n)\Delta_{n+m} + r(n+m-1, n)\Delta_{n+m+1}$, so $\Delta_{n_i-1} = \mathcal{O}(r(n_i-1, n_{i-1})\Delta_{n_{i-1}})$. The property (9) implies $\Delta_{n_i-1}^{1+\delta+\kappa/2} = \mathcal{O}(\Delta_{n_i}\Delta_{n_i-1}^{\kappa/2})$, hence

$$\left(\frac{\Delta_{n_i-1}}{r(n_i-1, n_{i-1})}\right)^{1+\delta+\kappa/2} \le \Delta_{n_i} \tag{24}$$

for large enough i.

Notice, that $0 < \Delta_n \le l_n < 1$ for all n. It follows from (23) and (24) that

$$\frac{\log l_{n_i}}{\log \Delta_{n_i}} \ge \frac{1+\delta+\kappa}{1+\delta+\kappa/2} \cdot \frac{\log l_{n_i-1} - \log r(n_i-1, n_{i-1})}{\log \Delta_{n_i-1} - \log r(n_i-1, n_{i-1})} \ge K\frac{\log l_{n_i-1}}{\log \Delta_{n_i-1}} \tag{25}$$

for large enough i, with $K = \frac{1+\delta+\kappa}{1+\delta+\kappa/2} > 1$, so the sequence $\gamma_i = \frac{\log l_{n_i}}{\log \Delta_{n_i}} > 0$ is unbounded. But $\gamma_i \le 1$ due to Lemma 2. This contradiction proves that $k_{n+1}\varepsilon_n \le \lambda_0^n$ for large enough n. $\qquad\square$

3.5 C^1-smoothness of ϕ

There is more than one way to derive C^1-smoothness of the conjugacy from the convergence of the series $\sum_n k_{n+1}\varepsilon_n$. We will construct the continuous density $h : \mathbb{T}^1 \to (0, +\infty)$ of the invariant probability measure for T, as in [1].

Proposition 5. ϕ is a C^1-smooth diffeomorphism.

Proof. Consider arbitrary trajectory $\Xi = \{\xi_i, i \in \mathbb{Z}\}$, $\xi_i = T^i \xi_0$, and define a function $\gamma : \Xi \to \mathbb{R}$ by use of the following recurrent relation:

$$\gamma(\xi_0) = 0; \quad \gamma(\xi_{i+1}) = \gamma(\xi_i) - \log T'(\xi_i), \quad i \in \mathbb{Z}$$

As soon as $\xi_j \in \Delta_i^{(n)}$, $j > i$, we have

$$\gamma(\xi_i) - \gamma(\xi_j) = \mathcal{O}\left(\varepsilon_n + \sum_{s=n+1}^{+\infty} k_{s+1}\varepsilon_s\right) = \mathcal{O}(\lambda_0^n) \to 0, \quad n \to +\infty$$

due to combinatorics of a trajectory and Proposition 3. It follows that $\gamma \in C(\Xi)$. Since Ξ is dense in $\mathbb{T}^1$, the function γ can be extended continuously onto $\mathbb{T}^1$. The function $h(\xi) = e^{\gamma(\xi)} \left(\int_{\mathbb{T}^1} e^{\gamma(\eta)} d\eta\right)^{-1}$ is continuous and positive on $\mathbb{T}^1$, satisfies the *homological equation*

$$h(T\xi) = \frac{1}{T'(\xi)} h(\xi), \quad \xi \in \mathbb{T}^1, \tag{26}$$

and $\int_{\mathbb{T}^1} h(\xi) d\xi = 1$. It is easy to check that the C^1-smooth diffeomorphism

$$\phi(\xi) = \int_{\xi_0}^{\xi} h(\eta) d\eta, \quad \xi \in \mathbb{T}^1$$

conjugates T to R_ρ. $\qquad \square$

3.6 $C^{\alpha-\delta}$-smoothness of h

A straightforward corollary of Proposition 5 is that $l_n \sim \Delta_n$.

Lemma 8. $\varepsilon_n = \mathcal{O}(\Delta_n^{\frac{\alpha}{1+\delta}})$.

