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On Dirac Physical Measures for Transitive Flows

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Abstract: We discuss some examples of smooth transitive flows with physical measures supported at fixed points. We give some conditions under which stopping a flow at a point will create a Dirac physical measure at that indifferent fixed point. Using the Anosov-Katok method, we construct transitive flows on surfaces with the only ergodic invariant probabilities being Dirac measures at hyperbolic fixed points. When there is only one such point, the corresponding Dirac measure is necessarily the only physical measure with full basin of attraction. Using an example due to Hu and Young, we also construct a transitive flow on a three-dimensional compact manifold without boundary, with the only physical measure the average of two Dirac measures at two hyperbolic fixed points.

1. Introduction

Let M be a compact manifold, and $f : M \to M$ a continuous map. A probability measure μ on M is *invariant under* f if for any measurable set $A \subset M$ we have $\mu(f^{-1}(A)) = \mu(A)$. The *basin of attraction of* μ, denoted $\mathcal{B}(\mu)$, is the set of points $x \in M$ such that

$$\lim_{n\to\infty} \frac{1}{n}(\delta_x + \delta_{f(x)} + \delta_{f^2(x)} + \cdots + \delta_{f^{n-1}(x)}) = \mu,$$

where δ_y is the Dirac measure at the point $y \in M$, and the limit is with respect to the weak* topology. This means that for any continuous observable $\alpha : M \to \mathbb{R}$, the average of α along the orbit of x will converge to the integral of α with respect to μ, which makes this notion useful in applications. We say that the invariant measure μ is *physical* if $\mathcal{B}(\mu)$ has positive Lebesgue measure on M.

One has similar definitions for the case of flows. If ϕ is a continuous flow on the compact manifold M, then a probability measure μ is *invariant* if $\mu(\phi_t(A)) = \mu(A)$, for any $t \in \mathbb{R}$ and any measurable $A \subset M$. Denote by $m_t(x)$ the probability measure along the orbit of the flow of length t starting at x, i. e.

$$\int_M \alpha dm_t(x) = \frac{1}{t}\int_0^t \alpha(\phi_s(x))ds,$$

for any $\alpha \in C^0(M, \mathbb{R})$. The *basin of attraction* $\mathcal{B}(\mu)$ is in this case the set of points $x \in M$ such that

$$\lim_{t\to\infty} m_t(x) = \mu.$$

The measure μ is *physical* if the basin of attraction has again positive Lebesgue measure. Physical measures are important in dynamics, because they describe statistically the orbit of a significant set of points, and they can be detected experimentally.

There is a large amount of work in the area of physical measures, specially if they are also absolutely continuous with respect to Lebesgue measure, or along unstable manifolds (the SRB measures). Such measures generally have very good statistical properties. However, in this paper we would like to turn our attention to situations where the physical measures are rather bad, for example they are supported only on some fixed points. In order to avoid simple examples such as sinks, or the Bowen eye, we are interested in systems with some recurrence, we will require transitivity at least in the basins of attraction of the measures. In other words, even if a typical orbit will travel everywhere on the manifold (or at least some large subset), it will spend most of the time arbitrarily close to some given points.

The motivation comes from some well-known examples in one dimensional dynamics. There are examples of non-invertible maps of the interval (or the circle) such that the only physical measure is supported on an indifferent fixed point. Suppose f is a piecewise smooth expanding map of the interval (or the circle), with $f(x) = x + x^a + o(x^a)$ in a neighborhood of the origin, and the origin is the only indifferent fixed point of f ($a > 1$). If $a < 2$, then f has an absolutely continuous finite invariant measure, but if $a \geq 2$, then there is no absolutely continuous finite invariant measure, the Dirac measure at the origin is the only physical measure, and its basin of attraction has full Lebesgue measure (see [6]).

There are also more unexpected examples of maps in the logistic family which have the only physical measure supported on a repelling fixed point. In [3] it is shown that, in any full continuous family of S-unimodal maps, there are uncountable many parameters such that, for the corresponding maps, the Dirac measure at the positive repelling fixed point is the unique physical measure, and its basin of attraction has full Lebesgue measure.

These maps are transitive (at least on some subset). These maps are also not invertible; this phenomena cannot happen for invertible maps in dimension 1. We are interested in the existence of such examples in higher dimensions and for smooth invertible systems. The simplest systems to consider (and in the same time the most restrictive) are the flows on surfaces. Examples in higher dimensions and/or for maps should be easier to construct.

In Sect. 2 we discuss some transitive flows on surfaces with physical measures supported on indifferent fixed points. The existence of such flows is not surprising, they can be easily obtained by stopping a transitive flow at a point for example. We give some conditions under which a reparametrization of a given transitive flow creates a physical measure at an indifferent fixed point.

In Sect. 3 we give examples of transitive flows on surfaces with the only physical measure supported on a hyperbolic fixed point. The construction of these flows is more difficult, and requires the construction of diffeomorphisms of the circle with 'bad' invariant measures - this is done in Sect. 4, and uses a method due to Anosov and Katok. Unfortunately the flows we obtain are either transitive only inside the basin of attraction of the Dirac measure at the fixed point, or the manifold has boundary.

Finally, in the last section we discuss some other examples with hyperbolic fixed points. We construct flows on surfaces of genus two with only two ergodic invariant measures supported at two hyperbolic fixed points, but we cannot conclude that a combination of them must be a physical measure, or there is no physical measure at all. We also give an example of a transitive flow on a three dimensional compact manifold without boundary with the only physical measure being the average of two Dirac measures at two hyperbolic fixed points.

2. Indifferent Fixed Points

In this section we will discuss about 'singular' reparametrizations of flows which create physical measures supported on indifferent fixed points. Here by 'singular' reparametrizations we mean that we allow the creation of fixed points, so the new flow is not exactly equivalent to the initial one, but the orbits of the initial flow are unions of orbits of the new flow (eventually with fixed points). The following discussion is more general, but the example to have in mind is an irrational linear flow on the torus, modified such that it has a fixed point, while the trajectories are the same straight lines. Some of the following facts about time-changes can be also found in other papers like [5 or 8], even in more general conditions; however we include them here because they are simple and intuitive in the smooth case.

Suppose we have a smooth flow ϕ on a compact manifold M. Let X be the vector field generating ϕ, and $f : M \to [0, \infty)$ a smooth function. Let $Y = fX$ and let ψ be the flow generated by Y. Let $Z = \{x \in M : f(x) = 0\}$ be the zero set of f. We would like to investigate if the invariant measures of ϕ will 'survive' when we modify the flow to ψ, and how this depends on f.

