Introduction to vibration of continuous systems

PEF 6000 - Special topics on dynamics of structures

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- To introduce to basic aspects related to the dynamics of continuous systems;
- Focus of the classes: Vibrations of beams;
- Examples of references

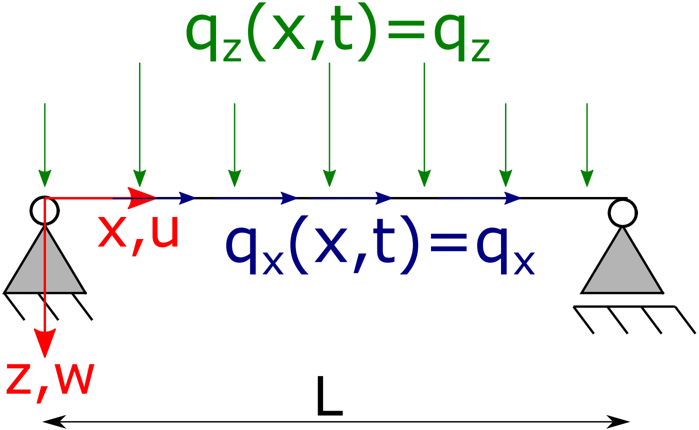
1. Blevins, R., 2001. *Formulas for natural frequency and mode shape*. Krieger Publishing Company.
2. Rao, S. 2009, *Mechanical vibrations*. Pearson Prentice Hall.
3. Lanczos, C., 1986. *The variational principles of mechanics*. Dover publications.
4. Meirovitch, L., 2003. *Methods of Analytical Dynamics*. Dover publications.
5. Thomson, W.T. & Dahleh, M.D., 2005. *Theory of vibration with application*. Pearson education.

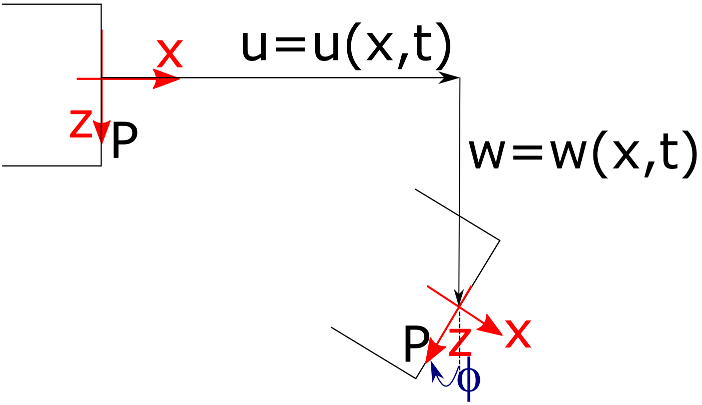
Nomenclature

- Transverse and longitudinal displacements of the centerline: $w = w(x,t)$ and $u = u(x,t)$, respectively;
- Transverse and longitudinal loads: $q_z = q_z(x,t)$ and $q_x = q_x(x,t)$, respectively;
- $\frac{\partial}{\partial t}(\ ) = (\dot{\ })$; $\frac{\partial}{\partial x}(\ ) = (\ )'$;
- $(\ )_P$: stands for a quantity calculated at a point P pertaining to the cross-section;
- $(\ )_{L^*} = (\ )(L^*, t)$, L^* being a certain point along the beam axis;

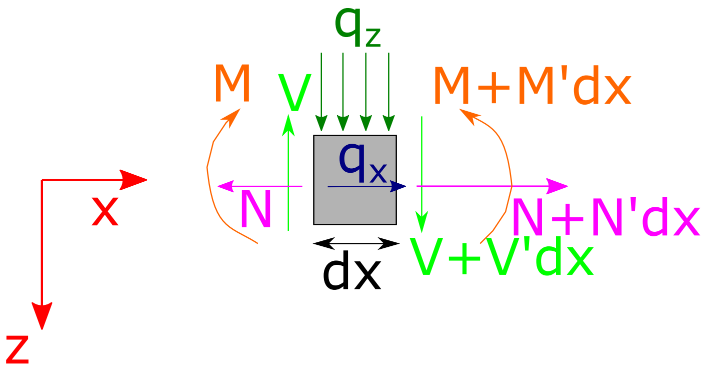
Hypotheses

- H_1: Bernoulli-Euler beam model;
- H_2: Small displacements and rotations;
- H_3: Linear-elastic material behavior;
- H_4: Planar vibrations


qz(x,t)=qz
x,u
qx(x,t)=qx
z,w
L



x
u=u(x,t)
z
P
w=w(x,t)
P
z
x
ϕ


x
z
M
V
q_z
M+M'dx
q_x
N
N+N'dx
dx
V+V'dx

The beam has mass *per* unit length μ, bending stiffness EI and is subjected to transverse and longitudinal loads q_z and q_x. The second Newton's law applied to the differential element reads (see figure above):

$$\sum F_x = \mu dx \ddot{u} \rightarrow -N + q_x dx + \left(N + N' dx\right) = \mu dx \ddot{u} \leftrightarrow \mu \ddot{u} - N' = q_x \tag{1}$$

$$\sum F_z = \mu dx \ddot{w} \rightarrow -V + q_z dx + \left(V + V' dx\right) = \mu dx \ddot{w} \leftrightarrow \mu \ddot{w} - V' = q_z \tag{2}$$

Differential equations of equilibrium and generalized constitutive equations:

$$N = EA\varepsilon = EAu' \tag{3}$$

$$M = -EI\kappa = -EIw'' \tag{4}$$

$$M' = V \tag{5}$$

Using Eqs. 3 - 5 in Eqs. 1 and 2, one obtains the equations of longitudinal and transverse motion of the prismatic beam (Eqs. 6 and 7, respectively).

$$\mu \ddot{u} - EAu'' = q_x \tag{6}$$

$$\mu \ddot{w} + EIw'''' = q_z \tag{7}$$

- Vibrations of beams are governed by partial differential equations of second order (longitudinal vibrations) or fourth order (transverse vibrations).
- In the investigated problem, the boundary conditions are:

$$w_0 = w_L = u_0 = 0 \tag{8}$$

$$w_0'' = w_L'' = 0 \tag{9}$$

$$u_L' = 0 \tag{10}$$

Equation 8 indicates null displacements at $x = 0$ and null transverse displacements at $x = L$. Equation 9 indicates null curvature (bending moment) at the ends of the pinned-pinned beam. Finally, Eq. 10 is associated with the null normal force at $x = L$. As it will be seen in this notes, the boundary conditions define the natural frequencies of the beam.

The equations of motion are now derived using the extended Hamilton's principle. Using H_1 and H_2, the longitudinal and transverse displacements of a point P of the cross-section are:

$$u_P = u - z \sin\phi = u - zw' \tag{11}$$

$$w_P = w - z(1 - \cos\phi) = w \tag{12}$$

The longitudinal strain is $\varepsilon_P = u'_P = u' - zw''$ and its variation is $\delta\varepsilon_P = \delta u' - z\delta w''$. Using H_3, the normal stress is $\sigma_P = E\varepsilon_P$, E being the Young's modulus. Following, the potential strain energy reads:

$$\mathcal{U} = \iiint_{\forall} \frac{1}{2}\sigma_P \varepsilon_P d\forall = \iiint_{\forall} \frac{1}{2} E\varepsilon_P^2 d\forall \tag{13}$$

$$\begin{aligned} \delta\mathcal{U} &= \iiint_{\forall} E\varepsilon_P \delta\varepsilon_P d\forall = \\ &= \int_0^L \iint_A E(u'\delta u' - z(u'\delta w'' + w''\delta u') + z^2 w''\delta w'')dAdx = \\ &= \int_0^L (EAu'\delta u' + EIw''\delta w'')dx \end{aligned} \tag{14}$$

By integrating twice Eq. 14 by parts, one obtains:

$$\delta\mathcal{U} = \left(EAu'\delta u\right)\big|_0^L + \left(EIw''\delta w'\right)\big|_0^L - \left(EIw'''\delta w\right)\big|_0^L - \\ - \int_0^L (EAu''\delta u - EIw''''\delta w)dx \tag{15}$$

