4

Design of Discrete-Time Control Systems by Conventional Methods

4-1 INTRODUCTION

In this chapter we first present mapping from the s plane to the z plane and then discuss stability of closed-loop control systems in the z plane. Next we treat three different design methods for single-input–single-output discrete-time or digital control systems. The first method is based on the root-locus technique using pole–zero configurations in the z plane. The second method is based on the frequency-response method in the w plane. The third method is an analytical method in which we attempt to obtain a desired behavior of the closed-loop system by manipulating the pulse transfer function of the digital controller.

Design techniques for continuous-time control systems based on conventional transform methods (the root-locus and frequency-response methods) have become well established since the 1950s. Conventional transform methods are especially useful for designing industrial control systems. In fact, in the past, many industrial digital control systems were successfully designed on the basis of conventional transform methods. Both familiarity with the root-locus and frequency-response techniques and experiences gained in the design of analog controllers are immensely valuable in designing discrete-time control systems.

Outline of the Chapter. Section 4–1 has presented introductory material. Section 4–2 treats mapping from the s plane to the z plane. Section 4–3 discusses the Jury stability criterion for closed-loop control systems in the z plane. Section 4–4 summarizes transient and steady-state response characteristics of discrete-time control systems. The design technique based on the root-locus method is presented in Section 4–5. Section 4–6 first reviews the frequency-response method and then presents frequency-response techniques using the w transformation for designing discrete-time control systems. Section 4–7 treats an analytical design method.

4-2 MAPPING BETWEEN THE s PLANE AND THE z PLANE

The absolute stability and relative stability of the linear time-invariant continuous-time closed-loop control system are determined by the locations of the closed-loop poles in the s plane. For example, complex closed-loop poles in the left half of the s plane near the $j\omega$ axis will exhibit oscillatory behavior, and closed-loop poles on the negative real axis will exhibit exponential decay.

Since the complex variables z and s are related by $z = e^{Ts}$, the pole and zero locations in the z plane are related to the pole and zero locations in the s plane. Therefore, the stability of the linear time-invariant discrete-time closed-loop system can be determined in terms of the locations of the poles of the closed-loop pulse transfer function. It is noted that the dynamic behavior of the discrete-time control system depends on the sampling period T. In terms of poles and zeros in the z plane, their locations depend on the sampling period T. In other words, a change in the sampling period T modifies the pole and zero locations in the z plane and causes the response behavior to change.

Mapping of the Left Half of the s Plane into the z Plane. In the design of a continuous-time control system, the locations of the poles and zeros in the s plane are very important in predicting the dynamic behavior of the system. Similarly, in designing discrete-time control systems, the locations of the poles and zeros in the z plane are very important. In the following paragraphs we shall investigate how the locations of the poles and zeros in the s plane compare with the locations of the poles and zeros in the z plane.

When impulse sampling is incorporated into the process, the complex variables z and s are related by the equation

$$z = e^{Ts}$$

This means that a pole in the s plane can be located in the z plane through the transformation $z = e^{Ts}$. Since the complex variable s has real part σ and imaginary part ω, we have

$$s = \sigma + j\omega$$

and

$$z = e^{T(\sigma + j\omega)} = e^{T\sigma}e^{jT\omega} = e^{T\sigma}e^{j(T\omega + 2\pi k)}$$

From this last equation we see that poles and zeros in the s plane, where frequencies differ in integral multiples of the sampling frequency $2\pi/T$, are mapped into the same locations in the z plane. This means that there are infinitely many values of s for each value of z.

Since σ is negative in the left half of the s plane, the left half of the s plane corresponds to

$$|z| = e^{T\sigma} < 1$$

The $j\omega$ axis in the s plane corresponds to $|z| = 1$. That is, the imaginary axis in the s plane (the line $\sigma = 0$) corresponds to the unit circle in the z plane, and the interior of the unit circle corresponds to the left half of the s plane

Primary Strip and Complementary Strips. Note that since $\angle z = \omega T$ the angle of z varies from $-\infty$ to ∞ as ω varies from $-\infty$ to ∞. Consider a representative point on the $j\omega$ axis in the s plane. As this point moves from $-j\frac{1}{2}\omega_s$ to $j\frac{1}{2}\omega_s$ on the $j\omega$ axis, where ω_s is the sampling frequency, we have $|z| = 1$, and $\angle z$ varies from $-\pi$ to π in the counterclockwise direction in the z plane. As the representative point moves from $j\frac{1}{2}\omega_s$ to $j\frac{3}{2}\omega_s$ on the $j\omega$ axis, the corresponding point in the z plane traces out the unit circle once in the counterclockwise direction. Thus, as the point in the s plane moves from $-\infty$ to ∞ on the $j\omega$ axis, we trace the unit circle in the z plane an infinite number of times. From this analysis, it is clear that each strip of width ω_s in the left half of the s plane maps into the inside of the unit circle in the z plane. This implies that the left half of the s plane may be divided into an infinite number of periodic strips as shown in Figure 4–1. The primary strip extends from $j\omega = -j\frac{1}{2}\omega_s$ to $j\frac{1}{2}\omega_s$. The complementary strips extend from $j\frac{1}{2}\omega_s$ to $j\frac{3}{2}\omega_s$, $j\frac{3}{2}\omega_s$ to $j\frac{5}{2}\omega_s$, . . . , and from $-j\frac{1}{2}\omega_s$ to $-j\frac{3}{2}\omega_s$, $-j\frac{3}{2}\omega_s$ to $-j\frac{5}{2}\omega_s$, ...

In the primary strip, if we trace the sequence of points 1–2–3–4–5–1 in the s plane as shown by the circled numbers in Figure 4–2(a), then this path is mapped into the unit circle centered at the origin of the z plane, as shown in Figure 4–2(b). The corresponding points 1, 2, 3, 4, and 5 in the z plane are shown by the circled numbers in Figure 4–2(b).

The area enclosed by any of the complementary strips is mapped into the same unit circle in the z plane. This means that the correspondence between the z plane

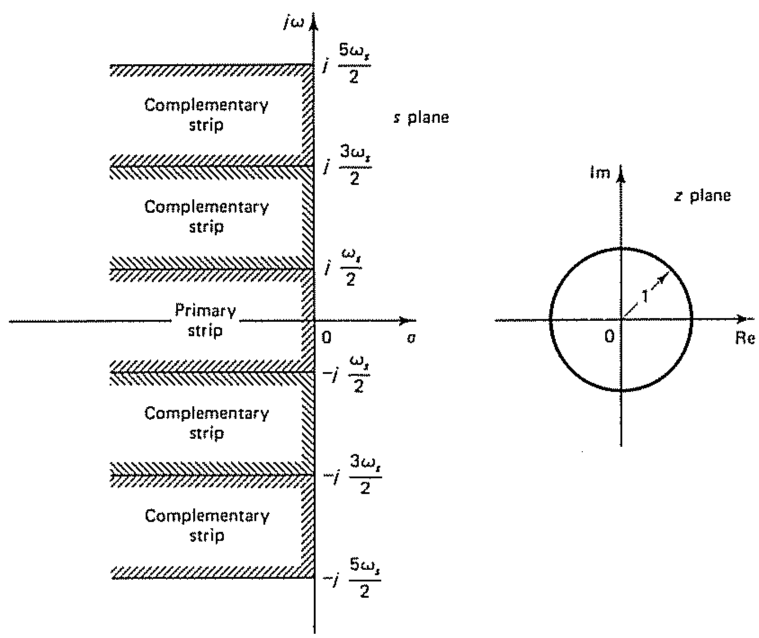


Figure 4–1 Periodic strips in the s plane and the corresponding region (unit circle centered at the origin) in the z plane.

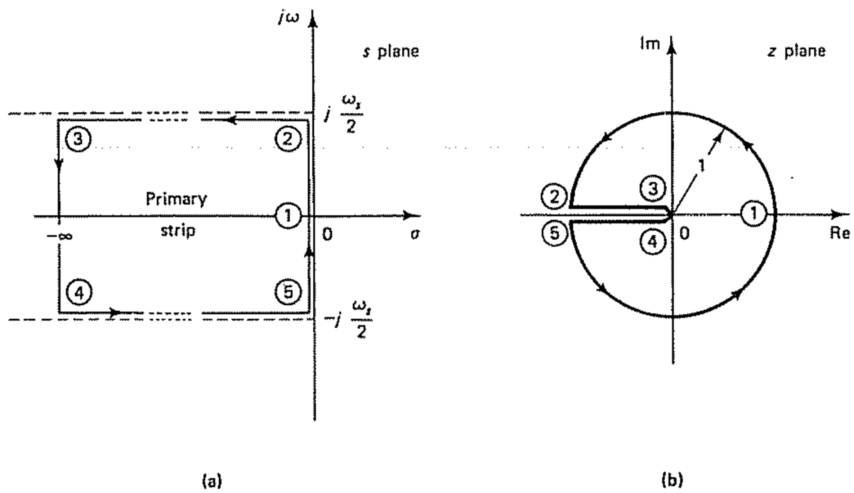


Figure 4–2 Diagrams showing the correspondence between the primary strip in the s plane and the unit circle in the z plane: (a) a path in the s plane; (b) the corresponding path in the z plane

and the s plane is not unique. A point in the z plane corresponds to an infinite number of points in the s plane, although a point in the s plane corresponds to a single point in the z plane.