The function f will induce a linear continuous operator $T : \mathcal{M}(M) \to \mathcal{M}(M)$ on the set of signed finite Borel measures on M in the following way: given a finite measure ν on M, let $T\nu$ be the unique finite measure such that f is the Radon-Nikodym derivative of $T\nu$ with respect to ν. In other words we have $T\nu(A) = \int_A f d\nu$ for any measurable set A, or $\int_M g dT\nu = \int_M gf d\nu$ for any $g \in C^0(M, \mathbb{R})$. $T\nu$ is absolutely continuous with respect to ν, and ν is supported on the zero set of f iff $T\nu$ is the zero measure, so the kernel of the operator is the set of finite measures supported on Z. The image is the set of finite measures μ such that $\int_M \frac{1}{f} d|\mu| < \infty$. If μ satisfies this condition, then there is a 'distinguished' element in $T^{-1}(\mu)$: it is the measure ν with the property that $\nu|_Z = 0$; we will denote this as $S\mu$ (this is some kind of partial inverse, defined only on the image of T). This can also be defined by $S\mu(A) = \int_A \frac{1}{f} d\mu$, or $\int_M g dS\mu = \int_M \frac{g}{f} d\mu$.

If ν is an invariant measure for ψ, then

$$\int_M g d\nu = \int_M g \circ \psi_t d\nu, \quad \forall t \in \mathbb{R}, \forall g \in C^0(M, \mathbb{R}),$$

and from here we get

$$\lim_{t \to 0} \int_M \frac{1}{t}(g \circ \psi_t - g) d\nu = \int_M Y(g) d\nu = \int_M fX(g) d\nu = 0, \quad \forall g \in C^1(M, \mathbb{R}),$$

and then

$$\int_M X(g)dT\nu = 0, \quad \forall g \in C^1(M, \mathbb{R}).$$

If we denote $I(t) = \int_M g \circ \phi_t dT\nu$, for some fixed $g \in C^1(M, \mathbb{R})$, then we have

$$I'(t) = \int_M X(g \circ \phi_t)dT\nu = 0.$$

This means that $I(t)$ is constant for every $g \in C^1(M, \mathbb{R})$, or

$$\int_M gdT\nu = \int_M g \circ \phi_t dT\nu, \quad \forall t \in \mathbb{R}, \forall g \in C^1(M, \mathbb{R}).$$

But $C^1(M, \mathbb{R})$ is dense in $C^0(M, \mathbb{R})$ in the C^0 topology, so we get that $T\nu$ must be an invariant measure for ϕ. In other words $T\mathcal{M}_\psi \subset \mathcal{M}_\phi$, T sends invariant measures of ψ to invariant measures of ϕ. In a similar way one can prove that S sends invariant measures of ψ to invariant measures of ϕ.

If ν is an ergodic invariant probability for ψ, then $T\nu$ is either zero (if ν is supported on Z) or an ergodic invariant measure for ϕ, and it can be made an ergodic probability after rescaling. Conversely, if μ from the image of T is an ergodic probability for ϕ, then $S(\mu)$ is an ergodic invariant measure for ψ and it can be made a probability after rescaling.

We also have $\mathrm{supp}(T(\nu)) = \mathrm{cl}(\mathrm{supp}(\nu) \setminus Z)$ and for μ in the image of T we have $\mathrm{supp}(S(\mu)) = \mathrm{supp}(\mu)$.

Regarding the basins of attraction, let us denote first

$$\mathcal{B}_\psi(Z) = \{x \in M : m_{t,\psi}(x) \text{ has all the weak limit measures supported in Z}\},$$

where $M_{t,\psi}(x)$ is the measure given by the piece of orbit of ψ staring at x and for time t, and $m_{t,\psi}(x)$ is the corresponding probability measure obtained by rescaling. Let $\mathcal{M}_\psi(x)$ be the limit set of $m_{t,\psi}(x)$ as t tends to infinity. We also have the same notations for ϕ.

For any $x \in M$ and $t \geq 0$ there exist $s(t) \geq 0$ such that $\phi_{s(t)}(x) = \psi_t(x)$ (this is because the speed of ϕ is greater or equal to a constant times the speed of ψ; the other direction is not always true). Then

$$\frac{d\psi_t(x)}{dt} = Y(\psi_t(x)) = f(\psi_t(x))X(\psi_t(x)) = \frac{d\phi_{s(t)}(x)}{dt} = s'(t)X(\psi_t(x))$$

or $ds = f(\psi_t(x))dt$. Also

$$TM_{t,\psi}(x)(g) = \int_0^t g(\psi_u(x))f(\psi_u(x))du = \int_0^{s(t)} g(\phi_v(x))dv = M_{s(t),\phi}(x)(g),$$

for any $g \in C^1(M, \mathbb{R})$, by using the change of variables $v = s(u)$. So, because $C^1(M, \mathbb{R})$ is dense in $C^0(M, \mathbb{R})$, we have $TM_{t,\psi}(x) = M_{s(t),\phi}(x)$, or $Tm_{t,\psi}(x) = \frac{s(t)}{t}m_{s(t),\phi}(x)$. From this we can conclude that the measure ν is in $\mathcal{M}_\psi(x)$ iff $\mu = T\nu$ is a limit of $\frac{s(t)}{t}m_{s(t),\phi}(x)$. We get the following result regarding physical measures for ϕ and ψ.

Proposition 1. *Let ϕ be the flow on the compact manifold M generated by the smooth vector field X, and ψ the flow generated by the vector field fX, where $f : M \to [0,\infty)$ is smooth. If $x \in M$ is in the basin of attraction of the invariant measure μ for ϕ, and $\int_M \frac{1}{f} d\mu = \infty$, then $x \in \mathcal{B}(Z)$. If the zero set of f consists of a single point, $Z = \{p\}$, then $\mathcal{B}(Z) = \mathcal{B}(\delta_p)$, or the basin of of attraction (w.r.t. ψ) of the Dirac measure at p contains the basin of attraction of μ (w.r.t. ϕ).*

Proof. Assume that $x \in \mathcal{B}(\mu)$ for some invariant measure μ of ϕ, then we have $\lim_{s\to\infty} m_{s,\phi}(x) = \mu$. If $\int_M \frac{1}{f} d\mu = \infty$, then μ is not in the image of T. If $\lim_{t\to\infty} \frac{s(t)}{t} \neq 0$, then there exist $t_i \to \infty$ such that $\lim_{i\to\infty} \frac{s(t_i)}{t_i} = c > 0$, and eventually passing to a subsequence there exists a measure ν such that $\nu = \lim_{i\to\infty} m_{t_i,\psi}(x)$. But then from the continuity of T one gets that $T\nu = c\mu$, which is a contradiction because μ is not in the image of T. So we must have $\frac{s(t)}{t} \to 0$, and this in turn will imply that $Tm_{t,\psi}(x) \to 0$, and again from the continuity of T we get $x \in \mathcal{B}(Z)$.