Notice that if the beam is not prismatic, the derivatives of EI and EA must be properly considered in the integration by parts. For the investigated problem, the essential boundary conditions are $\delta u_0 = \delta w_0 = \delta w_L = 0$. Hence, Eq. 15 becomes:

$$\delta\mathcal{U} = EAu_L'\delta u_L + EIw_L''\delta w_L' - EIw_0''\delta w_0' - \\ - \int_0^L (EAu''\delta u - EIw''''\delta w)dx \tag{16}$$

In this model, the rotary inertia is not considered in the kinetic energy $\mathcal{T}$, given by:

$$\mathcal{T} = \int_0^L \frac{1}{2}\mu(\dot{u}^2 + \dot{w}^2)dx \tag{17}$$

From Eq. 17, we have:

$$\delta\mathcal{T} = \int_0^L \mu(\dot{u}\delta\dot{u} + \dot{w}\delta\dot{w})dx \rightarrow \int_{t_1}^{t_2} \delta\mathcal{T}dt = \left[\int_0^L \mu\dot{u}\delta u dx\right]_{t_1}^{t_2} + \left[\int_0^L \mu\dot{w}\delta w dx\right]_{t_1}^{t_2} - \\ - \int_{t_1}^{t_2}\int_0^L \mu(\ddot{u}\delta u + \ddot{w}\delta w)dxdt \tag{18}$$

Provided δu and δw vanish at t_1 and t_2:

$$\int_{t_1}^{t_2} \delta\mathcal{T}dt = -\int_{t_1}^{t_2}\int_0^L \mu(\ddot{u}\delta u + \ddot{w}\delta w)dxdt \tag{19}$$

For the virtual work of the non-conservative forces, we consider a linear structural damping model and the external loads. Mathematically, we have:

$$\delta W_{nc} = \int_0^L ((-c\dot{u} + q_x)\delta u + (-c\dot{w} + q_z)\delta w)dx \tag{20}$$

The extended Hamilton's principle (EHP) reads:

$$\int_{t_1}^{t_2} (\delta \mathcal{T} - \delta \mathcal{U} + \delta W_{nc})dt = 0 \tag{21}$$

Now, we substitute Eqs. 15, 19 and 20 into Eq. 21. After some algebraic work, one obtains:

$$\begin{aligned}&\int_{t_1}^{t_2}\int_0^L ((-\mu\ddot{u} + EAu'' - c\dot{u} + q_x)\delta u + (-\mu\ddot{w} - c\dot{w} - EIw'''' + q_z)\delta w)dxdt + \\ &\int_{t_1}^{t_2} (-EAu_L'\delta u_L - EIw_L''\delta w_L' + EIw_0''\delta w_0')dt = 0\end{aligned} \tag{22}$$

As the EHP holds for arbitrary virtual displacements and t_1, t_2, we obtain the equations of longitudinal and transverse motions (Eqs. 23 and 24, respectively) and the natural boundary conditions (Eqs. 25-27).

$$\mu\ddot{u} + c\dot{u} - EAu'' = q_x \tag{23}$$

$$\mu\ddot{w} + c\dot{w} + EIw'''' = q_z \tag{24}$$

$$EAu'_L = N_L = 0 \tag{25}$$

$$EIw''_L = M_L = 0 \tag{26}$$

$$EIw''_0 = M_0 = 0 \tag{27}$$

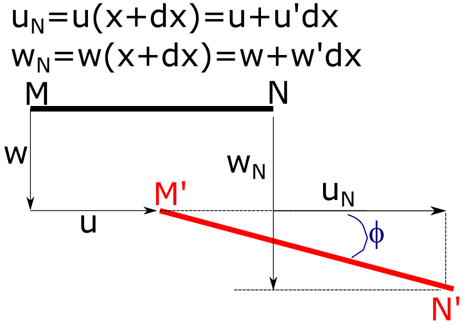
It is clear the agreement between Eqs. 23- 24 and 6 - 7. The use of Analytical Mechanics easily allows including the structural damping into the model.

- The exact displacements of the point P are given by:

$$u_P = u - z \sin \phi \tag{28}$$

$$w_P = w - z(1 - \cos \phi) \tag{29}$$

- Let M and N be two points at the centerline of the beam in the reference configuration. M and N define the infinitesimal fiber of length $d\ell_0 = dx$.



- From the above figure, the stretched length of this fiber is $d\ell = dx\sqrt{(1+u')^2+(w')^2}$. The stretch of this fiber is $\lambda = \frac{d\ell}{d\ell_0} = \frac{d\ell}{dx} = \sqrt{(1+u')^2+(w')^2}$;
- The same figure also reveals the following geometric quantities:

$$\tan\phi = \frac{w'}{1+u'}; \sin\phi = \frac{w'}{\lambda}; \cos\phi = \frac{1+u'}{\lambda} \tag{30}$$

- The stretch of a longitudinal fiber passing through P is $\lambda_P = \sqrt{(1+u'_P)^2+(w'_P)^2} = \sqrt{(1+u'-z\phi'\cos\phi)^2+(w'-z\phi'\sin\phi)^2}$. Using Eqs. 28 - 30 and after some manipulations, we obtain:

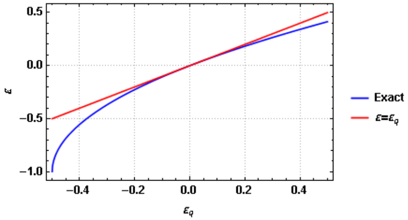
$$\lambda_P = \sqrt{(1+u'_P)^2+(w'_P)^2} = \sqrt{(1+u'-z\phi'\cos\phi)^2+(w'-z\phi'\sin\phi)^2} =$$
$$= \lambda - z\phi' \tag{31}$$

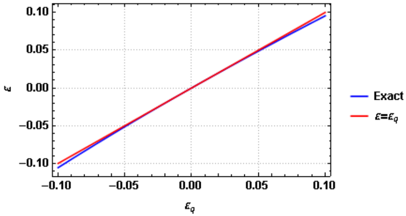
- The above expressions are exact. Now, we see some simplifications;
- We expand the expression for λ neglecting the terms of order equal or higher than linear in the quadratic strain, which is a good approximation for small strain conditions. This expansion leads to:

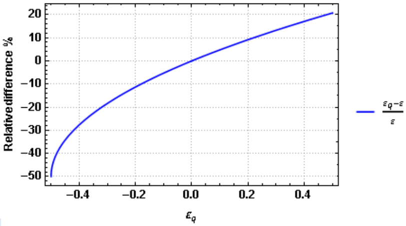
$$\lambda = \sqrt{1 + 2\varepsilon_q} \approx 1 + \varepsilon_q = 1 + u' + \frac{1}{2}(u')^2 + \frac{1}{2}(w')^2 \tag{32}$$

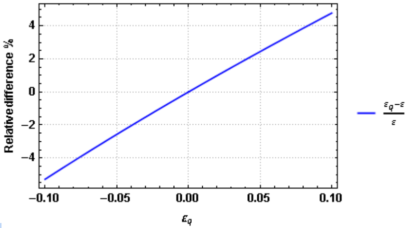
- The linear strain is $\varepsilon = \lambda - 1$. On the other hand, the quadratic strain is $\varepsilon_q = \frac{1}{2}(\lambda^2 - 1) = u' + \frac{1}{2}(u')^2 + \frac{1}{2}(w')^2$. Using Eq. 32 it is possible to conclude that for small strain, $\varepsilon \approx \varepsilon_q = u' + \frac{1}{2}(u')^2 + \frac{1}{2}(w')^2$.
- The procedure of expanding λ in terms of ε_q is necessary to avoid modelling mistakes

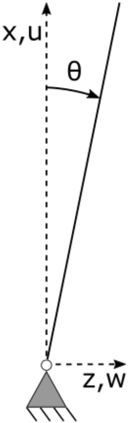
- The following graphics show the quality of the approximation $\varepsilon = \varepsilon_q$ as function of ε_q.