Since the entire left half of the s plane is mapped into the interior of the unit circle in the z plane, the entire right half of the s plane is mapped into the exterior of the unit circle in the z plane. As mentioned earlier, the $j\omega$ axis in the s plane maps into the unit circle in the z plane. Note that, if the sampling frequency is at least twice as fast as the highest-frequency component involved in the system, then every point in the unit circle in the z plane represents frequencies between $-\frac{1}{2}\omega_s$ and $\frac{1}{2}\omega_s$.

In what follows we shall investigate the mapping of some of the commonly used contours in the s plane into the z plane. Specifically, we shall map constant-attenuation loci, constant-frequency loci, and constant-damping-ratio loci.

Constant-Attenuation Loci. A constant-attenuation line (a line plotted as σ = constant) in the s plane maps into a circle of radius $z = e^{T\sigma}$ centered at the origin in the z plane, as shown in Figure 4–3.

Settling Time t_s. The settling time is determined by the value of attenuation σ of the dominant closed-loop poles. If the settling time is specified, it is possible to draw a line $\sigma = -\sigma_1$ in the s plane corresponding to a given settling time. The region to the left of the line $\sigma = -\sigma_1$ in the s plane corresponds to the inside of a circle with radius $e^{-\sigma_1 T}$ in the z plane, as shown in Figure 4–4.

Constant-Frequency Loci. A constant-frequency locus $\omega = \omega_1$ in the s plane is mapped into a radial line of constant angle $T\omega_1$ (in radians) in the z plane, as shown in Figure 4–5. Note that constant-frequency lines at $\omega = \pm\frac{1}{2}\omega_s$ in the left half of the s plane correspond to the negative real axis in the z plane between 0 and -1, since

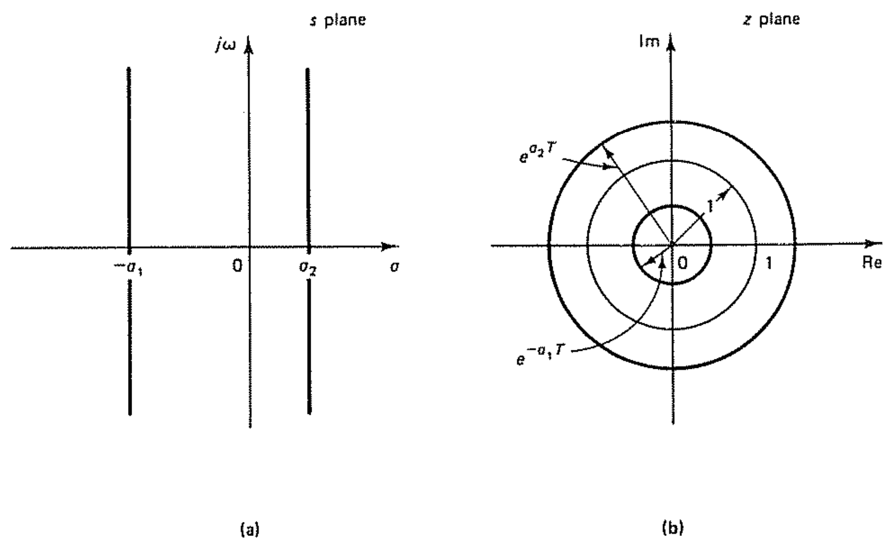


Figure 4–3 (a) Constant-attenuation lines in the s plane; (b) the corresponding loci in the z plane

$T(\pm\frac{1}{2}\omega_s) = \pm\pi$. Constant-frequency lines at $\omega = \pm\frac{1}{2}\omega_s$ in the right half of the s plane correspond to the negative real axis in the z plane between -1 and $-\infty$. The negative real axis in the s plane corresponds to the positive real axis in the z plane between 0 and 1. And constant frequency lines at $\omega = \pm n\omega_s$ $(n = 0, 1, 2, \ldots)$ in the right half of the s plane map into the positive real axis in the z plane between 1 and ∞.

The region bounded by constant-frequency lines $\omega = \omega_1$ and $\omega = -\omega_2$ (where both ω_1 and ω_2 lie between $-\frac{1}{2}\omega_s$ and $\frac{1}{2}\omega_s$) and constant-attenuation lines $\sigma = -\sigma_1$ and $\sigma = -\sigma_2$, as shown in Figure 4–6(a), is mapped into a region bounded by two radial lines and two circular arcs, as shown in Figure 4–6(b).

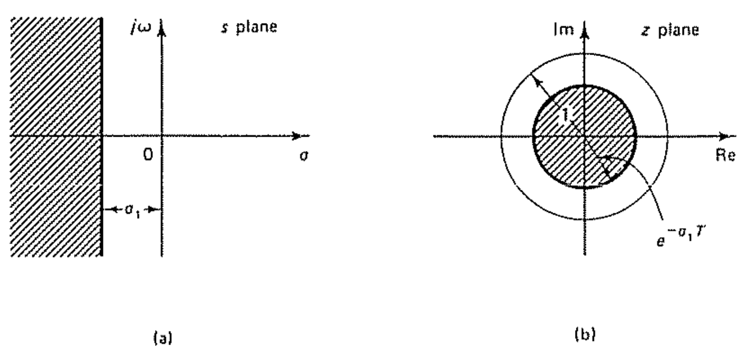


Figure 4–4 (a) Region for settling time T_s less than $4/\sigma_1$ in the s plane; (b) region for settling time T_s less than $4/\sigma_1$ in the z plane

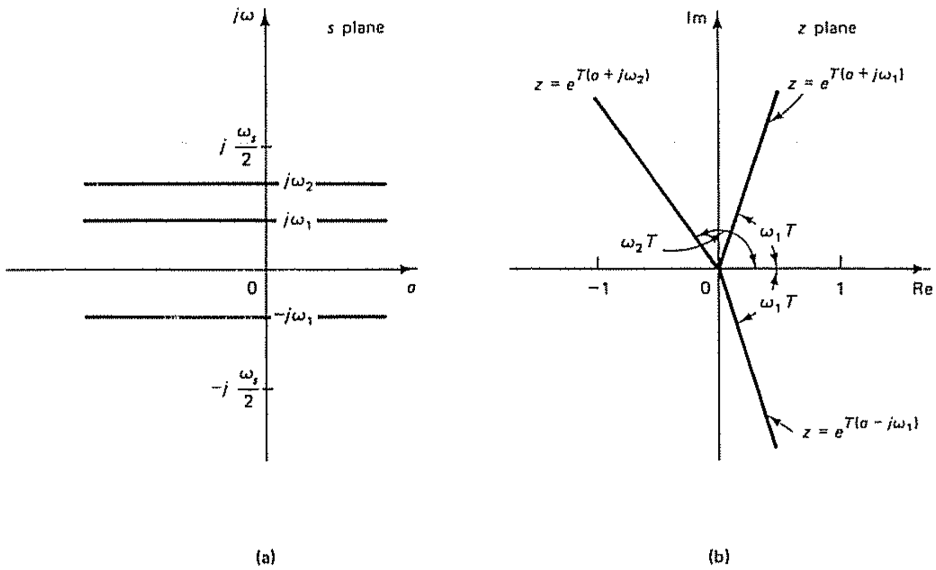


Figure 4–5 (a) Constant-frequency loci in the s plane; (b) the corresponding loci in the z plane.

Constant-Damping-Ratio Loci. A constant-damping-ratio line (a radial line) in the s plane is mapped into a spiral in the z plane. This can be seen as follows. In the s plane a constant-damping-ratio line can be given by

$$s = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} = -\zeta\omega_n + j\omega_d$$

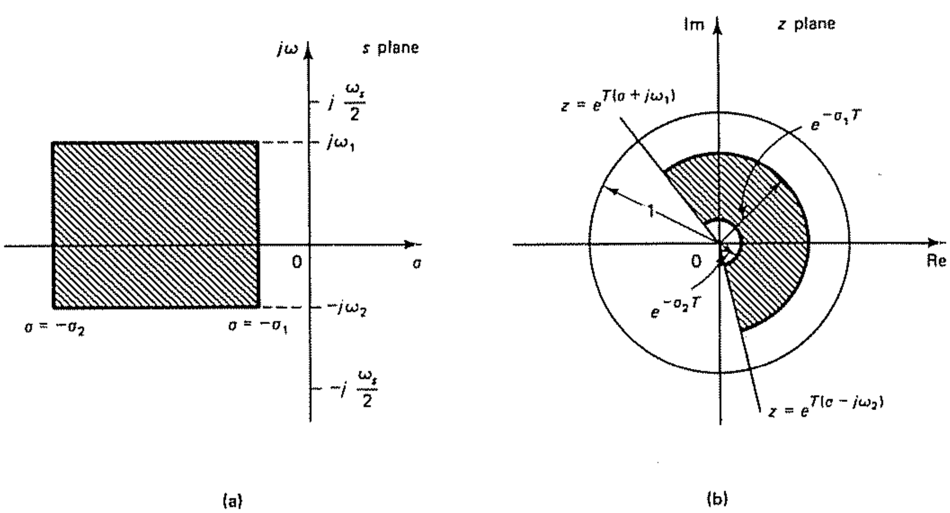


Figure 4–6 (a) Region bounded by lines $\omega = \omega_1$, $\omega = -\omega_2$, $\sigma = -\sigma_1$, and $\sigma = -\sigma_2$ in the s plane; (b) the corresponding region in the z plane

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ [see Figure 4–7(a)]. In the z plane this line becomes

$$z = e^{Ts} = \exp(-\zeta\omega_n T + j\omega_d T)$$
$$= \exp\left(-\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\frac{\omega_d}{\omega_s} + j2\pi\frac{\omega_d}{\omega_s}\right)$$

Hence,

$$|z| = \exp\left(-\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\frac{\omega_d}{\omega_s}\right) \tag{4–1}$$

and

$$\angle z = 2\pi\frac{\omega_d}{\omega_s} \tag{4–2}$$

Thus, the magnitude of z decreases and the angle of z increases linearly as ω_d increases, and the locus in the z plane becomes a logarithmic spiral, as shown in Figure 4–7(b).