Obviously if $Z = \{p\}$ then $\mathcal{B}(Z) = \mathcal{B}(\delta_p)$ and the conclusion follows. □

As an example consider the case when ϕ_t is an irrational translation of the torus, generated by the vector field X. Then ϕ is minimal, transitive, not mixing, uniquely ergodic, the Lebesgue measure being the unique invariant measure. Let f be a function on the torus with a unique zero at the origin, such that around the origin $f(x) \sim |x|^a$, and let ψ be the flow generated by fX. The new flow ψ is not minimal anymore, but it is however topologically mixing. If $a \geq 2$, then it is uniquely ergodic, the only invariant probability measure is δ_0, and its basin of attraction is the whole manifold. If $a < 2$ then ψ will have another ergodic probability measure ν, supported on the entire manifold, with density proportional to $\frac{1}{f}$ with respect to Lebesgue measure. This new measure will have a basin of attraction with full Lebesgue measure (but not the entire manifold).

We remark that when there is a global cross section to the flow ϕ, then one can consider the return map to the section, and then relate the invariant measures of the map with the ones of the flow with the help of the return time map to the transversal. For the modified flow ψ the return map is the same, but the return time will have some singularities corresponding to the zeros of f, and in order to see whether invariant measures survive one has to look at the integral of the return time with respect to the invariant measures on the transversal. We will discuss more about this in the next section.

We should also mention here the example in [4] of a transitive map on the two-torus with the physical measure at a point (see Theorem 1). The fixed point in their example has a contracting direction and a weakly expanding (indifferent) one. The map cannot be made the time one map of a flow because it is not homotopic to the identity, however we will use it in the last section to create flows on dimension three with the unique physical measure supported on hyperbolic fixed points.

3. Hyperbolic Fixed Points

In this section we will investigate the existence of transitive flows on surfaces with physical measures supported on hyperbolic fixed points. This turns out to be more difficult, and a reason for that is the fact that the hyperbolic fixed points create a singularity of the return time map of logarithmic type (unlike indifferent points, where it is usually power-like), and thus it is still integrable with respect to many invariant measures (the ones coming from a Hőlder conjugacy with a circle rotation for example). Consequently

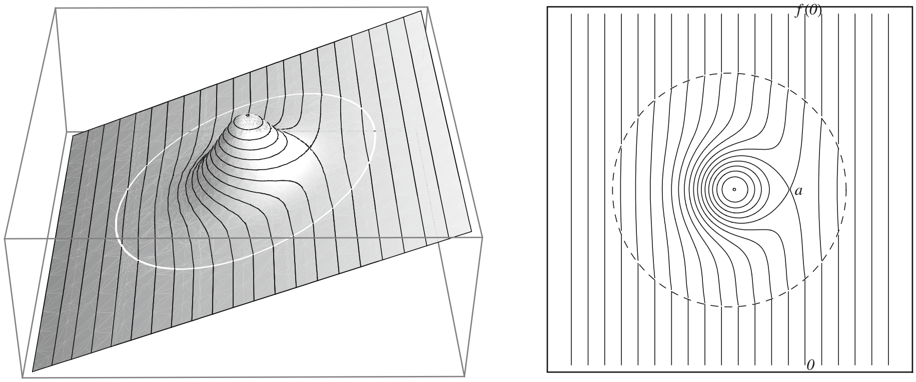

Fig. 1. Saddle-Node bifurcation

we will also need to create a map on the circle (the transversal) with a 'bad' invariant measure (or 'good' for our purpose) - this will be done in the next section.

We are not aware of any other known example of this type. In [4] the fixed point which supports the physical measure is only weakly expanding (although it has a contracting direction). There is also an example in [2], but in that case the map is not transitive, the basin of attraction contains a wandering domain.

Let $f : \mathbb{T}^1 \to \mathbb{T}^1$ be a C^∞ diffeomorphism of the circle conjugated to an irrational rotation, i. e. transitive. Let ϕ be the suspension flow over f with constant roof function 1, defined on a torus $\mathbb{T}^2$. We will consider $\mathbb{T}^1 = \mathbb{R}(\text{mod } \mathbb{Z})$, and we denote $\mathbb{T} = \mathbb{T}^1 \times \{0\} \subset \mathbb{T}^2$. Then the Poincaré return function of the flow ϕ to the circle $\mathbb{T}$ will be $\phi_1|_{\mathbb{T}} = f$. We will make an abuse of notation and consider that $\mathbb{T} = \mathbb{R}(\text{mod } \mathbb{Z})$. One can also define directly the flow on the flat torus, using interpolation between the identity and f, because f is isotopic to the identity.

We will make a saddle-node bifurcation to the suspension flow in the following way. By the Flow Box Theorem, there is a C^∞ chart on some small open domain U where the suspension flow is linear, let's say it is the hamiltonian flow given by the hamiltonian $h_0(x, y) = x$ (the trajectories are vertical lines). We can modify this hamiltonian by adding a C^∞ bump function α supported on some smaller disk inside U, $h_1(x, y) = x + t\alpha(x, y)$. If $t > 0$ is large enough, then h_1 will have two critical points, a saddle and a local maximum. Then the hamiltonian flow associated to h_1 will have a saddle with a homoclinic loop and a center, while it coincides with the initial flow outside the support of α. This can be seen in Fig. 1, where we show the graph of h_1 and some level curves, which represent trajectories of the new hamiltonian flow, together with the disk where the perturbation is supported.

Let ψ be the new C^∞ flow, which coincides with ϕ outside U and is constructed as above inside U. Let also a be the hyperbolic point and γ_0 the homoclinic loop. Without loss of generality we can assume that 0 belongs to the stable manifold of a, while $f(0) = \phi_1(0)$ belongs to the unstable manifold of a. We remark that the return map to the transversal $\mathbb{T}$ is the same f (considering that 0 returns to $f(0)$), because the level curves of h_1 coincide with the ones of h outside the support of α. The return time will change, and there will be a singularity at 0. This flow is not transitive, inside the homoclinic loop it has invariant circles. It is however transitive, even topologically mixing, outside of the loop. To see this let $A, B \subset \mathbb{T}^2$ be two open sets, disjoint with the interior of the homoclinic loop. Then the stable manifold of a intersects A because it is dense outside the homoclinic loop, and in the same way the unstable manifold of a intersects B. Then $\psi_t(A)$ accumulates on the unstable manifold of a when t tends to infinity, so for t sufficiently large it has to intersect B, q.e.d..