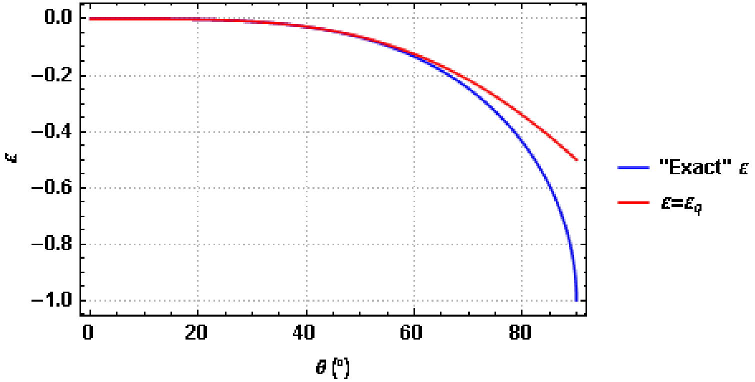
- Aiming at exemplifying the difference of expanding λ in terms of displacements rather than strain, consider the expansion correct up to second order in the displacements:

$$\lambda \approx 1 + u' + \frac{1}{2}(w')^2 \tag{33}$$

- For a rigid body rotation of the structure, as in the figure, one obtains:

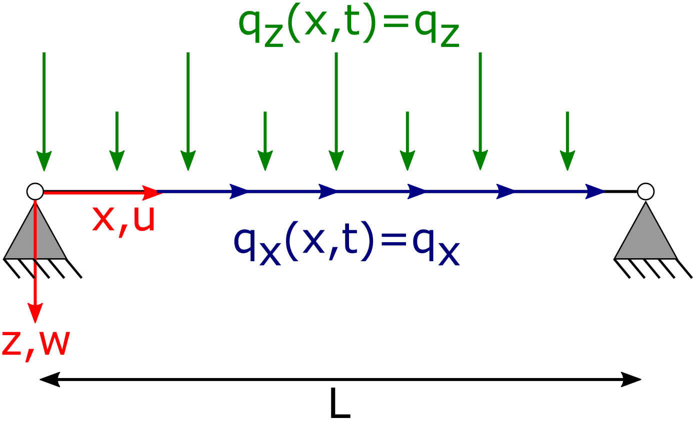
$$u' = \cos\theta - 1$$
$$w' = \sin\theta$$

- Equation 32 leads to $\lambda = 1$ while Eq. 33 leads to $\lambda = 1 + (\cos\theta - 1 + \frac{1}{2}\sin^2\theta)$, the latter containing an inherent mistake since for a rigid body motion $\lambda = 1$ and $\varepsilon = 0$.



- The expressions of Eqs. 32 and 33 can be taken as equivalent when and only when H_2 holds. This is because small strain can occur even in the presence of large displacements.
- Attaining to the example at hand, where $(u')^2$ may be taken out as a term of higher order, the condition for non-extensible beam reads: $\varepsilon = u' + \frac{1}{2}(w')^2 = 0 \leftrightarrow u = -\int_0^x \frac{1}{2}(w')^2 dx$. In terms of variation, the inextensibility condition reads $\delta u' = -w'\delta w'$.
- For problems were the displacements are large, $\varepsilon = u' + \frac{1}{2}(u')^2 + \frac{1}{2}(w')^2 = 0$. In this case, the following expressions can be obtained:

$$1 + u' = \sqrt{1 - (w')^2} \quad , \quad u > -x$$
$$1 + u' = -\sqrt{1 - (w')^2} \quad , \quad u < -x$$


$q_z(x,t)=q_z$
x,u
$q_x(x,t)=q_x$
z,w
L

- We will obtain the equation for the vibrations of an extensible and tensioned straight cable of mass *per* unit length μ and axial stiffness EA;
- Hypotheses: H_2, H_3 and H_4. It is also considered that there is no static term in q_z;
- Asides the axial loading, it is considered that a pretension $\overline{T}$ exists. The strain measurement is then $\varepsilon = u' + \frac{1}{2}(w')^2 + \frac{\overline{T}}{EA}$;

$$\mathcal{T} = \int_0^L \frac{1}{2}\mu\left(\dot{u}^2 + \dot{w}^2\right) dx \rightarrow \int_{t_1}^{t_2} \delta\mathcal{T} dt = -\int_{t_1}^{t_2}\int_0^L \mu\left(\ddot{u}\delta u + \ddot{w}\delta w\right) dxdt \tag{34}$$

$$\mathcal{U} = \iiint_{\forall} \frac{E\varepsilon^2}{2} d\forall \rightarrow$$

$$\delta\mathcal{U} = \int_0^L \iint_A E\left(u' + \frac{1}{2}(w')^2 + \frac{\overline{T}}{EA}\right)\left(\delta u' + w'\delta w'\right) dAdx =$$

$$= \int_0^L \left(EA\varepsilon\delta u' + EA\varepsilon w'\delta w'\right) dx \tag{35}$$

- Applying integration by parts in the equation for $\mathcal{U}$ and recalling the essential boundary conditions $\delta w_0 = \delta w_L = \delta u_0 = \delta u_L = 0$:

$$\delta\mathcal{U} = -\int_0^L \left[EA\left(u' + \frac{1}{2}(w')^2 + \frac{\overline{T}}{EA}\right)\right]' \delta u dx - \int_0^L \left[EAw'\left(u' + \frac{1}{2}(w')^2 + \frac{\overline{T}}{EA}\right)\right]' \delta w dx \tag{36}$$

- Virtual work of the non-conservative forces:

$$\delta W_{nc} = \int_0^L (q_x \delta u + q_z \delta w) dx \tag{37}$$

- Extended Hamilton's principle

$$\int_{t_1}^{t_2} (\delta\mathcal{T} - \delta\mathcal{U} + \delta W_{nc})dt = 0 \rightarrow$$

$$\int_{t_1}^{t_2}\int_0^L \left(-\mu\ddot{u} + \left[EA\left(u' + \frac{1}{2}(w')^2 + \frac{\overline{T}}{EA}\right)\right]' + q_x\right)\delta u dxdt$$
$$+\int_{t_1}^{t_2}\int_0^L \left(-\mu\ddot{w} + \left[EAw'\left(u' + \frac{1}{2}(w')^2 + \frac{\overline{T}}{EA}\right)\right]' + q_z\right)\delta w dxdt = 0 \quad (38)$$

- Equations of motion

$$\mu\ddot{u} - EAu'' - EAw'w'' - q_x = 0 \quad (39)$$
$$\mu\ddot{w} - \overline{T}w'' - EA\left(w'u'\right)' - \frac{3EA}{2}\left(w'\right)^2 w'' - q_z = 0 \quad (40)$$

- There may be terms in q_x, here called $q_{x,s}$, that are independent of time, such as the self weight of a vertical string. Those terms contribute to the linear geometrical stiffness of the transversal vibrations. In order to account for it, let us consider $u(x,t) = u_s(x) + u_d(x,t)$. The static equilibrium requires:

$$-EAu_s'' - q_{x,s} = 0 \tag{41}$$

- While the equation for transversal motion reads:

$$\mu \ddot{w} - \overline{T} w'' - EA\left(w' u_s'\right)' - EA\left(w' u_d'\right)' - \frac{3EA}{2}\left(w'\right)^2 w'' - q_z = 0 \tag{42}$$

- Notice now that $\overline{T} + EAu_s' = N = N(x)$ is the static normal force and $\overline{T} w'' = (\overline{T} w')'$. Then the equation for transversal motion may be written as

$$\mu \ddot{w} - \left(N w'\right)' - EA\left(w' u_d'\right)' - \frac{3EA}{2}\left(w'\right)^2 w'' - q_z = 0 \tag{43}$$

- Now the linear equation for transversal vibrations may be properly written as

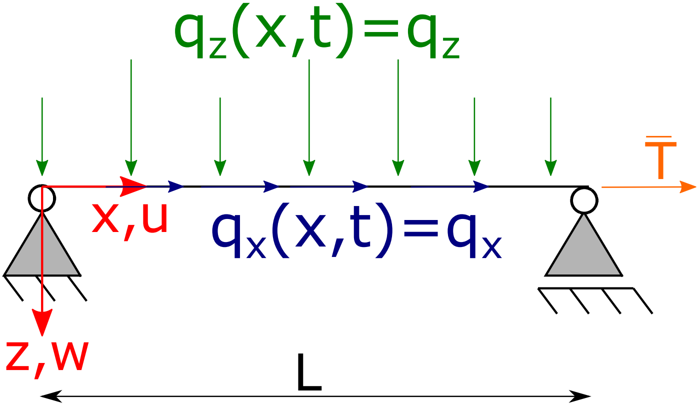
$$\mu \ddot{w} - \left(N w'\right)' = qz \tag{44}$$