Notice that for a given ratio of ω_d/ω_s the magnitude $|z|$ becomes a function only of ζ, and the angle of z becomes a constant. For example, if the damping ratio is specified as 0.3, or $\zeta = 0.3$, then for $\omega_d = 0.25\omega_s$ we have

$$|z| = \exp\left(-\frac{2\pi \times 0.3}{\sqrt{1-0.3^2}} \times 0.25\right) = 0.610$$
$$\angle z = 2\pi \times 0.25 = 0.5\pi = 90°$$

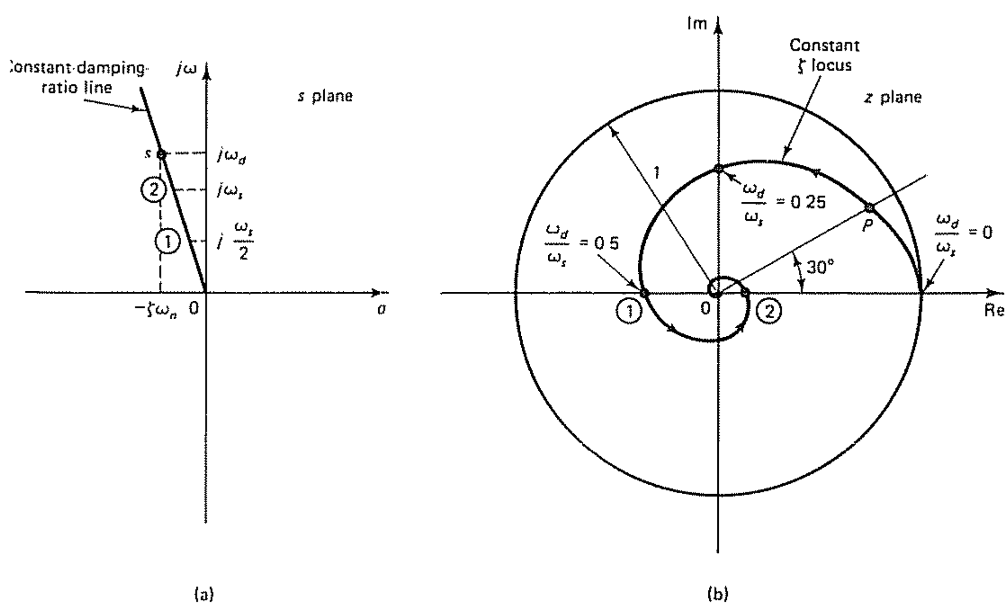


Figure 4–7 (a) Constant-damping-ratio line in the s plane; (b) the corresponding locus in the z plane.

For $\omega_d = 0.5\omega_s$,

$$|z| = \exp\left(-\frac{2\pi \times 0.3}{\sqrt{1 - 0.3^2}} \times 0.5\right) = 0.3725$$

$$\angle z = 2\pi \times 0.5 = \pi = 180°$$

Thus, the spiral can be graduated in terms of a normalized frequency ω_d/ω_s [see Figure 4–7(b)]. Once the sampling frequency ω_s is specified, the numerical value of ω_d at any point on the spiral can be determined. For example, at point P in Figure 4–7(b), ω_d can be determined as follows. If, for example, the sampling frequency is specified as $\omega_s = 10\pi$ rad/sec, then at point P

$$\angle z = \frac{\pi}{6} = 2\pi\frac{\omega_d}{\omega_s}$$

Hence, ω_d at point P is

$$\omega_d = \tfrac{1}{12}\omega_s = \tfrac{5}{6}\pi \text{ rad/sec}$$

Note that if a constant-damping-ratio line is in the second or third quadrant in the s plane then the spiral decays within the unit circle in the z plane. However, if a constant-damping-ratio line is in the first or fourth quadrant in the s plane (which corresponds to negative damping), then the spiral grows outside the unit circle. Figure 4–8 shows constant-damping-ratio loci for $\zeta = 0$, $\zeta = 0.2$, $\zeta = 0.4$, $\zeta = 0.6$, $\zeta = 0.8$, and $\zeta = 1$. The $\zeta = 1$ locus is a horizontal line between points $z = 0$ and $z = 1$. (Note that Figure 4–8 shows only the loci in the upper half of the z plane, which correspond to $0 \le \omega \le \frac{1}{2}\omega_s$. The loci corresponding to $-\frac{1}{2}\omega_s \le \omega \le 0$ are the mirror images of the loci in the upper half of the z plane about the horizontal axis.)

Notice that the constant ζ loci are normal to the constant ω_n loci in the s plane, as shown in Figure 4–9(a). In the z plane mapping, constant ω_n loci intersect constant ζ spirals at right angles, as shown in Figure 4–9(b). A mapping such as this, which preserves both the size and the sense of angles, is called a *conformal mapping*.

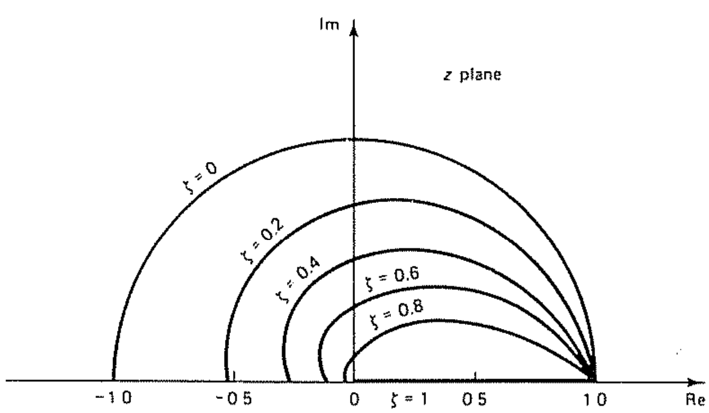


Figure 4–8 Constant-damping-ratio loci in the z plane

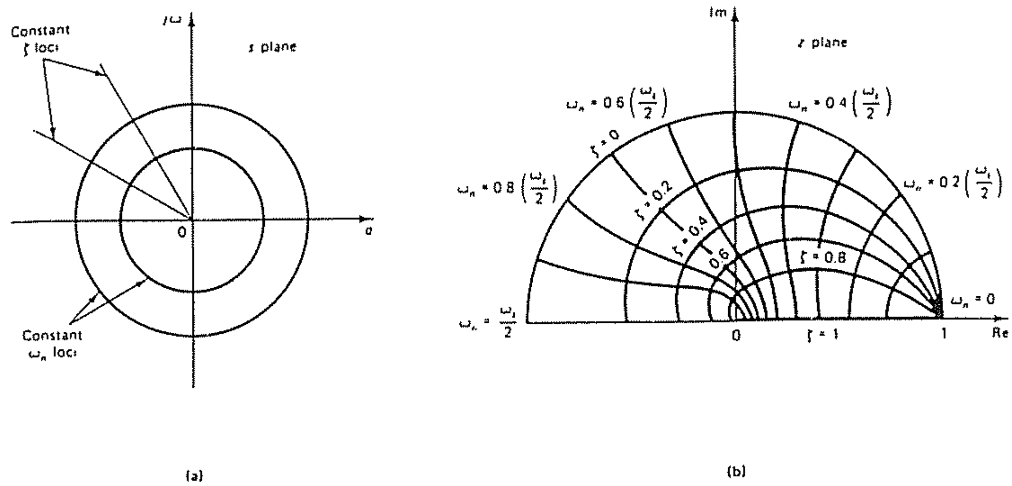


Figure 4–9 (a) Diagram showing orthogonality of the constant ζ loci and constant ω_n loci in the s plane; (b) the corresponding diagram in the z plane.

s Plane and z Plane Regions for $\zeta > \zeta_1$. Figure 4–10 shows constant ζ loci ($\zeta = \zeta_1$) in both the s plane and the z plane. Note that the logarithmic spirals shown correspond to the primary strip in the s plane. (If the sampling theorem is satisfied, then we need to consider only the primary strip in the s plane.)