Let M be the surface with boundary and a corner obtained by removing the interior of the homoclinic loop. We will denote with $[x, y]$ (or $[x, y)$, $(x, y]$, (x, y)) the piece of the trajectory of the flow between x and y (eventually without one or both endpoints). In this case x or y can also be the saddle, in which case the trajectories will be pieces of the invariant manifolds. Let $\tau : \mathbb{T} \to \mathbb{R}$ be the return time of ψ to $\mathbb{T}$. Because ψ is a C^∞ flow in dimension two, there exist a smooth conjugacy h^L with the linearized flow ψ^L from a neighborhood V of the saddle a to a square $S = [-\delta, \delta]^2$ in $\mathbb{R}^2$. Suppose that the linear flow is $\psi^L(u, v) = (ue^{\lambda t}, ve^{-\lambda t})$, where $\lambda > 0$. If an orbit enters S at some point $(u, \pm\delta)$ and leaves at a point $(\pm\delta, v)$, then the time spent in S will be $t(u) = \frac{\log\delta - \log|u|}{\lambda}$.

Let $T_1 = \{h^{L^{-1}}(u, \delta), -\delta \leq u \leq \delta\}$ and $T_2 = \{h^{L^{-1}}(u, -\delta), -\delta \leq u \leq \delta\}$. Because h^L is smooth, T_1 and T_2 are smooth curves in M transversal to the flow, so the holonomy between $\mathbb{T}$ and T_1 (and T_2) is bi-Lipschitz. From this, the fact that h^L is smooth so also bi-Lipschitz, and the fact that the time spent outside U by a trajectory going from $\mathbb{T}$ back to itself is uniformly bounded from above, we can conclude that on a neighborhood around 0 we have $\tau(x) \sim -\log|x|$. Here $u(x) \sim v(x)$ means that $\frac{u(x)}{v(x)}$ is uniformly bounded away from zero and infinity. Actually one can show that there also exist the two one-sided limits $\lim_{x\to 0^\pm} \frac{\tau(x)}{-\log|x|} \in (0, \infty)$, in this case they are different (a non-symmetric singularity).

We assume that f is conjugated to an irrational rotation R_ρ by the homeomorphism h, i. e. $h \circ f = R_\rho \circ h$. Then f is uniquely ergodic, with the only invariant probability $\nu = h^* Leb$, meaning that for any measurable subset A of $\mathbb{T}$ we have $\nu(A) = Leb(h(A))$, where Leb is the Lebesgue measure on the circle.

We have the following result.

Proposition 2. *Let $\mathbb{T}$, M, f, ν, ψ and τ as before. Then we have the following dichotomy:*

- *if $\int_{\mathbb{T}} -\log(x)d\nu = \infty$, then ψ has only one ergodic probability measure, the Dirac measure at the fixed point a; consequently δ_a is the only physical measure for ψ, with the basin of attraction the whole manifold;*
- *if $\int_{\mathbb{T}} -\log(x)d\nu < \infty$, then there exists also exactly one other ergodic probability measure μ of ψ, fully supported on M.*

Here by $\int_{\mathbb{T}} g(x)d\nu$ we mean $\int_{-\frac{1}{2}}^{\frac{1}{2}} g(x)d\nu'$, where $\nu' = i^*\nu$, $i : \left(-\frac{1}{2}, \frac{1}{2}\right) \to \mathbb{T}$ is the natural inclusion.

Proof. Let $\tilde{\psi}$ be the suspension semi-flow over f with roof function τ, defined on the non-compact manifold N. Let $g : N \to M$ be the map given by $g(\tilde{\psi}_t(x)) = \psi_t(x)$, $\forall t \geq 0$, $\forall x \in \mathbb{T}$. Then g is diffeomorphic to its image, which is $M \setminus (\gamma_0 \cup [a, f(0)))$, and from the definition it is a conjugacy between $\tilde{\psi}$ and ψ restricted to this set.

Let μ be an invariant probability measure (ipm) for ψ. Then $\mu(A) = 0$ for any wandering set A. It is easy to see that γ_0 and $(a, f(0))$ are a countable union of wandering segments, so their measure must be zero. The Dirac measure at a is clearly ipm for ψ. Suppose that $\mu(\{a\}) = 0$. Then $\tilde{\mu} = g^*(\mu)$ is an ipm for $\tilde{\psi}$. Because $\tilde{\psi}$ is a suspension semi-flow for f, for every ipm $\tilde{\mu}$ for $\tilde{\psi}$, there exists an unique invariant measure $\tilde{\nu}$ for f, such that $\mu = \tilde{\nu} \times Leb$, meaning that for any measurable set $A \subset N$ we have

$$\tilde{\mu}(A) = \int_{\mathbb{T}} Leb(A \cap [x, \tilde{\psi}_{\tau(x)}(x)]d\tilde{\nu}(x).$$

Here *Leb* is the Lebesgue measure on $[x, \tilde{\psi}_{\tau(x)}(x)] \sim [0, \tau(x)]$. But f is uniquely ergodic, so $\tilde{\nu}$ must be a multiple of ν. From this we see that there exists an ipm for $\tilde{\psi}$ if and only if

$$\int_{\mathbb{T}} \tau(x) d\nu \sim \int_{\mathbb{T}} -\log|x| d\nu < \infty.$$

In conclusion, if $\int_{\mathbb{T}} -\log|x|d\nu = \infty$, then there is no invariant probability for $\tilde{\psi}$, so the only invariant probability for ψ is δ_a. In this case δ_a must be also the only physical measure of ψ, with the basin of attraction the whole manifold M, because δ_a is the only possible limit of a sequence of probabilities supported on a sequence of pieces of orbits of increasing length. If $\int_{\mathbb{T}} -\log|x|d\nu < \infty$, then $\tilde{\psi}$ will have exactly one ipm, the rescaling of $\nu \times Leb$; then ψ will have two ergodic ipm's, δ_a and μ, which is the rescaling of $g_*(\nu \times Leb)$. Obviously μ is fully supported in M, because ν is fully supported on $\mathbb{T}$. □

We remark that if the rotation number ρ of f is Diophantine, and f is C^∞, then the conjugacy h is smooth, so $\nu = h' Leb$, and consequently

$$\int_{\mathbb{T}} \tau(x) d\nu \sim \int_{\mathbb{T}} -\log|x| d\nu = \int_{\mathbb{T}} -h'(x) \log|x| d\nu < \infty.$$

In this case we are in the second situation of the proposition above, the flow ψ has two ergodic ipm's, δ_a and μ which is fully supported. Actually we will have the same situation whenever the conjugacy h between f and the rigid rotation is only Hőlder continuous, because $x^{\alpha-1} \log|x|$ is integrable if $\alpha > 0$.

If the rotation number of f is Liouville, we have two possibilities. If the conjugacy is nice enough, then we have again the two ergodic ipm's. But it is also possible that the conjugacy is bad, and for the ipm ν for f the integral $\int_{\mathbb{T}} \tau(x) d\nu$ is divergent, and in this case ψ has only one ipm, the Dirac measure at a, and this is the physical measure for ψ. An example is given in the next section.

4. A C^∞ Diffeomorphism of the Circle with 'Bad' Invariant Measure

In this section we will prove the following result.