- For a vertical cable, $q_x = -\gamma$ (γ is the weight *per* unit length) and $N = N(x) = \bar{T} + \gamma(x - \frac{L}{2}) \rightarrow N' = \gamma$

$$\mu \ddot{w} - (\bar{T} + \gamma(x - \frac{L}{2}))w'' - \gamma w' = 0 \tag{45}$$

- For a horizontal cable, $N = N(x) = \bar{T}$

$$\mu \ddot{w} - \bar{T} w'' = 0 \tag{46}$$



qz(x,t)=qz
T
x,u
qx(x,t)=qx
z,w
L

- We will obtain the equation for the transverse vibration of an non-nextensible and tensioned prismatic beam of mass *per* unit length μ and bending stiffness EI;
- Hypotheses: H_1, H_2, H_3 and H_4. Strain measurement $\varepsilon_P = u' + \frac{1}{2}(w')^2 - zw'' = \varepsilon - zw''$ ($\varepsilon = u' + \frac{1}{2}(w')^2$ is the strain of the beam axis);
- Inextensiblity condition: $u' = -\frac{1}{2}(w')^2 \rightarrow \dot{u}' = -w'\dot{w}' \rightarrow \dot{u} = -\int_0^x w'\dot{w}'ds$
- Kinetic energy: $\mathcal{T} = \int_0^L \frac{1}{2}(\dot{u}^2 + \dot{w}^2)dx$. For a linear mathematical model, $\mathcal{T}$ must not contain non-linearities of order higher than quadratic. Hence

$$\mathcal{T} = \int_0^L \frac{1}{2}\mu\dot{w}^2 dx \rightarrow \int_{t_1}^{t_2} \delta\mathcal{T} dt = -\int_{t_1}^{t_2}\int_0^L \mu\ddot{w}\delta w dx dt \tag{47}$$

$$\mathcal{U} = \iiint_\forall \frac{E\varepsilon_P^2}{2} d\forall \rightarrow$$

$$\delta\mathcal{U} = \int_0^L \iint_A E\left(u' + \frac{1}{2}(w')^2 - zw''\right)(\delta u' + w'\delta w' - z\delta w'')dA dx =$$

$$= \int_0^L (EA\varepsilon\delta u' + EA\varepsilon w'\delta w' + EIw''\delta w'')dx \tag{48}$$

- Inextensibility condition $\delta u' = -w'\delta w'$. Using this result in the above equation for $\mathcal{U}$ and recalling the essential boundary conditions $\delta w_0 = \delta w_L = 0$:

$$\delta\mathcal{U} = \int_0^L EIw''\delta w''dx = [EIw''\delta w']_0^L - \underbrace{[EIw'''\delta w]_0^L}_{0} + \int_0^L EIw''''\delta w dx \quad (49)$$

- Virtual work of the non-conservative forces:

$$\delta W_{nc} = \int_0^L (q_x\delta u + q_z\delta w)dx + \bar{T}\delta u_L \quad (50)$$

- "Static normal force": $N = N(x) = T_0 - \int_0^x q_x ds = \bar{T} + \int_0^L q_x ds - \int_0^x q_x ds$. Notice also that $\int_0^L \delta u' dx = \delta u_L$.
- Using the above result and the integration by parts

$$\int_0^L N\delta u' dx = \int_0^L \left[\bar{T} + \int_0^L q_x dx\right]\delta u' dx - \int_0^L \left[\int_0^x q_x ds\right]\delta u' dx =$$
$$= \left[\bar{T} + \int_0^L q_x dx\right]\delta u_L - \left(\left[\int_0^x q_x ds\right]\delta u\right)_0^L + \int_0^L q_x\delta u dx = \bar{T}\delta u_L + \int_0^L q_x\delta u dx \quad (51)$$

- By comparing Eqs. 50 and 51 and using the inextensibility condition $\delta u' = -w'\delta w'$, we have:

$$\delta W_{nc} = \int_0^L q_z \delta w dx + \int_0^L N \delta u' dx = \int_0^L q_z \delta w dx - \int_0^L N w' \delta w' dx =$$
$$= \underbrace{(N w' \delta w)_0^L}_{0} + \int_0^L (q_z + (N w')') \delta w dx \tag{52}$$

- Extended Hamilton' principle

$$\int_{t_1}^{t_2} (\delta \mathcal{T} - \delta \mathcal{U} + \delta W_{nc}) dt = 0 \rightarrow \int_{t_1}^{t_2} \int_0^L (-\mu \ddot{w} - EIw'''' + (Nw')' + q_z) \delta w dx dt +$$
$$\int_{t_1}^{t_2} (EIw_0'' \delta w_0' - EIw_L'' \delta w_L') dt = 0 \tag{53}$$

- Since the virtual displacement field is arbitrary, Eq. 54 governs the transverse dynamics of the inextensible prismatic beam and Eqs. 55 and 56 give the essential boundary conditions (null curvature at the supports).

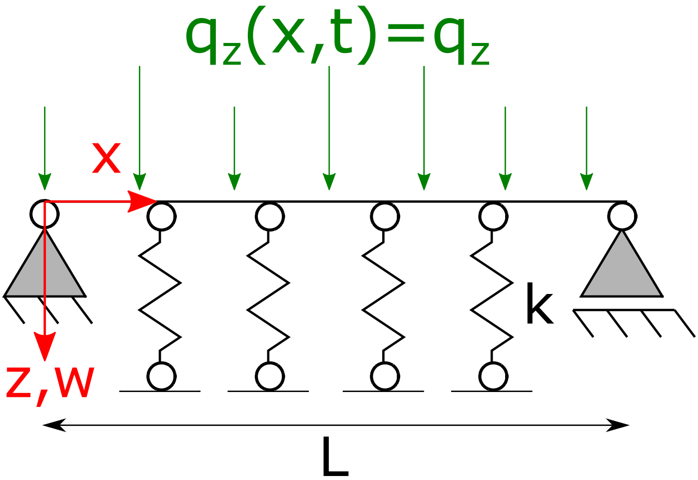
$$\mu\ddot{w} + EIw'''' - (Nw')' = q_z \tag{54}$$

$$w_0'' = 0 \tag{55}$$

$$w_L'' = 0 \tag{56}$$

- For a vertical beam, $q_x = -\gamma$ (γ is the weight *per* unit length) and $N = N(x) = \bar{T} + \gamma(x - L) \rightarrow N' = \gamma$

$$\mu\ddot{w} + EIw'''' - (\bar{T} + \gamma(x - L))w'' - \gamma w' = 0 \tag{57}$$


$q_z(x,t)=q_z$
x
z,w
k
L

Transverse vibration of a pinned-pinned beam on elastic (Winkler) foundation. Prismatic beam of mass *per* unit length μ and bending stiffness EI;

- Essential boundary conditions: $\delta w_0 = \delta w_L = 0$;
- Terms associated with kinetic energy:

$$\mathcal{T} = \int_0^L \frac{1}{2}\mu \dot{w}^2 dx \tag{58}$$

$$\int_{t_1}^{t_2} \delta\mathcal{T} dt = -\int_{t_1}^{t_2}\int_0^L \mu\ddot{w}\delta w dx dt \tag{59}$$

- Terms associated with potential energy:

$$\mathcal{U} = \iiint_{\forall} \frac{1}{2}E\varepsilon_P^2 d\forall + \int_0^L \frac{1}{2}kw^2 dx \tag{60}$$

$$\delta\mathcal{U} = \iiint_{\forall} E\varepsilon_P \delta\epsilon_P d\forall + \int_0^L kw\delta w dx = \int_0^L EIw''\delta w'' dx + \int_0^L kw\delta w dx =$$

$$= (EIw''\delta w')_0^L - \underbrace{(EIw'''\delta w)_0^L}_{0,\delta w_0=\delta w_L=0} + \int_0^L (EIw'''' + kw)\delta w dx \tag{61}$$

- Virtual work of the non-conservative force:

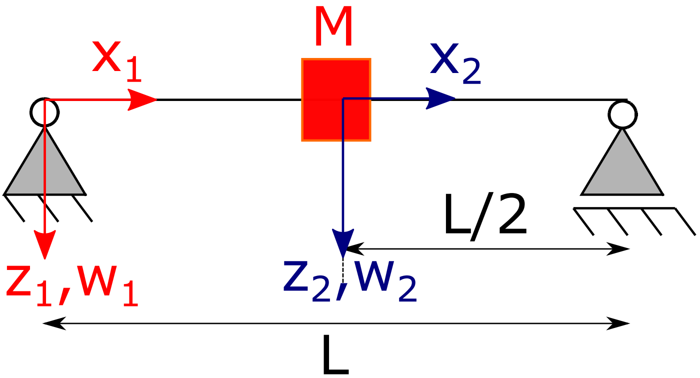
$$\delta W_{nc} = \int_0^L q_z \delta w dx \tag{62}$$

- Using extended Hamilton's principle:

$$\mu \ddot{w} + kw + EIw'''' = q_z \tag{63}$$

$$w_0'' = w_L'' = 0 \tag{64}$$

- Equation 63 is the equation of motion and Eq. 64 indicates the natural boundary conditions at $x = 0$ and $x = L$.