If all the poles in the s plane are specified as having a damping ratio not less than a specified value ζ_1, then the poles must lie to the left of the constant-damping ratio line in the s plane (the shaded region). In the z plane, the poles must lie in the region bounded by logarithmic spirals corresponding to $\zeta = \zeta_1$ (the shaded region).

Example 4–1

Specify the region in the z plane that corresponds to a desirable region (shaded region) in the s plane bounded by lines $\omega = \pm\omega_1$, lines $\zeta = \zeta_1$, and a line $\sigma = -\sigma_1$, as shown in Figure 4–11(a).

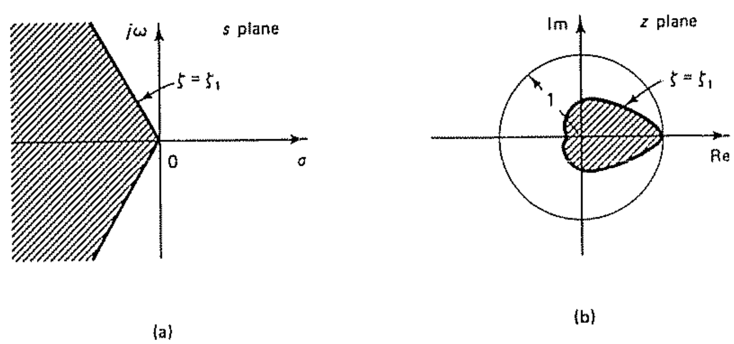


Figure 4–10 (a) Region for $\zeta > \zeta_1$ in the s plane; (b) region for $\zeta > \zeta_1$ in the z plane

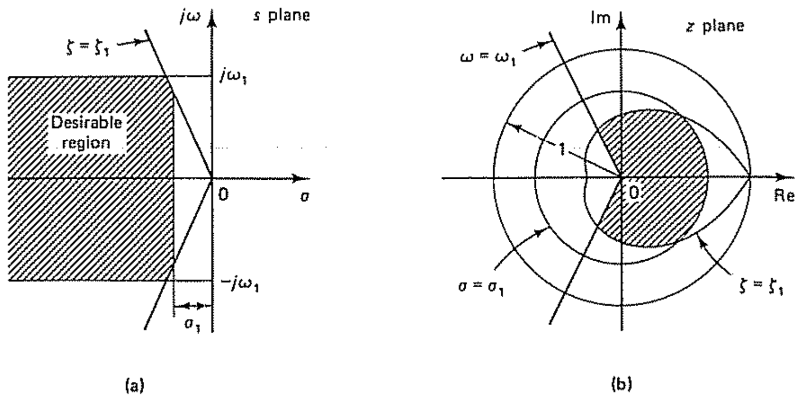


Figure 4–11 (a) A desirable region in the s plane for closed-loop pole locations; (b) corresponding region in the z plane.

On the basis of the preceding discussions on mapping from the s plane to the z plane, the desirable region can be mapped to the z plane as in Figure 4–11(b).

Note that if the dominant closed-loop poles of the continuous-time control system are required to be in the desirable region specified in the s plane, then the dominant closed-loop poles of the equivalent discrete-time control system must lie inside the region in the z plane that corresponds to the desirable region in the s plane. Once the discrete-time control system is designed, the system response characteristics must be checked by experiments or simulation. If the response characteristics are not satisfactory, then closed-loop pole and zero locations must be modified until satisfactory results are obtained.

Comments. For discrete-time control systems, it is necessary to pay particular attention to the sampling period T. This is because, if the sampling period is too long and the sampling theorem is not satisfied, then frequency folding occurs and the effective pole and zero locations will be changed.

Suppose a continuous-time control system has closed-loop poles at $s = -\sigma_1 \pm j\omega_1$ in the s plane. If the sampling operation is involved in this system and if $\omega_1 > \frac{1}{2}\omega_s$, where ω_s is the sampling frequency, then frequency folding occurs and the system behaves as if it had poles at $s = -\sigma_1 \pm j(\omega_1 \pm n\omega_s)$, where $n = 1, 2, 3, \ldots$. This means that the sampling operation folds the poles outside the primary strip back into the primary strip, and the poles will appear at $s = -\sigma_1 \pm j(\omega_1 - \omega_s)$; see Figure 4–12(a). On the z plane those poles are mapped into one pair of conjugate complex poles, as shown in Figure 4–12(b). When frequency folding occurs, oscillations with frequency $\omega_s - \omega_1$, rather than frequency ω_1, are observed.

4–3 STABILITY ANALYSIS OF CLOSED-LOOP SYSTEMS IN THE z PLANE

Stability Analysis of a Closed-Loop System. In what follows we shall discuss the stability of linear time-invariant single-input–single-output discrete-time control systems. Consider the following closed-loop pulse-transfer function system:

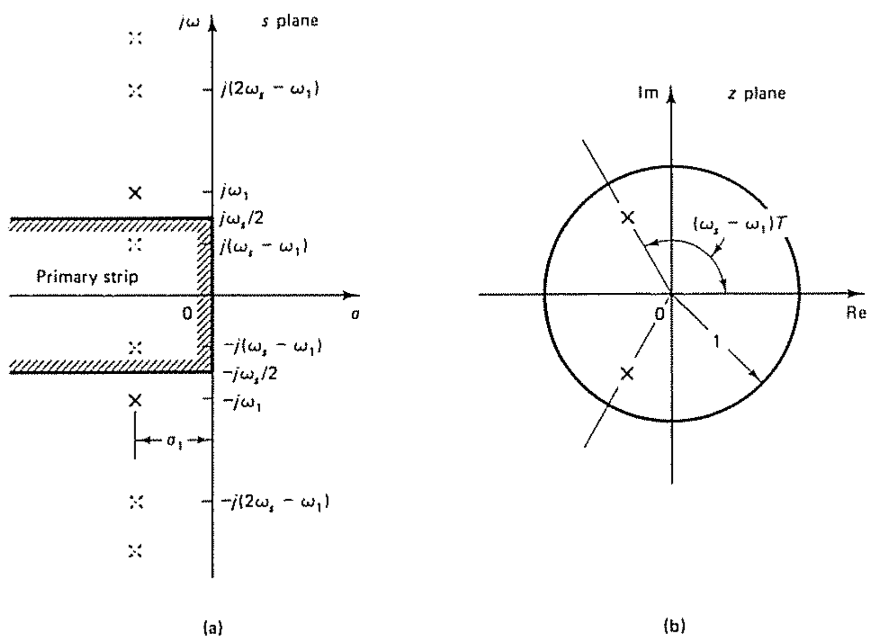


Figure 4–12 (a) Diagram showing s plane poles at $-\sigma_1 \pm j\omega_1$ and folded poles appearing at $-\sigma_1 \pm j(\omega_1 \pm \omega_s)$, $-\sigma_1 \pm j(\omega_1 \pm 2\omega_s)$, ... ; (b) z plane mapping of s plane poles at $-\sigma_1 \pm j\omega_1$, $-\sigma_1 \pm j(\omega_1 \pm \omega_s)$, $-\sigma_1 \pm j(\omega_1 \pm 2\omega_s)$, ...

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + GH(z)} \tag{4–3}$$

The stability of the system defined by Equation (4–3), as well as of other types of discrete-time control systems, may be determined from the locations of the closed-loop poles in the z plane, or the roots of the characteristic equation

$$P(z) = 1 + GH(z) = 0$$

as follows:

1. For the system to be stable, the closed-loop poles or the roots of the characteristic equation must lie within the unit circle in the z plane. Any closed-loop pole outside the unit circle makes the system unstable.
2. If a simple pole lies at $z = 1$, then the system becomes critically stable. Also, the system becomes critically stable if a single pair of conjugate complex poles lies on the unit circle in the z plane. Any multiple closed-loop pole on the unit circle makes the system unstable.
3. Closed-loop zeros do not affect the absolute stability and therefore may be located anywhere in the z plane.

Thus, a linear time-invariant single-input–single-output discrete-time closed-loop control system becomes unstable if any of the closed-loop poles lies outside the unit circle and/or any multiple closed-loop pole lies on the unit circle in the z plane.

Example 4–2

Consider the closed-loop control system shown in Figure 4–13. Determine the stability of the system when $K = 1$. The open-loop transfer function $G(s)$ of the system is

$$G(s) = \frac{1 - e^{-s}}{s} \frac{1}{s(s + 1)}$$

Referring to Equation (3–58), the z transform of $G(s)$ is

$$G(z) = \mathcal{Z}\left[\frac{1 - e^{-s}}{s} \frac{1}{s(s + 1)}\right] = \frac{0.3679z + 0.2642}{(z - 0.3679)(z - 1)} \tag{4–4}$$

Since the closed-loop pulse transfer function for the system is

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)}$$

the characteristic equation is

$$1 + G(z) = 0$$

which becomes

$$(z - 0.3679)(z - 1) + 0.3679z + 0.2642 = 0$$

or

$$z^2 - z + 0.6321 = 0$$

The roots of the characteristic equation are found to be

$$z_1 = 0.5 + j0.6181, \qquad z_2 = 0.5 - j0.6181$$

Since

$$|z_1| = |z_2| < 1$$

the system is stable.