Proposition 3. *There exist a C^∞ diffeomorphism of the circle f, with an irrational rotation number, and a unique invariant measure ν, such that*

$$\int_T -\log|x| d\nu = \infty.$$

Proof. The construction of the diffeomorphism uses a method due to Anosov and Katok (see [1]). We construct a sequence of diffeomorphisms f_n conjugated to the rational rotations R_{ρ_n} by the diffeomorphisms h_n. Then ρ_n will converge to the irrational number ρ, h_n will converge to the homeomorphism h in the C^0 topology, and f_n will converge to the diffeomorphism f in the C^∞ topology. The construction is done by induction, h_{n+1} is chosen such that

$$h_n^{-1} \circ R_{\rho_n} \circ h_n = h_{n+1}^{-1} \circ R_{\rho_n} \circ h_{n+1},$$

and then ρ_{n+1} is chosen such that

$$\|f_n - f_{n+1}\|_{C^n}, \|f_n^{-1} - f_{n+1}^{-1}\|_{C^n} < 2^{-n}$$

(here the choice of h_{n+1} plays an important role). This condition implies that f_n is convergent in the C^∞ topology. Now one has the freedom to choose the sequence h_n to converge in the C^0 topology to a homeomorphism h, which has the property that it maps a sequence of small intervals near 0 to some relatively large intervals. Thus the ν-measure of these intervals is very large compared with their Lebesgue size, and this will imply that the integral is divergent.

Let $\rho_1 = 0, h_1 = f_1 = Id$. Now suppose that $\rho_n = \frac{p_n}{q_n}$ and the diffeomorphism h_n of $\mathbb{T}$ are fixed. We will construct h_{n+1} of the form $h_{n+1} = A_n \circ h_n$, where A_n is a diffeomorphism of $\mathbb{T}$ of the form $A_n = Id + a_n$, for some C^∞ map $a_n : \mathbb{T} \to \mathbb{R}$ which is periodic with period $\frac{1}{q_n}$. Then

$$R_{\rho_n}(A_n(x)) = (x + a_n(x)) + \rho_n = (x + \rho_n) + a_n(x + \rho_n) = A_n(R_{\rho_n}(x)),\ \forall x \in \mathbb{T},$$

so R_{ρ_n} and A_n commute. This implies that

$$h_{n+1}^{-1} \circ R_{\rho_n} \circ h_{n+1} = h_n^{-1} \circ A_n^{-1} \circ R_{\rho_n} \circ A_n \circ h_n = h_n^{-1} \circ R_{\rho_n} \circ h_n = f_n.$$

We choose the C^∞ function a_n such that it satisfies the following conditions:

- a_n is periodic with period $\frac{1}{q_n}$;
- $a_n(x) = c_n x - x$ for $x \in [0, \frac{1}{2c_n q_n}]$;
- $\|a_n\|_{C^0} < \frac{1}{q_n}$;
- $a_n' > -1$.

The first condition implies that A_n and R_{ρ_n} commute, a fact needed in order to prove that f_n is convergent in the C^∞ topology. The second condition (together with the right choice of the constants c_n) is used to prove that f will have a 'bad' invariant measure. The third condition is used to prove that h_n converges in the C^0 topology, and the fourth condition says that h_n are strictly increasing, a fact used to deduce that h is a homeomorphism.

The sequence of numbers $c_n > 1$ will be chosen later. Then A_n is a C^∞ diffeomorphism of $\mathbb{T}$ which commutes with R_{ρ_n}, and $A_n(x) = c_n x$ for $x \in [0, \frac{1}{2c_n q_n}]$. We will use the following result (Lemma 3.2 in [7]).

Lemma 1. *There exists $C_n > 0$ (depending on the natural number n), such that if g is a C^{n+1} diffeomorphism of the circle and t_1 and t_2 are two real numbers in $[0, 1)$, then*

$$\|g^{-1} \circ R_{t_1} \circ g - g^{-1} \circ R_{t_2} \circ g\|_{C^n} \le C_n \max\{\|g\|_{C^{n+1}}^{n+1}, \|g^{-1}\|_{C^{n+1}}^{n+1}\}|t_1 - t_2|.$$

Now we choose the rational number $\rho_{n+1} = \frac{p_{n+1}}{q_{n+1}}$ such that

- $q_{n+1} > 2^{n+1}$;
- q_{n+1} is a multiple of $2q_n$;
- $|\rho_{n+1} - \rho_n| < \frac{1}{2^n q_n}$ and $q_{n+1} > q_n$;
-

$$|\rho_{n+1} - \rho_n| < \frac{1}{C_n \max\{\|h_{n+1}\|_{C^{n+1}}^{n+1}, \|h_{n+1}^{-1}\|_{C^{n+1}}^{n+1}\}} 2^{-n}.$$

The first condition is again used to prove that h_n is convergent in the C^0 topology. The second condition says that h_n will preserve some small intervals for all large enough n, a fact used to prove that the invariant measure of f is 'bad'. The third condition will imply that ρ is irrational, while the fourth condition is used again to prove that f_n converges to f in the C^∞ topology.

Step 1. f_n converges in the C^∞ topology to the C^∞ diffeomorphism f. For any $n \geq 1$ we have

$$\|f_{n+1} - f_n\|_{C^n} = \|h_{n+1}^{-1} \circ R_{\rho_{n+1}} \circ h_{n+1} - h_{n+1}^{-1} \circ R_{\rho_n} \circ h_{n+1}\|_{C^n} < 2^{-n},$$

then for any fixed $m \geq 1$ and any $n_1 > n_2 > n \geq m$ we have

$$\begin{aligned}\|f_{n_1} - f_{n_2}\|_{C^m} &\leq \|f_{n_1} - f_{n_1-1}\|_{C^m} + \cdots + \|f_{n_2+1} - f_{n_2}\|_{C^m} \\ &\leq \|f_{n_1} - f_{n_1-1}\|_{C^{n_1-1}} + \cdots + \|f_{n_2+1} - f_{n_2}\|_{C^{n_2}} \\ &< 2^{-n_1+1} + \cdots + 2^{-n_2} < 2^{-n_2+1} \leq 2^{-n},\end{aligned}$$

or for any $m \geq 1$ the sequence f_n is Cauchy in the C^m topology, which means that f_n is convergent in the C^∞ topology to some C^∞ map f. A similar argument can be made for the sequence f_n^{-1} which again must converge in the C^∞ topology to the C^∞ map f^{-1}, so f must be a C^∞ diffeomorphism.

Step 2. ρ is irrational. Obviously the sequence ρ_n must be convergent to some real number ρ in [0, 1]. We recall that in the choice of the sequence ρ_n we required that $q_{n+1} > q_n$ and $|\rho_{n+1} - \rho_n| < \frac{1}{2^n q_n}$. If we suppose that $\rho = \frac{p}{q}$ is rational, we obtain

$$\frac{1}{qq_n} \leq |\rho - \rho_n| \leq \sum_{k=n}^{\infty} |\rho_{k+1} - \rho_k| < \sum_{k=n}^{\infty} \frac{1}{2^k q_k} < \frac{1}{2^{n-1} q_n},$$

or $q > 2^{n-1}$ for any natural number n, which is a contradiction. Consequently ρ must be an irrational number (actually it will be a Liouville number).