M
x1
x2
L/2
z1,w1
z2,w2
L

- Focus on transverse vibrations of a prismatic beam with μ and EI known;
- Essential boundary conditions: $w_{1_0} = w_{2_{L/2}} = 0$; $w_{1_{L/2}} = w_{2_0}$ and $w'_{1_{L/2}} = w'_{2_0}$. In terms of virtual displacements: $\delta w_{1_0} = \delta w_{2_{L/2}} = 0; \delta w_{1_{L/2}} = \delta w_{2_0}; \ \delta w'_{1_{L/2}} = \delta w'_{2_0};$
- Terms associated with kinetic energy:

$$\mathcal{T} = \int_0^{L/2} \frac{1}{2}\mu \dot{w_1}^2 dx_1 + \int_0^{L/2} \frac{1}{2}\mu \dot{w_2}^2 dx_2 + \frac{1}{2} M \dot{w}^2_{1_{L/2}} \tag{65}$$

$$\int_{t_1}^{t_2} \delta\mathcal{T} dt = -\int_{t_1}^{t_2}\int_0^{L/2} \mu \ddot{w}_{1_{L/2}} \delta w_1 \, dx_1 \, dt - \int_{t_1}^{t_2}\int_0^{L/2} \mu \ddot{w}_{2_{L/2}} \delta w_2 \, dx_2 \, dt - \\ - \int_{t_1}^{t_2} M \ddot{w}_{1_{L/2}} \delta w_{1_{L/2}} dt \tag{66}$$

- Terms associated with potential energy:

$$\mathcal{U} = \iint_{\forall} \frac{1}{2} E \varepsilon_P^2 d\forall \tag{67}$$

$$\delta\mathcal{U} = \iiint_{\forall} E\varepsilon_P \delta\varepsilon_P d\forall = \int_0^{L/2} EIw_1'' \delta w_1'' dx_1 + \int_0^{L/2} EIw_2'' \delta w_2'' dx_2 \tag{68}$$

- Integrating by parts twice and using the essential boundary conditions:

$$\begin{aligned}
\delta\mathcal{U} &= (EIw_1''\delta w_1')_0^L + (EIw_2''\delta w_2')_0^L - EIw_{1_{L/2}}'''\delta w_{1_{L/2}} + EIw_2'''\delta w_{2_0} + \\
&+ \int_0^{L/2} EIw_1''''\delta w_1\,dx_1 + \int_0^{L/2} EIw_2''''\delta w_2\,dx_2 = \\
&= (EIw_{1_{L/2}}'' - EIw_{2_0}'')\delta w_{1_{L/2}}' - EIw_{1_0}''\delta w_{1_0}' + EIw_{2_{L/2}}''\delta w_{2_{L/2}}' - \\
&- (EIw_{1_{L/2}}''' - EIw_2''')\delta w_{1_{L/2}} + \int_0^{L/2} EIw_1''''\delta w_1\,dx_1 + \int_0^{L/2} EIw_2''''\delta w_2\,dx_2
\end{aligned} \tag{69}$$

- $\delta W_{nc} = 0$;
- Using EHP, we have the following equations of motion:

$$\mu\ddot{w}_1 + EIw_1'''' = 0 \tag{70}$$

$$\mu\ddot{w}_2 + EIw_2'''' = 0 \tag{71}$$

- The use of EHP also leads to the natural boundary conditions:

$$w_{1_{L/2}}'' = w_{2_0}'' \tag{72}$$

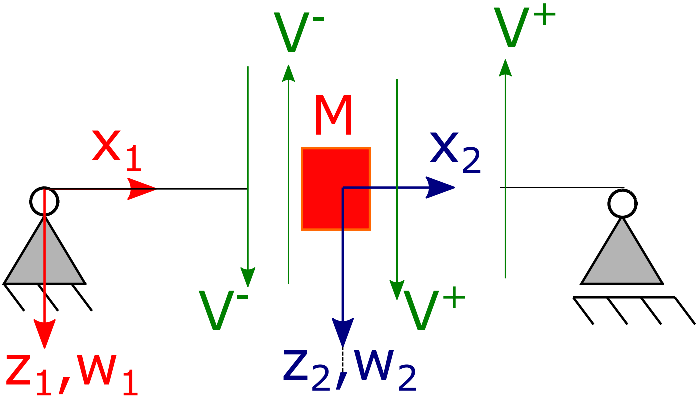
$$w_{1_0}'' = 0 \tag{73}$$

$$w_{2_{L/2}}'' = 0 \tag{74}$$

$$M\ddot{w}_{1_{L/2}} - EIw_{1_{L/2}}''' + EIw_{2_0}''' = 0 \tag{75}$$

- Equation 72: Curvature is continuous at midspan;
- Equations 73 and 74: Curvature is null at the supports;
- Second Newton's law applied to the mass:

$$M\ddot{w}_{1_{L/2}} = V^+ - V^- = -EIw_{2_0}''' - (-EIw_{1_{L/2}}''') \tag{76}$$


V-
V+
M
x1
x2
V-
V+
z1,w1
z2,w2

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- Now, we determine the natural modes and frequencies associated with the transverse direction of the pinned-pinned prismatic beam:
- Homogeneous equation of motion $\ddot{w} + \frac{EI}{\mu} w'''' = 0$. The boundary conditions are $w_0 = w_L = 0$ and $w_0'' = w_L'' = 0$;
- Separation of variables: $w = A(t)\psi(x) = A\psi$. Substituting into the equation of motion:

$$\ddot{A}\psi + \frac{EI}{\mu} A\psi'''' = 0 \leftrightarrow \frac{\ddot{A}}{A} + \frac{EI}{\mu}\frac{\psi''''}{\psi} = 0 \tag{77}$$

- Equation 77 holds if there is a real constant ω^2 that leads to:

$$\frac{\ddot{A}}{A} = -\frac{EI}{\mu}\frac{\psi''''}{\psi} = -\omega^2 \tag{78}$$

- From Eq. 78, we have $\ddot{A} + \omega^2 A = 0 \rightarrow A(t) = \rho\cos(\omega t - \theta)$, ρ and θ depending on the initial conditions. ω is a natural frequency of the transverse vibration of the beam.