It is important to note that in the absence of the sampler a second-order system is always stable. In the presence of the sampler, however, a second-order system such as this can become unstable for large values of gain. In fact, it can be shown that the second-order system shown in Figure 4–13 will become unstable if $K > 2.3925$. (See Example 4–7.)

Methods for Testing Absolute Stability. Three stability tests can be applied directly to the characteristic equation $P(z) = 0$ without solving for the roots. Two of them are the Schur–Cohn stability test and the Jury stability test. These two tests

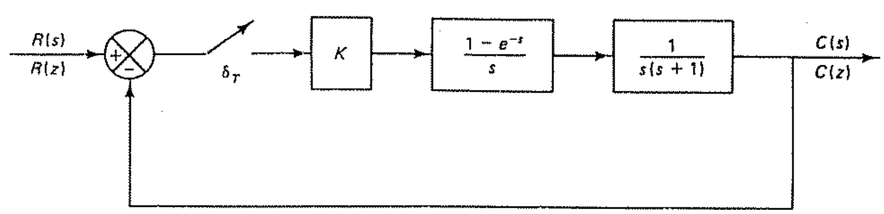


Figure 4–13 Closed-loop control system of Example 4–2

reveal the existence of any unstable roots (the roots that lie outside the unit circle in the z plane). However, these tests neither give the locations of unstable roots nor indicate the effects of parameter changes on the system stability, except for the simple case of low-order systems. (See Example 4–7.) The third method is based on the bilinear transformation coupled with the Routh stability criterion, which will be outlined later in this section. (In Chapter 5 we discuss Liapunov stability analysis, which is applicable to control systems defined in state space.)

Both the Schur–Cohn stability test and the Jury stability test may be applied to polynomial equations with real or complex coefficients. The computations required in the Jury test, when the polynomial equation involves only real coefficients, are much simpler than those required in the Schur–Cohn test. Since the coefficients of the characteristic equations corresponding to physically realizable systems are always real, the Jury test is preferred to the Schur–Cohn test.

The Jury Stability Test. In applying the Jury stability test to a given characteristic equation $P(z) = 0$, we construct a table whose elements are based on the coefficients of $P(z)$. Assume that the characteristic equation $P(z)$ is a polynomial in z as follows:

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n \tag{4-5}$$

where $a_0 > 0$. Then the Jury table becomes as given in Table 4–1.

Notice that the elements in the first row consist of the coefficients in $P(z)$ arranged in the ascending order of powers of z. The elements in the second row consist of the coefficients of $P(z)$ arranged in the descending order of powers of z. The elements for rows 3 through $2n - 3$ are given by the following determinants:

TABLE 4–1 GENERAL FORM OF THE JURY STABILITY TABLE

Row	z^0	z^1	z^2	z^3	$\cdots$	z^{n-2}	z^{n-1}	z^n
1	a_n	a_{n-1}	a_{n-2}	a_{n-3}	$\cdots$	a_2	a_1	a_0
2	a_0	a_1	a_2	a_3	$\cdots$	a_{n-2}	a_{n-1}	a_n
3	b_{n-1}	b_{n-2}	b_{n-3}	b_{n-4}	$\cdots$	b_1	b_0	
4	b_0	b_1	b_2	b_3	$\cdots$	b_{n-2}	b_{n-1}	
5	c_{n-2}	c_{n-3}	c_{n-4}	c_{n-5}	$\cdots$	c_0		
6	c_0	c_1	c_2	c_3	$\cdots$	c_{n-2}		
$2n-5$	p_3	p_2	p_1	p_0				
$2n-4$	p_0	p_1	p_2	p_3				
$2n-3$	q_2	q_1	q_0					

$$b_k = \begin{vmatrix} a_n & a_{n-1-k} \\ a_0 & a_{k+1} \end{vmatrix}, \qquad k = 0, 1, 2, \ldots, n-1$$

$$c_k = \begin{vmatrix} b_{n-1} & b_{n-2-k} \\ b_0 & b_{k+1} \end{vmatrix}, \qquad k = 0, 1, 2, \ldots, n-2$$

$$\vdots$$

$$q_k = \begin{vmatrix} p_3 & p_{2-k} \\ p_0 & p_{k+1} \end{vmatrix}, \qquad k = 0, 1, 2$$

Note that the last row in the table consists of three elements. (For second-order systems, $2n - 3 = 1$ and the Jury table consists only of one row containing three elements.) Notice that the elements in any even-numbered row are simply the reverse of the immediately preceding odd-numbered row.

Stability Criterion by the Jury Test. A system with the characteristic equation $P(z) = 0$ given by Equation (4–5), rewritten as

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

where $a_0 > 0$, is stable if the following conditions are all satisfied:

1. $|a_n| < a_0$
2. $P(z)|_{z=1} > 0$
3. $P(z)|_{z=-1} \begin{cases} >0 & \text{for } n \text{ even} \\ <0 & \text{for } n \text{ odd} \end{cases}$
4. $|b_{n-1}| > |b_0|$
   $|c_{n-2}| > |c_0|$
   $\vdots$
   $|q_2| > |q_0|$

Example 4–3

Construct the Jury stability table for the following characteristic equation:

$$P(z) = a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4$$

where $a_0 > 0$. Write the stability conditions.

Referring to the general case of the Jury stability table given by Table 4–1, a Jury stability table for the fourth-order system may be constructed as shown in Table 4–2. This table is slightly modified from the standard form and is convenient for the computations of the b's and c's. The determinant given in middle of each row gives the value of b or c written on the right-hand side of the same row.

The stability conditions are as follows:

1. $|a_4| < a_0$
2. $P(1) = a_0 + a_1 + a_2 + a_3 + a_4 > 0$
3. $P(-1) = a_0 - a_1 + a_2 - a_3 + a_4 > 0, \qquad n = 4 = \text{even}$
4. $|b_3| > |b_0|$
   $|c_2| > |c_0|$

TABLE 4–2 JURY STABILITY TABLE FOR THE FOURTH-ORDER SYSTEM

Row	z^0	z^1	z^2	z^3	z^4	
	a_4 a_0				a_0 a_4	$= b_3$
	a_4 a_0			a_1 a_3		$= b_2$
	a_4 a_0		a_2 a_2			$= b_1$
1 2	a_4 a_0	a_3 a_1				$= b_0$
	b_3 b_0			b_0 b_3		$= c_2$
	b_3 b_0		b_1 b_2			$= c_1$
3 4	b_3 b_0	b_2 b_1				$= c_0$
5	c_2	c_1	c_0			

It is noted that the value of c_1 (or, in the case of the nth-order system, the value of q_1) is not used in the stability test, and therefore the computation of c_1 (or q_1) may be omitted.

Example 4–4

Examine the stability of the following characteristic equation:

$$P(z) = z^4 - 1.2z^3 + 0.07z^2 + 0.3z - 0.08 = 0$$

Notice that for this characteristic equation

$$a_0 = 1$$
$$a_1 = -1.2$$
$$a_2 = 0.07$$
$$a_3 = 0.3$$
$$a_4 = -0.08$$

Clearly, the first condition, $|a_4| < a_0$, is satisfied. Let us examine the second condition for stability:

$$P(1) = 1 - 1.2 + 0.07 + 0.3 - 0.08 = 0.09 > 0$$

The second condition is satisfied. The third condition for stability becomes

$$P(-1) = 1 + 1.2 + 0.07 - 0.3 - 0.08 = 1.89 > 0, \qquad n = 4 = \text{even}$$

Hence the third condition is satisfied.

We now construct the Jury stability table. Referring to Example 4–3, we compute the values of b_3, b_2, b_1, and b_0 and c_2 and c_0. The result is shown in Table 4–3. (Although the value of c_1 is shown in the table, c_1 is not needed in the stability test and therefore need not be computed.) From this table, we get

$$|b_3| = 0.994 > 0.204 = |b_0|$$

$$|c_2| = 0.946 > 0.315 = |c_0|$$

Thus both parts of the fourth condition given in Example 4–3 are satisfied. Since all conditions for stability are satisfied, the given characteristic equation is stable, or all roots lie inside the unit circle in the z plane.

As a matter of fact, the given characteristic equation $P(z)$ can be factored as follows:

$$P(z) = (z - 0.8)(z + 0.5)(z - 0.5)(z - 0.4)$$

As a matter of course, the result obtained above agrees with the fact that all roots are within the unit circle in the z plane.