Step 3. h_n converges in the C^0 topology to the homeomorphism h. So we know that R_{ρ_n} converges in the C^∞ topology to the irrational rotation R_ρ. We also have

$$\|h_{n+1} - h_n\|_{C^0} = \|A_n \circ h_n - h_n\|_{C^0} = \|A_n - Id\|_{C^0} = \|a_n\|_{C^0} < 2^{-n},$$

from the choice of a_n and ρ_n, which implies that h_n is convergent in the C^0 topology to the map h. Then h must be a semi-conjugacy between f and R_ρ, and because h_n are all increasing homeomorphisms, h must also be increasing. If h is not a homeomorphism, then it must map an interval to a point, and consequently we would get a wandering interval for f, but this would contradict the Denjoy Theorem because f is a C^∞ diffeomorphism and the rotation number is irrational. So h is a homeomorphism and f is then conjugated by h to the irrational rotation R_ρ.

Step 4. $\int_{\mathbb{T}} -\log(x) d\nu = \infty$. Now we will choose the sequence c_n in order to obtain the desired conclusion. Let $I_n = \left[0, \frac{1}{2q_n}\right]$ and $J_n = h_{n+1}^{-1}(I_n)$. If we denote $d_n = \prod_{i=1}^n c_i$, then, from the definition of a_n (and h_{n+1}), we have $J_n = [0, \frac{1}{2q_n d_n}]$. Because in the choice of the sequence ρ_n we required that q_{n+1} is a multiple of $2q_n$ for all $n > 0$, we obtain that for any $m > n > 0$ we have $h_m(J_n) = I_n$ (this is because for $m > n$ the perturbations a_m are periodic with period $\frac{1}{2q_n}$, and then A_m will leave invariant the interval I_n). By taking the limit we get that $h(J_n) = I_n$.

Let ν be the invariant measure for f. Then

$$\int_{J_n} -\log(x)d\nu \geq -\nu(J_n)\log\frac{1}{2q_nd_n} = Leb(I_n)\log(2q_nd_n) = \frac{\log(2q_nd_n)}{2q_n}.$$

Now it is enough to choose the sequence c_n such that the sequence $\frac{\log(2q_nd_n)}{2q_n}$ does not converge to zero as n tends to infinity, and this would imply that the integral $\int_{\mathbb{T}} -\log(x)d\nu$ is divergent. Such a sequence is for example $c_n = 2^{q_n}$, so we get $d_n = 2^{\sum_{i=1}^n q_i}$, and then

$$\frac{\log(2q_nd_n)}{2q_n} = \frac{\log(2q_n)}{2q_n} + \frac{\sum_{i=1}^n q_i}{q_n}\log 2,$$

is clearly not convergent to zero. □

5. Other Examples and Remarks

We constructed in the previous section a transitive flow with the only physical measure supported at a hyperbolic fixed point, but the inconvenience of that example is that the flow is defined on a manifold with boundary and with a corner. An interesting question to ask is whether the presence of this phenomena – physical measures at hyperbolic points (or even sets) for transitive flows – would impose some restrictions on the topology of the support of the measure and/or its basin of attraction. We believe that in higher dimensions there are no such restrictions, but in lower dimensions it is unclear. In this section we will present two more related examples, this time with transitive flows on manifolds without boundary.

The first example is a transitive flow ψ on a surface of genus two M with two hyperbolic fixed points. Let $f : [-1, 1] \to \mathbb{R}$ be a continuous, C^∞ on $(-1, 1)$, even, strictly convex function such that $f(0) = 1$ is a minimum, $f(-1) = f(1) = 2$, and all the derivatives of the (local) inverse of f at 1 and -1 are zero. Rotate the graph $\{(x, f(x)), -1 \leq x \leq 1\}$ around the x-axis to obtain a surface of revolution S. Let R_1 and D_1 be the square, respectively the disk, inside the plane $x = -1$, centered at $(-1, 0, 0)$ and with the side equal to 8, respectively the radius equal to 2. Define similarly R_2 and D_2 for $x = 1$. Let N be the C^∞ surface $S \cup (R_1 \setminus D_1) \cup (R_2 \setminus D_2)$. On N we consider the gradient semi-flow given by the height function z. This semi-flow will have two saddles $a = (0, 0, -1)$ and $b = (0, 0, 1)$, connected by two heteroclinic loops.

Next identify the lateral sides of R_1 (and R_2) corresponding to $x = -1$, $y = -4$ and $x = -1$, $y = 4$ (respectively $x = 1$, $y = -4$ and $x = 1$, $y = 4$) to obtain the surface in Fig. 2, Part a. Then identify C_2 with C_3 using a rotation by π (c goes to d), in order to create a heteroclinic loop from b to a, and identify C_4 and C_1 using again a rotation by π composed with the C^∞ circle homeomorphism f (e goes to $f(0)$).

In this way we obtain a C^∞ flow ψ on a surface of genus two M. This flow is also shown in Fig. 2, Part b, with the two circles identified. The Poincaré return map to $\mathbb{T} = C_1$ is f, with the convention that the return point of 0 is $f(0)$, with the return time being infinity. The return time τ will have again a logarithmic singularity at zero, this time symmetric. The flow is topologically mixing on the whole manifold without boundary M, and the forward orbit of any point which does not belong to the heteroclinic connections or the stable manifold of a is dense on M, while the backward orbit of any point which does not belong to the heteroclinic connections or the unstable manifold of b is dense in M.

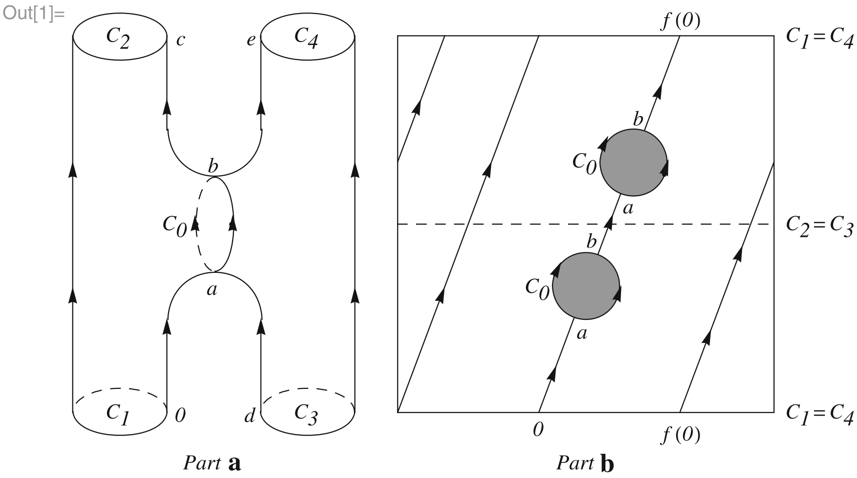

Fig. 2. Flow with two saddles

If ν is the invariant measure for f, and $\int_{\mathbb{T}} -\log|x|d\nu = \infty$, then from Proposition 2 we can conclude that ψ has only two ergodic ipm's, the Dirac measures at a and b. However we can't conclude that a combination of them is a physical measure for the system. A priori it is possible that there is no physical measure at all, meaning that for Lebesgue almost every $x \in M$ the sequence $m_{t,\psi}(x)$ is not convergent, the weak limits may form an interval of measures between δ_a and $\frac{1}{2}(\delta_a + \delta_b)$. In conclusion we have the following result.