- Also from Eq. 78 and defining $\beta^4 = \frac{\mu\omega^2}{EI}$, we have $\psi'''' = \beta^4\psi$.
- The above EDO has solution of form $\psi(x) = e^{\lambda x}$. Using this definition, we have $\lambda^4 e^{\lambda x} = \beta^4 e^{\lambda x} \rightarrow \lambda = \pm\beta; \pm i\beta$.
- The general solution of the "spatial" ODE is:

$$\psi(x) = a_1 e^{i\beta x} + a_2 e^{-i\beta x} + a_3 e^{\beta x} + a_4 e^{-\beta x} \tag{79}$$

  with a_1, a_2, a_3 and a_4 possibly complex constants.
- Since $\psi(x)$ is a real function, we must have $a_2 = a_1^*$ and a_3 and a_4 real-valued constants. In addition, we recall that $\cos\beta x = \frac{e^{i\beta x}+e^{-i\beta x}}{2}$, $i\sin\beta x = \frac{e^{i\beta x}-e^{-i\beta x}}{2}$, $\cosh\beta x = \frac{e^{\beta x}+e^{-\beta x}}{2}$ and $\sinh\beta x = \frac{e^{\beta x}-e^{-\beta x}}{2}$ $\rightarrow e^{i\beta x} = \cos\beta x + i\sin\beta x$, $e^{-i\beta x} = \cos\beta x - i\sin\beta x$, $e^{\beta x} = \cosh\beta x + \sinh\beta x$, $e^{-\beta x} = \cosh\beta x - \sinh\beta x$
- The above definitions lead to:

$$\psi(x) = (a_1 + a_2)\cos\beta_x + i(a_1 - a_2)\sin\beta x + (a_3 + a_4)\cosh\beta x + (a_3 - a_4)\sinh\beta x \tag{80}$$

- Since $a_2 = a_1^*$ and a_3 and a_4 are real constants, we rewrite $\psi(x)$ in terms of real constants c_1, c_2, c_3 and c_4 as

$$\psi(x) = c_1 \cos \beta x + c_2 \sin \beta x + c_3 \cosh \beta x + c_4 \sinh \beta x \tag{81}$$

- Constants c_1, c_2, c_3 and c_4 are determined by imposing the boundary conditions;
- For a pinned-pinned beam: $w_0 = w_L = 0 \rightarrow \psi(0) = \psi(L) = 0$, $w_0'' = w_L'' = 0 \rightarrow \psi''(0) = \psi''(L) = 0$. In matrix form, these equations read:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ \cos \beta L & \sin \beta L & \cosh \beta L & \sinh \beta L \\ -1 & 0 & 1 & 0 \\ -\cos \beta L & -\sin \beta L & \cosh \beta L & \sinh \beta L \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{82}$$

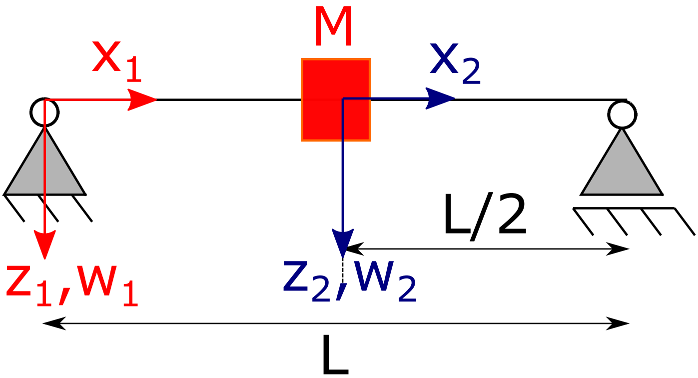
- Non-trivial solutions of Eq. 82 are obtained if

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ \cos\beta L & \sin\beta L & \cosh\beta L & \sinh\beta L \\ -1 & 0 & 1 & 0 \\ -\cos\beta L & -sin\beta L & \cosh\beta L & \sinh\beta L \end{vmatrix} =$$

$$= 4\sinh(\beta L)\sin(\beta L) = 0 \rightarrow \beta_n L = n\pi, n = 1, 2, 3... \tag{83}$$

- Recalling that $\beta^4 = \frac{\mu\omega^2}{EI}$, the undamped natural frequencies are $\omega_n = \beta_n^2\sqrt{\frac{EI}{\mu}} = \frac{(n\pi)^2}{L^2}\sqrt{\frac{EI}{\mu}}, n = 1, 2, 3, \ldots$
- By substituting the values of β_n into Eq. 82, we find that $c_1 = c_3 = c_4 = 0$ and $c_2 \neq 0$. Taking $c_2 = 1$, the natural modes of the pinned-pinned beam are

$$\psi_n(x) = \sin\left(\frac{n\pi}{L}x\right) \tag{84}$$


M
x1
x2
L/2
z1,w1
z2,w2
L

- Now, we will find the natural frequency of the beam fitted with a mass at midspan;
- For each part of the beam, the modal functions are given by ($0 \leq x_1 \leq L/2$ and $0 \leq x_2 \leq L/2$):

$$\psi_{(1)}(x_1) = c_1 \cos \beta_{(1)} x_1 + c_2 \sin \beta_{(1)} x_1 + c_3 \cosh \beta_{(1)} x_1 + c_4 \sinh \beta_{(1)} x_1 \quad (85)$$

$$\psi_{(2)}(x_2) = d_1 \cos \beta_{(2)} x_2 + d_2 \sin \beta_{(2)} x_2 + d_3 \cosh \beta_{(2)} x_2 + d_4 \sinh \beta_{(2)} x_2 \quad (86)$$

- $\beta_{(1)}^4 = \frac{\mu \omega_{(1)}^2}{EI}$ and $\beta_{(2)}^4 = \frac{\mu \omega_{(2)}^2}{EI}$. Since for a given natural frequency, the parts at left and at right of midpsan oscillate with the same frequency, $\beta = \beta_{(1)} = \beta_{(2)}$, with $\beta^4 = \frac{\mu \omega^2}{EI}$;
- We determine $c_1, \ldots, c_4$ and $d_1, \ldots, d_4$ from eight boundary conditions.

- The boundary conditions are (already obtained):

$$\psi_{(1)}(0) = 0 \tag{87}$$

$$\psi_{(2)}(L/2) = 0 \tag{88}$$

$$\psi_{(1)}(L/2) - \psi_{(2)}(0) = 0 \tag{89}$$

$$\psi'_{(1)}(L/2) - \psi'_{(2)}(0) = 0 \tag{90}$$

$$\psi''_{(1)}(0) = 0 \tag{91}$$

$$\psi''_{(2)}(L/2) = 0 \tag{92}$$

$$\psi''_{(1)}(L/2) - \psi''_{(2)}(0) = 0 \tag{93}$$

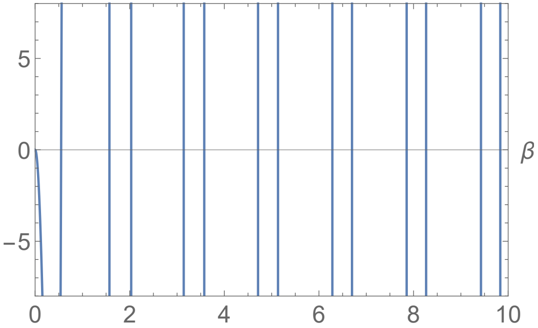
$$M\omega^2\psi_{(1)}(L/2) + EI\psi'''_{(2)}(0) - EI\psi'''_{(1)}(0) = 0 \tag{94}$$

- The above boundary conditions can be written in the form of a matrix equation

$$\boldsymbol{A}\left\{\begin{array}{c} c_1 \\ \vdots \\ d_4 \end{array}\right\} = \left\{\begin{array}{c} 0 \\ \vdots \\ 0 \end{array}\right\} \tag{95}$$

- The existence of non-trivial solutions implies that $det(A) = 0$ (transcedental equation). This equation leads to the different values of β, each of them associated with the corresponding natural frequencies $\omega_1, \omega_2 \ldots$.
- "Manually" solving this determinant is a cumbersome task. Symbolic/computational algebra are of value;
- A numerical example: Square cross-section of side 100 mm; Young's modulus $E = 200$ GPa; specific mass $\rho = 8000$ kg/m^3; length: $L = 4$ m, lumped-mass $M = 500$ kg $\rightarrow \mu = 80$ kg/m and $EI = \frac{5\times10^6}{3}$ Nm^2.