TABLE 4–3 JURY STABILITY TABLE FOR THE SYSTEM OF EXAMPLE 4–4

Row	z^0	z^1	z^2	z^3	z^4	
	−0.08				1	$= b_3 = -0.994$
	1				−0.08	
	−0.08			−1.2		$= b_2 = 1.176$
	1			0.3		
	−0.08		0.07			$= b_1 = -0.0756$
	1		0.07			
1	−0.08	0.3				$= b_0 = -0.204$
2	1	−1.2				
	−0.994			−0.204		$= c_2 = 0.946$
	−0.204			−0.994		
	−0.994		−0.0756			$= c_1 = -1.184$
	−0.204		1.176			
3	−0.994	1.176				$= c_0 = 0.315$
4	−0.204	−0.0756				
5	0.946	−1.184	0.315			

Example 4-5

Examine the stability of the characteristic equation given by

$$P(z) = z^3 - 1.1z^2 - 0.1z + 0.2 = 0$$

First we identify the coefficients:

$$a_0 = 1$$
$$a_1 = -1.1$$
$$a_2 = -0.1$$
$$a_3 = 0.2$$

The conditions for stability in the Jury test for the third-order system are as follows:

1. $|a_3| < a_0$
2. $P(1) > 0$
3. $P(-1) < 0, \qquad n = 3 = \text{odd}$
4. $|b_2| > |b_0|$

The first condition, $|a_3| < a_0$, is clearly satisfied. Now we examine the second condition of the Jury stability test:

$$P(1) = 1 - 1.1 - 0.1 + 0.2 = 0$$

This indicates that at least one root is at $z = 1$. Therefore, the system is at best critically stable. The remaining tests determine whether the system is critically stable or unstable. (If the given characteristic equation represents a control system, critical stability will not be desired. The stability test may be stopped at this point.)

The third condition of the Jury test gives

$$P(-1) = -1 - 1.1 + 0.1 + 0.2 = -1.8 < 0, \qquad n = 3 = \text{odd}$$

The third condition is satisfied. Now we examine the fourth condition of the Jury test. Simple computations give $b_2 = -0.96$ and $b_0 = -0.12$. Hence,

$$|b_2| > |b_0|$$

The fourth condition of the Jury test is satisfied.

From the above analysis we conclude that the given characteristic equation has one root on the unit circle ($z = 1$) and its other two roots within the unit circle in the z plane. Hence, the system is critically stable.

Example 4-6

A control system has the following characteristic equation:

$$P(z) = z^3 - 1.3z^2 - 0.08z + 0.24 = 0$$

Determine the stability of the system.

We first identify the coefficients:

$$a_0 = 1$$
$$a_1 = -1.3$$
$$a_2 = -0.08$$
$$a_3 = 0.24$$

Clearly, the first condition for stability, $|a_3| < a_0$, is satisfied. Next, we examine the second condition for stability:

$$P(1) = 1 - 1.3 - 0.08 + 0.24 = -0.14 < 0$$

The test indicates that the second condition for stability is violated. The system is therefore unstable. We may stop the test here.

Example 4–7

Consider the discrete-time unity-feedback control system (with sampling period $T = 1$ sec) whose open-loop pulse transfer function is given by

$$G(z) = \frac{K(0.3679z + 0.2642)}{(z - 0.3679)(z - 1)}$$

Determine the range of gain K for stability by use of the Jury stability test.

The closed-loop pulse transfer function becomes

$$\frac{C(z)}{R(z)} = \frac{K(0.3679z + 0.2642)}{z^2 + (0.3679K - 1.3679)z + 0.3679 + 0.2642K}$$

Thus, the characteristic equation for the system is

$$P(z) = z^2 + (0.3679K - 1.3679)z + 0.3679 + 0.2642K = 0$$

Since this is a second-order system, the Jury stability conditions may be written as follows:

1. $|a_2| < a_0$
2. $P(1) > 0$
3. $P(-1) > 0, \qquad n = 2 = \text{even}$

We shall now apply the first condition for stability. Since $a_2 = 0.3679 + 0.2642K$ and $a_0 = 1$, the first condition for stability becomes

$$|0.3679 + 0.2642K| < 1$$

or

$$2.3925 > K > -5.1775 \tag{4–6}$$

The second condition for stability becomes

$$P(1) = 1 + (0.3679K - 1.3679) + 0.3679 + 0.2642K = 0.6321K > 0$$

which gives

$$K > 0 \tag{4–7}$$

The third condition for stability gives

$$P(-1) = 1 - (0.3679K - 1.3679) + 0.3679 + 0.2642K = 2.7358 - 0.1037K > 0$$

which yields

$$26.382 > K \tag{4–8}$$

For stability, gain constant K must satisfy inequalities (4–6), (4–7), and (4–8). Hence,

$$2.3925 > K > 0$$

The range of gain constant K for stability is between 0 and 2.3925.

If gain K is set equal to 2.3925, then the system becomes critically stable (meaning that sustained oscillations exist at the output). The frequency of the sustained oscillations can be determined if 2.3925 is substituted for K in the characteristic equation and the resulting equation is investigated. With $K = 2.3925$, the characteristic equation becomes

$$z^2 - 0.4877z + 1 = 0$$

The characteristic roots are at $z = 0.2439 \pm j0.9698$. Noting that the sampling period T is equal to 1 sec, from Equation (4-2) we have

$$\omega_d = \frac{\omega_s}{2\pi} \angle z = \frac{2\pi}{2\pi} \angle z = \tan^{-1} \frac{0.9698}{0.2439} = 1.3244 \text{ rad/sec}$$

The frequency of the sustained oscillations is 1.3244 rad/sec.

Stability Analysis by Use of the Bilinear Transformation and Routh Stability Criterion. Another method frequently used in the stability analysis of discrete-time control systems is to use the bilinear transformation coupled with the Routh stability criterion. The method requires transformation from the z plane to another complex plane, the w plane. Those who are familiar with the Routh–Hurwitz stability criterion will find the method simple and straightforward. However, the amount of computation required is much more than that required in the Jury stability criterion.

The bilinear transformation defined by

$$z = \frac{w + 1}{w - 1}$$

which, when solved for w, gives

$$w = \frac{z + 1}{z - 1}$$

maps the inside of the unit circle in the z plane into the left half of the w plane. This can be seen as follows. Let the real part of w be called σ and the imaginary part ω, so that

$$w = \sigma + j\omega$$

Since the inside of the unit circle in the z plane is

$$|z| = \left|\frac{w + 1}{w - 1}\right| = \left|\frac{\sigma + j\omega + 1}{\sigma + j\omega - 1}\right| < 1$$

or

$$\frac{(\sigma + 1)^2 + \omega^2}{(\sigma - 1)^2 + \omega^2} < 1$$

we get

$$(\sigma + 1)^2 + \omega^2 < (\sigma - 1)^2 + \omega^2$$

which yields

$$\sigma < 0$$

Thus, the inside of the unit circle in the z plane ($|z| < 1$) corresponds to the left half of the w plane. The unit circle in the z plane is mapped into the imaginary axis in the w plane, and the outside of the unit circle in the z plane is mapped into the right half of the w plane. (It is pointed out that, although the w plane is similar to the s plane in that it maps the inside of the unit circle to the left half-plane, it is by no means quantitatively equivalent to the s plane. Therefore, estimating the relative stability of the system from the pole locations in the w plane is difficult.)

In the stability analysis using the bilinear transformation coupled with the Routh stability criterion, we first substitute $(w + 1)/(w - 1)$ for z in the characteristic equation

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$$

as follows:

$$a_0\left(\frac{w + 1}{w - 1}\right)^n + a_1\left(\frac{w + 1}{w - 1}\right)^{n-1} + \cdots + a_{n-1}\frac{w + 1}{w - 1} + a_n = 0$$

Then, clearing the fractions by multiplying both sides of this last equation by $(w - 1)^n$, we obtain

$$Q(w) = b_0 w^n + b_1 w^{n-1} + \cdots + b_{n-1} w + b_n = 0$$

Once we transform $P(z) = 0$ into $Q(w) = 0$, it is possible to apply the Routh stability criterion in the same manner as in continuous-time systems.

It is noted that the bilinear transformation coupled with the Routh stability criterion will indicate exactly how many roots of the characteristic equation lie in the right half of the w plane and how many lie on the imaginary axis. However, such information about the exact number of unstable poles is usually not needed in control systems design, because unstable or critically stable control systems are not desired. As mentioned earlier, the amount of computation required in this approach is much more than that required in the Jury stability test. Therefore, we shall not go any further on this subject here. We refer the reader to Problem A–4–3, where the present method is used for stability analysis.

A Few Comments on the Stability of Closed-Loop Control Systems

1. If we are interested in the effect of a system parameter on the stability of a closed-loop control system, a root-locus diagram may prove to be useful. MATLAB may be used to compute and plot a root-locus diagram.
2. It is noted that in testing the stability of a characteristic equation it may be simpler, in some cases, to find the roots of the characteristic equation directly by use of MATLAB.
3. It is important to point out that stability has nothing to do with the system's ability to follow a particular input. The error signal in a closed-loop control system may increase without bound, even if the system is stable. (Refer to Section 4–4 for a discussion of error constants.)

4-4 TRANSIENT AND STEADY-STATE RESPONSE ANALYSIS

Absolute stability is a basic requirement of all control systems. In addition, good relative stability and steady-state accuracy are also required of any control system, whether continuous time or discrete time.

In this section we shall discuss transient response and steady-state response characteristics of closed-loop control systems. The transient response refers to that portion of the response due to the closed-loop poles of the system, and the steady-state response refers to that portion of the response due to the poles of the input or forcing function.