Proof. The Diophantine condition $\Delta_{n-1}^{1+\delta} = \mathcal{O}(\Delta_n)$ implies that

$$\varepsilon_n = \mathcal{O}\left(\sum_{m=0}^{n} \frac{\Delta_n}{\Delta_{n-m}} \Delta_{n-m-1}^{\alpha}\right) = \mathcal{O}\left(\Delta_n \sum_{m=0}^{n} \Delta_{n-m}^{\frac{\alpha}{1+\delta}-1}\right) =$$

$$\mathcal{O}\left(\Delta_n^{\frac{\alpha}{1+\delta}} \sum_{m=0}^{n} \left(\frac{\Delta_n}{\Delta_{n-m}}\right)^{\frac{1-\alpha+\delta}{1+\delta}}\right) = \mathcal{O}(\Delta_n^{\frac{\alpha}{1+\delta}}),$$

since $\frac{\Delta_n}{\Delta_{n-m}} = \mathcal{O}(\lambda^m)$ is exponentially small in m. $\qquad \square$

Remark 5. Since $k_{n+1}\Delta_n < \Delta_{n-1} = \mathcal{O}(\Delta_n^{\frac{1}{1+\delta}})$, Lemma 8 implies that

$$k_{n+1}\varepsilon_n = \mathcal{O}(\Delta_n^{\frac{\alpha-\delta}{1+\delta}}) = \mathcal{O}(\Delta_{n-1}^{\alpha-\delta})$$

This improves Proposition 4 a posteriori.

Proposition 6. $h \in C^{\alpha-\delta}(\mathbb{T}^1)$.

Proof. Consider two points $\xi_0, \xi \in \mathbb{T}^1$ and such n that $\Delta_n \leq |\phi(\xi) - \phi(\xi_0)| < \Delta_{n-1}$. Let k be the greatest positive integer such that $|\phi(\xi) - \phi(\xi_0)| \geq k\Delta_n$. (It follows that $1 \leq k \leq k_{n+1}$.) Due to the combinatorics of trajectories, continuity of h and the homologic equation (26), we have

$$|\log h(\xi) - \log h(\xi_0)| = \mathcal{O}\left(k\varepsilon_n + \sum_{s=n+1}^{+\infty} k_{s+1}\varepsilon_s\right)$$

The right-hand side here is bounded and so is h, hence the same estimate holds for $|h(\xi)-h(\xi_0)|$. By Lemma 8, we have

$$k\varepsilon_n = \mathcal{O}\left(k^{\alpha-\delta}\Delta_n^{\alpha-\delta}\left(\frac{k\Delta_n}{\Delta_{n-1}}\right)^{1-\alpha+\delta}\right) = \mathcal{O}((k\Delta_n)^{\alpha-\delta})$$

and

$$\sum_{s=n+1}^{+\infty} k_{s+1}\varepsilon_s = \mathcal{O}\left(\sum_{s=n+1}^{+\infty} \Delta_{s-1}^{\alpha-\delta}\right) = \mathcal{O}(\Delta_n^{\alpha-\delta}),$$

so $|h(\xi) - h(\xi_0)| = \mathcal{O}((k\Delta_n)^{\alpha-\delta}) = \mathcal{O}(|\phi(\xi) - \phi(\xi_0)|^{\alpha-\delta}) = \mathcal{O}(|\xi - \xi_0|^{\alpha-\delta})$. □

References

1. Ya. G. Sinai and K. M. Khanin, Smoothness of conjugacies of diffeomorphisms of the circle with rotations, *Uspekhi Mat. Nauk* **44**:1 (1989), 57-82 (in Russian); English transl., *Russian Math. Surveys* **44**:1 (1989), 69-99.

2. M.-R. Herman. Sur la conjugaison differentiable des diffeomorphismes du cercle a des rotations, *I. H. E. S. Publ. Math.*, **49**: 5–233, 1979.

3. J.-C. Yoccoz. Conjugaison differentiable des diffeomorphismes du cercle dont le nombre de rotation verifie une condition diophantienne. *Ann. Sci. Ecole Norm. Sup. (4)*, **17** (3): 333–359, 1984.

4. K. M. Khanin and Ya. G. Sinai. A new proof of M. Herman's theorem, *Comm. Math. Phys.*, **112** (1): 89–101, 1987.

5. Y. Katznelson and D. Ornstein. The differentiability of the conjugation of certain diffeomorphisms of the circle, *Ergodic Theory Dynam. Systems*, **9**:4 (1989), 643–680.

6. Y. Katznelson and D. Ornstein. The absolute continuity of the conjugation of certain diffeomorphisms of the circle, *Ergodic Theory Dynam. Systems*, **9** (4): 681–690, 1989.

7. Ya. G. Sinai. Topics in ergodic theory. Princeton Mathematical Series, 44. Princeton Univ. Press, Princeton, NJ, 1994.