Example: Beam with a lumped-mass at midspan



- From the above figure, $\beta_1 = 0.549944$ 1/m and, consequently, $\omega_1 = \beta_1^2 \sqrt{\frac{EI}{\mu}} = 43.65$ rad/s;
- For the case without the mass: $\omega_1 = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{\mu}} = 89.03$ rad/s

- We saw that $\psi_n(x)'''' = \beta_n^4 \psi_n(x)$, n integer and $\beta_n^4 = \frac{\mu\omega_n^2}{EI}$
- Consider two different modes n and k.
- It is possible to write the following identities:

$$\psi_n''''(x) = \frac{\mu\omega_n^2}{EI}\psi_n(x) \rightarrow \int_0^L \psi_k(x)\psi_n''''(x)dx = \frac{\mu\omega_n^2}{EI}\int_0^L \psi_n(x)\psi_k(x)dx \quad (96)$$

$$\psi_k''''(x) = \frac{\mu\omega_k^2}{EI}\psi_k(x) \rightarrow \int_0^L \psi_n(x)\psi_k''''(x)dx = \frac{\mu\omega_k^2}{EI}\int_0^L \psi_k(x)\psi_n(x)dx \quad (97)$$

- Integrating by parts twice, we have:

$$[\psi_k\psi_n''']_0^L - [\psi_k'\psi_n'']_0^L + \int_0^L \psi_n''(x)\psi_k''(x)dx = \frac{\mu\omega_n^2}{EI}\int_0^L \psi_n(x)\psi_k(x)dx \quad (98)$$

$$[\psi_n\psi_k''']_0^L - [\psi_n'\psi_k'']_0^L + \int_0^L \psi_k''(x)\psi_n''(x)dx = \frac{\mu\omega_k^2}{EI}\int_0^L \psi_k(x)\psi_n(x)dx \quad (99)$$

- At a free-end: $\psi_n''' = \psi_n'' = 0$; At a support: $\psi_n = \psi_n'' = 0$; At a clamp $\psi_n = \psi_n' = 0$

- The application of the above conditions to Eqs. 98 and 99 yields:

$$\frac{\mu(\omega_n^2 - \omega_k^2)}{EI} \int_0^L \psi_k(x)\psi_n(x)dx = 0 \tag{100}$$

- Since $k \neq n$, $\omega_k \neq \omega_n$. In this scenario, Eq. 100 holds if:

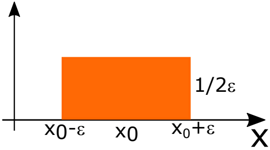
$$\int_0^L \psi_k(x)\psi_n(x)dx = 0 \tag{101}$$

- Equation 101 indicates the ortogonality of the vibration modes.

- The Dirac delta function satisfies $\delta_D(x - x_0) = 0$ for $x \neq x_0$ and the following identify:

$$\int_{-\infty}^{\infty} \delta_D(x - x_0)dx = 1 \tag{102}$$

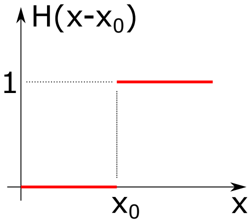
- Example: $f(x) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon}, x_0 - \epsilon \leq x \leq x_0 + \epsilon$ and $f(x) = 0$ outside this interval.



- The above figure helps understanding the important property (filtering property):

$$\int_{-\infty}^{\infty} g(x)\delta_D(x - x_0)dx = g(x_0) \tag{103}$$

- The Dirac delta function is usefull for representing lumped properties such as, for example, a point mass m placed at the tip of a beam of length L and mass *per* unit length μ;
- In this case, the equivalent mass *per* unit length is $\mu_{eq} = \mu + m\delta_D(x - L)$;
- Notice that $\int_0^L \mu_{eq} dx = \int_0^L \mu dx + m \int_0^{L^+} \delta_D(x - L) dx = \int_0^L \mu dx + m$, corresponding to the total mass of the system.



$$H(x-x_0) = \begin{cases} 0 & , x < x_0 \\ 1 & , x > x_0 \end{cases} \tag{104}$$

- It is easy to note the following relations:

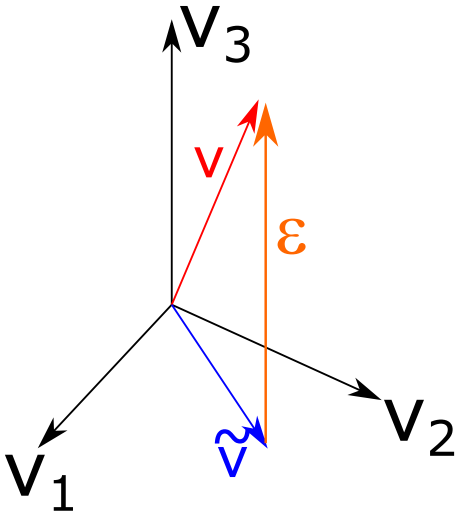
$$\delta_D(x-x_0) = \frac{d}{dx}H(x-x_0) \tag{105}$$

$$\int_{-\infty}^{x} \delta_D(s-x_0)ds = H(x-x_0) \tag{106}$$

$$h(x) = \int_{-\infty}^{x} H(s-x_0)g(s) =$$

$$= H(x-x_0)\int_{x_0}^{x} g(s)ds \tag{107}$$

- The Heaviside function can be used for stepped beams (i.e., the properties of the beam are constant during certain intervals).


V_3
v
ε
V_1
$\tilde{v}$
V_2

- This approach follows the enlightening discussions made with MSc Vítor Maciel and supported by his notes;
- Firstly, we consider the very simple linear algebra problem: What is the best approximation for the vector $\boldsymbol{v}$ pertaining to the plane generated by the orthonormal vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$;
- We define $\boldsymbol{v} = b_1 \boldsymbol{v}_1 + b_2 \boldsymbol{v}_2 + b_3 \boldsymbol{v}_3$ and the desired approximation vector $\tilde{\boldsymbol{v}} = a_1 \boldsymbol{v}_1 + a_2 \boldsymbol{v}_2$. We also define the error between the real vector and the desired approximation as $\boldsymbol{\varepsilon} = \boldsymbol{v} - \tilde{\boldsymbol{v}}$.
- From the above figure, it is clear that the error has minimum norm if it is orthogonal to the plane generated by $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$;
- Mathematically:
  $\boldsymbol{\varepsilon}.\boldsymbol{v}_1 = 0 \rightarrow (\boldsymbol{v} - \tilde{\boldsymbol{v}}).\boldsymbol{v}_1 = 0 \leftrightarrow \boldsymbol{v}.\boldsymbol{v}_1 - \tilde{\boldsymbol{v}}.\boldsymbol{v}_1 = 0 \leftrightarrow b_1 - a_1 = 0 \leftrightarrow a_1 = b_1$;
- Analogously: $a_2 = b_2$.

- Now, we will see how the Galerkin's method can be applied to the vibration of beams. Here, we consider the transverse dynamics of a pinned-pinned prismatic beam, governed by $\mu\ddot{w} + EIw'''' - q_z = 0$;
- We consider N_m modes in the expansion (approximation):
  $\tilde{w} = \tilde{w}(x,t) = \sum_{k=1}^{N_m} \psi_k(x)A_k(t) \rightarrow \ddot{\tilde{w}} = \sum_{k=1}^{N_m} \psi_k(x)\ddot{A}_k(t)$ and
  $\tilde{w}'''' = \sum_{k=1}^{N_m} \psi_k''''(x)A_k(t)$.
- The inner product between two functions $f(x)$ and $g(x)$ is $\int_0^L f(x)g(x)dx$. If we use $\tilde{w}$ in the equation of motion, the RHS is no longer zero, but a certain error ε.
- Minimum error is achieved if ε is orthogonal to the subspace spanned by $\psi_1(x), \psi_2(x), \ldots, \psi_k(x)$.
- Using the approximation $\tilde{w}$ and the definition of inner product, we have:

$$\int_0^L \left[\sum_{k=1}^{N_m} \left(\mu\psi_k(x)\ddot{A}_k(t) + EI\psi_k''''(x)A_k(t)\right) - q_z\right] \psi_m(x)dx = 0, m = 1, 2, \ldots, N_m \tag{108}$$

- For the pinned-pinned prismatic beam, the orthogonality condition implies that $\int_0^L \psi_k(x)\psi_m(x)dx = 0$ and $\int_0^L \psi_k''''(x)\psi_m(x)dx = 0$ for $k \neq m$;
- In this scenario, we have:

$$\underbrace{\left(\int_0^L \mu\psi_k^2(x)dx\right)}_{m_{\psi,k}}\ddot{A}_k(t) + \underbrace{\left(\int_0^L EI\psi_k''''(x)\psi_k(x)dx\right)}_{k_{\psi,k}}A_k(t) = \underbrace{\int_0^L q_z\psi_k(x)dx}_{p_{\psi,k}(t)} \tag{109}$$

- The PDE has been transformed into a set of uncoupled ODEs with the general form $m_{\psi,k}\ddot{A}_k(t) + k_{\psi,k}A_k(t) = p_{\psi,k}(t)$, $A_k(t)$ is the modal-amplitude time-history associated with the k-th mode. Modal oscillator, which can be solved with techniques already saw for 1-dof systems;
- $m_{\psi,k}$ is the modal mass, $k_{\psi,k}$ is the modal stiffness and $p_{\psi,k}(t)$ is the modal force. The associated natural frequency is $\omega_k = \sqrt{k_{\psi,k}/m_{\psi,k}}$.