Discrete-time control systems are very frequently analyzed with "standard" inputs such as step inputs, ramp inputs, or sinusoidal inputs. This is because the system's response to any arbitrary input may be estimated from its response to such standard inputs. In this section, we shall consider the response of the discrete-time control system to time-domain inputs such as step inputs.

Transient Response Specifications. In many practical cases, the desired performance characteristics of control systems, whether they are continuous time or discrete time, are specified in terms of time-domain quantities. This is because systems with energy storage cannot respond instantaneously and will always exhibit transient response whenever they are subjected to inputs or disturbances.

Frequently, the performance characteristics of a control system are specified in terms of the transient response to a unit-step input, since the unit-step input is easy to generate and is sufficiently drastic to provide useful information on both the transient response and the steady-state response characteristics of the system.

The transient response of a system to a unit-step input depends on the initial conditions. For convenience in comparing transient responses of various systems, it is a common practice to use the standard initial condition: the system is at rest initially and the output and all its time-derivatives are zero. The response characteristics can then be easily compared.

The transient response of a practical control system, where the output signal is continuous time, often exhibits damped oscillations before reaching the steady state. (This is true for the majority of discrete-time or digital control systems because the plants to be controlled are in most cases continuous time and, therefore, the output signals are continuous time.)

Consider, for example, the digital control system shown in Figure 4-14. The

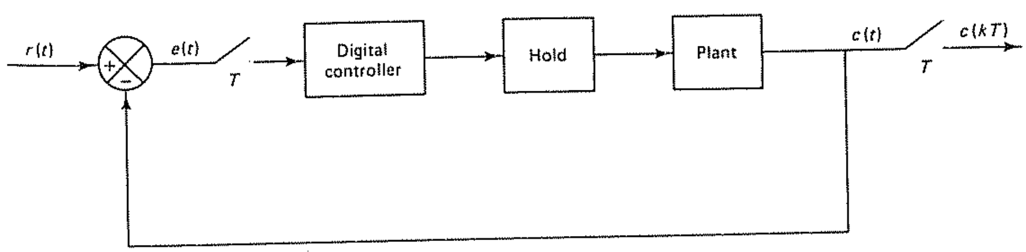
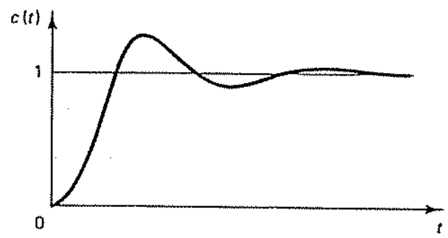
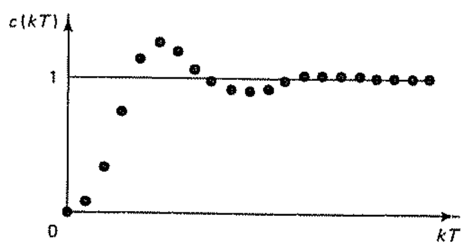


Figure 4-14 A digital control system.



(a)



(b)

Figure 4–15 (a) Unit-step response of the system shown in Figure 4–14; (b) discrete-time output in the unit-step response.

output $c(t)$ of such a system to a unit-step input may exhibit damped oscillations as shown in Figure 4–15(a). Figure 4–15(b) shows the discrete-time output $c(kT)$.

Just as in the case of continuous-time control systems, the transient response of a digital control system may be characterized not only by the damping ratio and damped natural frequency, but also by the rise time, maximum overshoot, settling time, and so forth, in response to a step input. In fact, in specifying such transient response characteristics, it is common to specify the following quantities:

TRANSIENT RESPONSE SPECIFICATIONS

1. Delay time t_d
2. Rise time t_r
3. Peak time t_p
4. Maximum overshoot M_p
5. Settling time t_s

The aforementioned transient response specifications in the unit-step response are defined in what follows and are shown graphically in Figure 4–16.

1. *Delay time* t_d. The delay time is the time required for the response to reach half the final value the very first time.

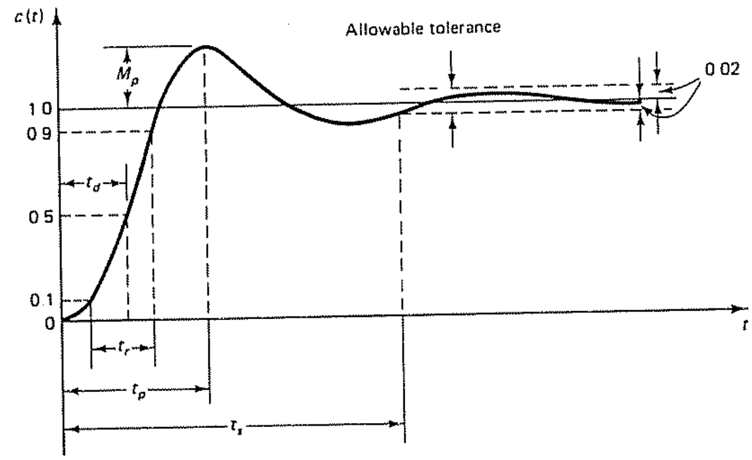


Figure 4–16 Unit-step response curve showing transient response specifications t_d, t_r, t_p, M_p, and t_s

2. *Rise time* t_r. The rise time is the time required for the response to rise from 10% to 90%, or 5% to 95%, or 0% to 100% of its final value, depending on the situation. For underdamped second-order systems, the 0% to 100% rise time is commonly used. For overdamped systems and systems with transportation lags, the 10% to 90% rise time is commonly used.
3. *Peak time* t_p. The peak time is the time required for the response to reach the first peak of the overshoot.
4. *Maximum overshoot* M_p. The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by the relation

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

The amount of the maximum (percent) overshoot directly indicates the relative stability of the system.

5. *Settling time* t_s. The settling time is the time required for the response curve to reach and stay within a range about the final value of a size specified as an absolute percentage of the final value, usually 2%. The settling time is related to the largest time constant of the control system.

The time-domain specifications just given are quite important since most control systems are time-domain systems; that is, they must exhibit acceptable time responses. (This means that the control system being designed must be modified until the transient response is satisfactory.)

Not all the specifications we have just defined necessarily apply to any given case. For example, for an overdamped system, the peak time and maximum overshoot terms do not apply. On the other hand, other specifications may be involved: for systems that yield steady-state errors for step inputs, the error must be kept within a specified percentage level. (Detailed discussions of steady-state errors will be given later in this section.)

Let us assume that the sampling theorem is satisfied and no frequency folding occurs. The nature of the transient response of a discrete-time control system to a given input depends on the actual locations of the closed-loop poles and zeros in the z plane. Consider the discrete-time control system defined by

$$\frac{C(z)}{R(z)} = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}{z^n + a_1 z^{n-1} + \cdots + a_n} \tag{4-9}$$

where $R(z)$ is the z transform of the input and $C(z)$ is the z transform of the output. The transient response of such a system to the Kronecker delta input, step input, ramp input, and so on, can be obtained easily by use of MATLAB. See, for example, Example 4–8.

Example 4–8

Consider the discrete-time control system defined by

$$\frac{C(z)}{R(z)} = \frac{0.4673z - 0.3393}{z^2 - 1.5327z + 0.6607} \tag{4-10}$$

Obtain the unit-step response of this system.

A MATLAB program for obtaining the unit-step response is shown in MATLAB Program 4–1. The resulting plot of $c(k)$ versus k is shown in Figure 4–17.

MATLAB Program 4–1

```
% ---------- Unit-step response ----------

num = [0  0.4673  -0.3393];
den = [1  -1.5327  0.6607];
r = ones(1,41);
v = [0  40  0  1.6];
axis(v);
k = 0: 40;
c = filter(num,den,r );
plot(k,c,'o')
grid
title('Unit-Step Response')
xlabel('k')
ylabel('c(k)')
```

Steady-State Error Analysis. An important feature associated with transient response is steady-state error. The steady-state performance of a stable control system is generally judged by the steady-state error due to step, ramp, and acceleration inputs. In what follows we shall investigate a type of steady-state error that is caused by the inability of a system to follow particular types of inputs. (It should be noted that, besides this type of steady-state error, there are errors that can be attributed to other causes, such as imperfections in system components, static

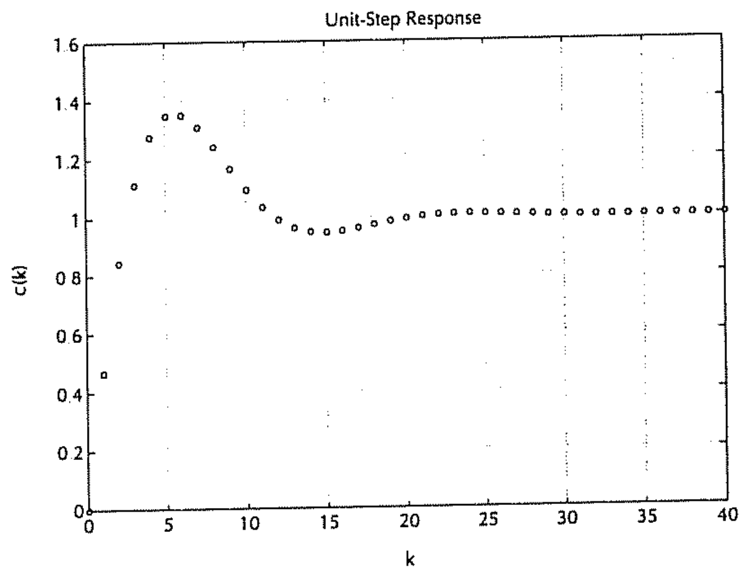


Figure 4–17 Unit-step response of the system defined by Equation (4–10)

friction, backlash, or deterioration or aging of components. In this section, however, we shall not discuss steady-state error due to such causes.)