Proposition 4. *There exist a transitive flow on a surface of genus two with two hyperbolic fixed points, and the only ergodic probability measures the Dirac measures at these two points.*

One can also create similar examples with hyperbolic points which are not conservative (the eigenvalues are not symmetric); however this will also modify the return map to the transversal, it will create critical points, with the criticality depending on the respective eigenvalues. Also one can create similar examples with more hyperbolic fixed points and/or less homoclinic connections, and in this case the return map to the transversal becomes a piecewise monotone piecewise continuous map. Unfortunately we know less about the ergodic theory of critical circle maps and piecewise monotone piecewise continuous maps.

The second example of this section is a three-dimensional version of the previous one. In this case we can construct a transitive flow on a three-dimensional compact manifold without boundary such that it has two hyperbolic fixed points, and this time the average of the two Dirac measures at the two hyperbolic fixed points is indeed the only physical measure, and the basin of attraction has full Lebesgue measure on the manifold.

Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a C^∞ diffeomorphism on the 2-torus with the following properties:

1. f has a fixed point p, i. e. $f(p) = p$;
2. There exists a dominated splitting $T\mathbb{T}^2 = E^s \oplus E^u$ for f such that:
   - $\|Df|_{E^s}\| < 1$;
   - $\|Df|_{E^u_x}\| > 1$ if $x \neq p$ and $\|Df|_{E^u_p}\| = 1$;
3. f is transitive.

Such a diffeomorphism can be obtained by deforming a linear Anosov automorphism of the torus, in order to make the expanding eigenvalue at the origin equal to 1. The following result can be found in [4].

Theorem 1. *The diffeomorphism f with the properties listed above has one unique physical measure supported on the fixed point p. The basin of attraction of the Dirac measure at p has full Lebesgue measure on $\mathbb{T}^2$: for almost every $x \in \mathbb{T}^2$, $\lim_{n\to\infty} \sum_{i=0}^{n-1} \delta_{f^i(x)} = \delta_p$, where the convergence is with respect to the weak* topology.*

We have the following result.

Proposition 5. *There exists a transitive flow ψ on a compact three-dimensional manifold M, which has two hyperbolic fixed points a and b, and a unique physical measure equal to $\frac{1}{2}(\delta_a + \delta_b)$, with the basin of attraction having full Lebesgue measure on M.*

Proof. Step 1. The construction of the flow. The construction is a three-dimensional version of the previous one, using now as the return map the function f introduced above, with the physical measure being the Dirac measure at the fixed point p. Let g be the function used in the previous example, and let $S = \{(x, y, z, u) \in \mathbb{R}^4, -1 \le x \le 1, y^2 + z^2 + u^2 = g^2(x)\}$. Let R_1 and D_1 be the three-dimensional solid cube, respectively solid sphere, inside the hyperplane $x = -1$, centered at $(-1, 0, 0, 0)$, and with the side equal to 8, respectively the radius equal to 2, and R_2 and D_2 defined similarly for $x = 1$. Let $N = S \cup (R_1 \setminus D_1) \cup (R_2 \setminus D_2)$ and ϕ be the gradient flow given by the function u. There are again two saddles a and b, and a heteroclinic sphere connecting them. Make similar identification as in the previous example, using the f from the theorem above, such that in the end we obtain a C^∞ transitive flow ψ on a C^∞ compact manifold without boundary M (this time $\mathbb{T} = C_1$ becomes a two-torus $\mathbb{T}^2$, and the circle C_0 becomes a heteroclinic two-dimensional sphere). The return map to the transversal $\mathbb{T}^2$ is f, with the convention that p returns to $f(p)$ and the return time of the flow is $\tau : \mathbb{T}^2 \to \mathbb{R}$, which has again a logarithmic singularity at p, or $\tau(x) \sim -\log(d(x, p))$ in a neighborhood of p, where $d(\cdot, \cdot)$ is the standard distance in $\mathbb{T}^2$. From the symmetry of the construction we can conclude that the time spent by a trajectory going from $\mathbb{T}^2$ to itself near a will be equal to the time spent near b.

Step 2. Almost all trajectories spend most of the time near a and b. This fact can be proved using the techniques from Sect. 3, but there is also a direct proof as follows. Let A be the set of points in the base $\mathbb{T}^2$ which are generic with respect to the invariant measure δ_p. This set is invariant with respect to f and has full Lebesgue measure. Let $B = \cup_{t\in\mathbb{R}} \psi_t(A)$, this set will have again full Lebesgue measure in M. We will prove that for every $x \in B$ and every neighborhood U of $\{a, b\}$, we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \chi_U(\psi_s(x))ds = 1, \tag{1}$$

where χ_U is the indicator function of U. This will imply that all the possible physical measures are of the form $u\delta_a + (1 - u)\delta_b$, for some $u \in [0, 1]$. The idea of the proof is simple, the iterates of x under f tend to accumulate close to p, while trajectories starting closer to p spend more and more time close to a and b.

The speed of the flow is bounded away from zero outside U, M is compact, and the length of the trajectories of the flow from y to $\psi_{\tau(y)}(y)$ is uniformly bounded for every

$y \in \mathbb{T}^2 \setminus \{p\}$, so $\int_0^{\tau(y)} \chi_{M\setminus U}(\psi_s(y))ds < L$ for some fixed positive number L and every $y \in \mathbb{T}^2 \setminus \{p\}$ (this can be extended for p too).