- Once the time-histories $A_k(t)$ are numerically or analytically obtained, approximate displacement is $\tilde{w}(x,t) = \sum_{k=1}^{N_m} \psi_k(x) A_k(t)$;
- The time-history of bending moment can be obtained as $M(x,t) = -EI\tilde{w}'' = -EI \sum_{k=1}^{N_m} \psi_k''(x) A_k(t)$;
- If the orthogonality condition is not satisfied, the ROM will be given by a set of coupled ODEs.

- A note: If the mathematical model is given by a set of ODEs, as the case of the dynamic response of a discrete system, the equation of motion is given by

$$\boldsymbol{M}\ddot{\boldsymbol{U}} + \boldsymbol{C}\dot{\boldsymbol{U}} + \boldsymbol{K}\boldsymbol{U} = \boldsymbol{P}(\boldsymbol{t}) \tag{110}$$

- Suppose that we are interested in obtaining a two dof approximation for Eq. 110. In this case, we assume

$$\boldsymbol{U} = \{\phi_m \quad \phi_k\} \left\{ \begin{array}{c} a_k(t) \\ a_m(t) \end{array} \right\} = \tilde{\phi}\tilde{\boldsymbol{a}}(t) \tag{111}$$

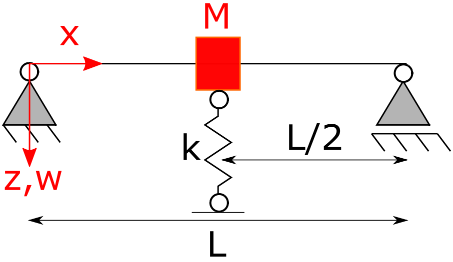
- Substituting Eq. 111 into Eq. 110, we have:

$$\boldsymbol{M}\tilde{\phi}\ddot{\tilde{a}} + \boldsymbol{C}\tilde{\phi}\dot{\tilde{a}} + \boldsymbol{K}\tilde{\phi}\tilde{\boldsymbol{a}} - \boldsymbol{P}(\boldsymbol{t}) = \varepsilon \tag{112}$$

- The error ε is minimum if

$$(\tilde{\phi}^T\boldsymbol{M}\tilde{\phi})\ddot{\tilde{a}} + (\tilde{\phi}^T\boldsymbol{C}\tilde{\phi})\dot{\tilde{a}} + (\tilde{\phi}^T\boldsymbol{K}\tilde{\phi})\tilde{\boldsymbol{a}} = \tilde{\phi}^T\boldsymbol{P}(\boldsymbol{t}) \tag{113}$$

- For linear systems, ϕ_k and ϕ_m can be taken as two vibration modes.
- For non-linear systems ϕ_k and ϕ_m can be, for example, two modes of the linearized problem.
- In the case of vibrations of beams with a lumped mass or spring, we can obtain a ROM by applying the Galerkin's method to the equation of motion written with the singularity functions and considering a projection set composed of functions that satisfy the essential boundary conditions.



- The beam is prismatic with mass *per* unit length μ and bending stiffness EI.

- The lumped mass and spring can be considered by using Dirac delta function in the form of equivalent mass *per* unit length and foundation stiffness (also *per* unit length) as:

$$\mu_{eq} = \mu + M\delta_D(x - L/2) \tag{114}$$

$$k_{eq} = k\delta_D(x - L/2) \tag{115}$$

$$\tag{116}$$

- Terms associated with kinetic energy:

$$\mathcal{T} = \int_0^L \mu_{eq}\dot{w}^2 dx \tag{117}$$

$$\int_{t_1}^{t_2} \delta\mathcal{T} dt = -\int_{t_1}^{t_2}\int_0^L \mu_{eq}\ddot{w}\delta w dx dt \tag{118}$$

- Terms associated with potential energy:

$$\mathcal{U} = \iiint_{\forall} \frac{1}{2} E\varepsilon_P d\forall + \int_0^L \frac{1}{2} k_{eq} w^2 dx \tag{119}$$

- After integrating by parts twice:

$$\delta\mathcal{U} = (EIw''\delta w')_0^L + \int_0^L (EIw'''' + k_{eq}w)\delta w dx \tag{120}$$

- The use of EHP lead to the equation of motion (Eq. 121) and the natural boundary conditions (Eq. 122):

$$\mu_{eq}\ddot{w} + EIw'''' + k_{eq}w = 0 \tag{121}$$

$$w_0'' = w_L'' = 0 \tag{122}$$

- Assume that we are interested in obtaining a ROM for the dynamics of the system in the first vibration mode ($w(x,t) = A_1(t)\psi_1(x) = A\psi$). In this case, we can use as a projection function in the Galerkin's method the vibration mode of a pinned-pinned beam without lumped mass and spring ($\psi(x) = \sin(\pi x/L)$).

$$\int_0^L \mu_{eq}\psi(x)^2 dx\ddot{A} + \int_0^L (EI\psi(x)'''' + k_{eq}\psi(x))\psi(x)dxA = 0 \tag{123}$$

- It is possible to prove that:

$$\int_0^L \psi^2(x)dx = \frac{L}{2} \tag{124}$$

$$\int_0^L \psi(x)\psi''''(x)dx = \frac{\pi^4}{2L^3} \tag{125}$$

- Using the above integrals, we have:

$$\int_0^L \mu_{eq}\psi^2(x)dx = m_\psi = \frac{\mu L}{2} + M\int_0^L \psi^2(x)\delta_D(x - L/2)dx =$$

$$= M\left(1 + \frac{\mu L}{2M}\right) \tag{126}$$

$$\int_0^L (EI\psi''''(x) + k_{eq}\psi(x))\psi(x)dx = k_\psi = \frac{EI\pi^4}{2L^3} + \int_0^L k_{eq}\psi(x)^2 dx =$$

$$= 48.7\frac{EI}{L^3} + k \tag{127}$$

- An approximate expression for the first natural frequency is

$$\omega_1 = \sqrt{\frac{k_\psi}{m_\psi}} = \sqrt{\frac{48.7\frac{EI}{L^3} + k}{M\left(1 + \frac{\mu L}{2M}\right)}} \tag{128}$$

- Physical interpretation: From strength of the materials, the vertical displacement of the midspan due to a concentrated load P is $\Delta = PL^3/48EI$.
- The contribution of the pinned-pinned beam to the equivalent stiffness is $k_{beam} = P/\Delta = 48\frac{EI}{L^3}$;
- As the spring is at the midspan and is parallel association, an analysis based on strength of materials leads to $k_{eq} = k_{beam} + k = 48\frac{EI}{L^3} + k$;
- If the mass of the beam is negligible when compared to the lumped mass, $m_\psi \approx M$ and the natural frequency, if computed with the stiffness obtained from a static analysis reads the value already discussed in PEF5916

$$\omega_1 = \sqrt{\frac{48\frac{EI}{L^3} + k}{M}} \tag{129}$$

- The modal mass is a way to consider the distributed mass of the beam in the 1-dof oscillator;

- Using the same numerical values employed in the example of the modal analysis of the beam with a lumped mass at midspan, we have, for $k = 0$,

$$\omega_1 = \sqrt{\frac{k_\psi}{m_\psi}} = \sqrt{\frac{48.7\frac{EI}{L^3} + k}{M\left(1 + \frac{\mu L}{2M}\right)}} = 43.52 \text{ rad/s} \tag{130}$$

- This approximated result is close to the analytical one $\omega_1 = 43.65$ rad/s. Notice, however, that the ROM does not allow obtaining the vibration modes.
- The books written by Rao (2009) and Blevins (2001) bring expressions for the natural modes of beams with different boundary conditions.