Let $\epsilon > 0$ fixed. There exists a neighborhood V of p such that for every $y \in V$ we have $\tau(y) > \frac{4L}{\epsilon}$. Because x is in the basin of δ_p, there exists n_0 such that for any $n \geq n_0$ we have $\frac{1}{n}Card\{k : 0 \leq k < n, f^k(x) \in V\} > \frac{1}{2}$. Let $t_0 = \sum_{k=0}^{n_0-1} \tau(f^k(x))$. For any $t > t_0$, there exist $n \geq n_0$ and $0 \leq t' < \tau(f^n(x))$ such that $t = \sum_{k=0}^{n-1} \tau(f^k(x)) + t'$. Then

$$\begin{aligned}\int_0^t \chi_U(\psi_s(x))ds &= t - \int_0^t \chi_{M\setminus U}(\psi_s(x))ds \\ &= t - \sum_{k=0}^{n-1} \int_0^{\tau(f^k(x))} \chi_{M\setminus U}(\psi_s(f^k(x))ds - \int_0^{t'} \chi_{M\setminus U}(\psi_s(f^n(x))ds \\ &> t - (n+1)L,\end{aligned}$$

so

$$\frac{1}{t}\int_0^t \chi_U(\psi_s(x))ds > 1 - \frac{(n+1)L}{t}.$$

But

$$t = \sum_{k=0}^{n-1} \tau(g^k(x)) + t' \geq \sum_{f^k(x)\in V} \tau(f^k(x)) \geq \frac{2nL}{\epsilon} \geq \frac{(n+1)L}{\epsilon},$$

so $\frac{(n+1)L}{t} < \epsilon$. Consequently

$$\frac{1}{t}\int_0^t \chi_U(\psi_s(x))ds > 1 - \epsilon, \quad \forall t > t_0,$$

which proves formula (1).

Step 3. $\frac{1}{2}(\delta_a + \delta_b)$ is the only physical measure. Let m_t be the probability measure given by the piece of trajectory $\psi_s(x)$ for $0 \leq s \leq t$, i. e. for any continuous function α on M we have $\int_M \alpha dm_t = \frac{1}{t}\int_0^t \alpha(\psi_s(x))ds$. We proved that if $x \in B$, then every weak limit measure of m_t as t tends to infinity must be in the span of $\{\delta_a, \delta_b\}$. Now we will prove that the only limit is $\frac{1}{2}(\delta_a + \delta_b)$. The rough idea of the proof is that the orbits cannot jump suddenly very close to p (or a and b), but they approach it gradually, at a maximal exponential rate given by the derivative of f.

We start with the remark that when the trajectory of x comes near p, then it will come close to a, b, a and b again and then go to $f(x)$. By the symmetry of the construction, we have that $\frac{1}{2}(\delta_a + \delta_b)$ must be a limit measure, and if there is another limit measure then it should be of the type $u\delta_a + (1-u)\delta_b$, for some $\frac{1}{2} \leq u \leq 1$ (this is because the trajectories come first to a). Let $t_n = \sum_{k=0}^{n-1} \tau(f^k(x))$, then for $t_n \leq t < t_{n+1}$ we have $t = t_n + t'$ with $0 \leq t' < \tau(f^n(x))$. Let $M_t = tm_t$, $\tilde{m}$ be the corresponding probability measure given by the piece of trajectory $\psi_s(f^n(x))$ for $0 \leq s \leq t'$, and $\tilde{M} = t'\tilde{m}$. Then

$$m_t = \frac{M_t}{t} = \frac{M_{t_n} + \tilde{M}}{t_n + t'} = \frac{t_n}{t_n + t'}m_{t_n} + \frac{t'}{t_n + t'}\tilde{m}.$$

As we remarked before, m_{t_n} will converge to $\frac{1}{2}(\delta_a + \delta_b)$, so if $\frac{t'}{t_n}$ converges to zero then we get the desired conclusion. Consequently it is enough to prove that

$$\lim_{n\to\infty} \frac{\tau(f^n(x))}{t_n} = 0. \tag{2}$$

By rescaling the metric if necessary we can assume that $d(x, p) > 1$. Let $d(f^n(x), p) = d$, $\lambda = sup_{y\in\mathbb{T}^2}\|Df_y^{-1}\| > 1$, let A be a neighborhood of p which contains a ball of radius $\frac{1}{2}$ in $\mathbb{T}$, and $C > 0$ such that $\frac{1}{C} \leq \frac{\tau(y)}{-\log(d(y,p))} \leq C$ for all y in A. We have that $d(f^{n-k}(x), p) \leq d\lambda^k$, for all $0 \leq k \leq n$. We remark that $\lambda^k d \leq \frac{1}{2}$ is equivalent to $k \leq -\frac{\log 2d}{\log\lambda}$; this means that if $k \leq -\frac{\log 2d}{\log\lambda}$ then $f^{n-k}(x)$ belongs to A. Let n_0 be the integer part of $-\frac{\log 2d}{\log\lambda}$. For $d < \frac{1}{2\lambda}$ we have $n_0 \geq 1$ and $f^{n-k}(x) \in A$ for $0 \leq k \leq n_0$, and then

$$\tau(f^n(x)) \leq -C\log d,$$

$$t_n \geq \sum_{k=1}^{n_0} \tau(f^{n-k}(x)) \geq -\frac{1}{C}\sum_{k=0}^{n_0} \log d\lambda^k = -\frac{n_0\log d}{C} - \frac{n_0(n_0+1)\log\lambda}{2C}.$$

By using the fact that $-\frac{\log 2d}{\log\lambda} - 1 < n_0 \leq -\frac{\log 2d}{\log\lambda}$ and plugging in the above inequality we get $t_n \geq \alpha(\log d)^2 + \beta\log d + \gamma$, where $\alpha = \frac{1}{2C\log\lambda} > 0$, β and γ are constants depending on C and λ. There exist a $D > 0$ such that if $\log d < -D$, or $d < e^{-D}$, then $t_n \geq \frac{\alpha}{2}(\log d)^2$, or $\log d \geq -C'\sqrt{t_n}$ for some positive constant C' (this is because $\log d$ is negative; we choose D such that $e^{-D} < \frac{1}{2\lambda}$). Consequently, if $d < e^{-D}$, we get

$$\tau(f^n(x)) \leq -C\log d \leq C''\sqrt{t_n},$$

for some positive constant C''.

To conclude, let $T = \sup\{\tau(y) : y \in \mathbb{T}^2, d(y, p) \geq e^{-D}\} < \infty$, where D is defined before. Then

$$\tau(f^n(x)) \leq \max\{T, C''\sqrt{t_n}\},$$

so

$$\frac{\tau(f^n(x))}{t_n} \leq \max\left\{\frac{T}{t_n}, \frac{C''}{\sqrt{t_n}}\right\}.$$

The fact that $\lim_{n\to\infty} t_n = \infty$ finishes the proof of (2). □

The flow is clearly transitive. Unlike the examples from the previous sections, it has infinitely many invariant measures and minimal sets (actually it has positive entropy and dense periodic orbits).

Question 1. Is it possible to construct a smooth transitive flow on a surface without boundary such that the physical measure supported on a union of hyperbolic points? What about diffeomorphisms (with the physical measure supported even on a uniformly hyperbolic set which is not the whole surface)?

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