Any physical control system inherently suffers steady-state error in response to certain types of inputs. That is, a system may have no steady-state error with step inputs, but the same system may exhibit nonzero steady-state error in response to ramp inputs. Whether or not a given system will exhibit steady-state error in response to a given type of input depends on the type of open-loop transfer function of the system.

Consider the continuous-time control system whose open-loop transfer function $G(s)H(s)$ is given by

$$G(s)H(s) = \frac{K(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1)}{s^N(T_1 s + 1)(T_2 s + 1)\cdots(T_p s + 1)}$$

The term s^N in the denominator represents a pole of multiplicity N at the origin. It is customary to classify the system according to the number of integrators in the open-loop transfer function.

A system is said to be of type 0, type 1, type 2, . . . , if $N = 0$, $N = 1$, $N = 2, \ldots$, respectively. Type 0 systems will exhibit finite steady-state errors in response to step inputs and infinite errors in response to ramp and higher-order inputs. Type 1 systems will exhibit no steady-state error in response to step inputs, finite steady-state errors in response to ramp inputs, and infinite steady-state errors in response to acceleration and higher-order inputs. As the type number is increased, accuracy is improved. However, increasing the type number aggravates the stability problem. A compromise between steady-state accuracy and relative stability (transient response characteristics) is always necessary.

The concepts of static error constants can be extended to the discrete-time control system, as discussed in what follows.

Discrete-time control systems can be classified according to the number of open-loop poles at $z = 1$. (An open-loop pole at $z = 1$ corresponds to an integrator in the loop.) Suppose the open-loop pulse transfer function is given by the equation

$$\text{Open-loop pulse transfer function} = \frac{1}{(z-1)^N}\frac{B(z)}{A(z)}$$

where $B(z)/A(z)$ contains neither a pole nor a zero at $z = 1$. Then the system can be classified as a type 0 system, a type 1 system, or a type 2 system according to whether $N = 0$, $N = 1$, or $N = 2$, respectively. The system type specifies the steady-state characteristics or steady-state accuracy.

The physical meaning of the static error constants for discrete-time control systems is the same as that for continuous-time control systems, except that the former transmit information only at the sampling instants.

Consider the discrete-time control system shown in Figure 4–18. We assume that the system is stable so that the final value theorem can be applied to find the steady-state values. From the diagram we have the actuating error

$$e(t) = r(t) - b(t)$$

We shall consider the steady-state actuating error at the sampling instants. Note that from the final value theorem we have

$$\lim_{k\to\infty} e(kT) = \lim_{z\to 1}[(1 - z^{-1})E(z)] \tag{4–11}$$

For the system shown in Figure 4–18, define

$$G(z) = (1 - z^{-1})\mathcal{Z}\left[\frac{G_p(s)}{s}\right]$$

and

$$GH(z) = (1 - z^{-1})\mathcal{Z}\left[\frac{G_p(s)H(s)}{s}\right]$$

Then we have

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + GH(z)}$$

and

$$E(z) = R(z) - B(z) = R(z) - GH(z)E(z)$$

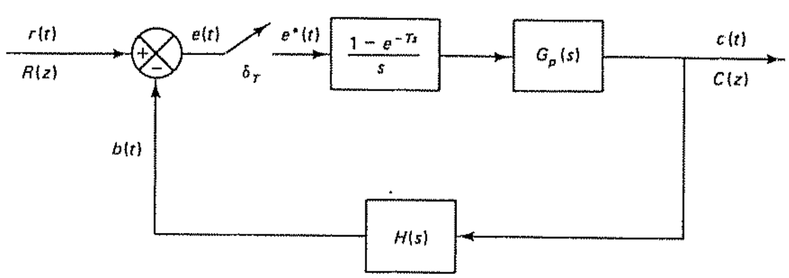


Figure 4–18 Discrete-time control system

or

$$E(z) = \frac{1}{1 + GH(z)} R(z) \tag{4–12}$$

By substituting Equation (4–12) into Equation (4–11), we obtain

$$e_{ss} = \lim_{z \to 1} \left[(1 - z^{-1}) \frac{1}{1 + GH(z)} R(z) \right] \tag{4–13}$$

As in the case of the continuous-time control system, we consider three types of inputs: unit-step, unit-ramp, and unit-acceleration inputs.

Static Position Error Constant. For a unit-step input $r(t) = 1(t)$, we have

$$R(z) = \frac{1}{1 - z^{-1}}$$

By substituting this last equation into Equation (4–13) the steady-state actuating error in response to a unit-step input can be obtained as follows:

$$e_{ss} = \lim_{z \to 1} \left[(1 - z^{-1}) \frac{1}{1 + GH(z)} \frac{1}{1 - z^{-1}} \right] = \lim_{z \to 1} \frac{1}{1 + GH(z)}$$

We define the static position error constant K_p as follows:

$$K_p = \lim_{z \to 1} GH(z) \tag{4–14}$$

Then the steady-state actuating error in response to a unit-step input can be obtained from the equation

$$e_{ss} = \frac{1}{1 + K_p} \tag{4–15}$$

The steady-state actuating error in response to a unit-step input becomes zero if $K_p = \infty$, which requires that $GH(z)$ have at least one pole at $z = 1$.

Static Velocity Error Constant. For a unit-ramp input $r(t) = t1(t)$, we have

$$R(z) = \frac{Tz^{-1}}{(1 - z^{-1})^2}$$

By substituting this last equation into Equation (4–13), we have

$$e_{ss} = \lim_{z \to 1} \left[(1 - z^{-1}) \frac{1}{1 + GH(z)} \frac{Tz^{-1}}{(1 - z^{-1})^2} \right] = \lim_{z \to 1} \frac{T}{(1 - z^{-1})GH(z)}$$

Now we define the static velocity error constant K_v as follows:

$$K_v = \lim_{z \to 1} \frac{(1 - z^{-1})GH(z)}{T} \tag{4–16}$$

Then the steady-state actuating error in response to a unit-ramp input can be given by

$$e_{ss} = \frac{1}{K_v} \tag{4–17}$$

If $K_v = \infty$, then the steady-state actuating error in response to a unit-ramp input is zero. This requires $GH(z)$ to possess a double pole at $z = 1$.

Static Acceleration Error Constant. For a unit acceleration input $r(t) = \frac{1}{2}t^2 1(t)$, we have

$$R(z) = \frac{T^2(1 + z^{-1})z^{-1}}{2(1 - z^{-1})^3}$$

By substituting this last equation into Equation (4–13), we obtain

$$e_{ss} = \lim_{z \to 1}\left[(1 - z^{-1})\frac{1}{1 + GH(z)}\frac{T^2(1 + z^{-1})z^{-1}}{2(1 - z^{-1})^3}\right] = \lim_{z \to 1}\frac{T^2}{(1 - z^{-1})^2 GH(z)}$$

We define the static acceleration error constant K_a as follows:

$$K_a = \lim_{z \to 1}\frac{(1 - z^{-1})^2 GH(z)}{T^2} \tag{4–18}$$

Then the steady-state actuating error becomes

$$e_{ss} = \frac{1}{K_a} \tag{4–19}$$

The steady-state actuating error in response to a unit-acceleration input becomes zero if $K_a = \infty$. This requires $GH(z)$ to possess a triple pole at $z = 1$.

Equations (4–15), (4–17), and (4–19) give the expressions for steady-state actuating errors of the discrete-time control system shown in Figure 4-18 at the sampling instants for a unit-step, unit-ramp, and unit-acceleration input, respectively.

Summary. It is important to emphasize that the actuating error is the difference between the reference input and the feedback signal, not the difference between the reference input and the output. From the foregoing analysis we see that a type 0 system will exhibit a constant steady-state actuating error in response to a step input and an infinite actuating error in response to ramp, acceleration, or higher-order inputs. A type 1 system will exhibit a zero steady-state actuating error in response to a step input, a constant steady-state error in response to a ramp input, and an infinite steady-state actuating error in response to acceleration or higher-order inputs.

TABLE 4–4 SYSTEM TYPES AND THE CORRESPONDING STEADY-STATE ERRORS IN RESPONSE TO STEP, RAMP, AND ACCELERATION INPUTS FOR THE DISCRETE-TIME CONTROL SYSTEM SHOWN IN FIGURE 4–18

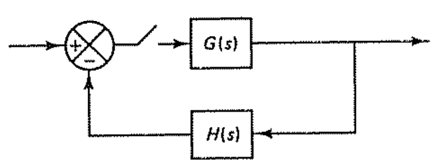
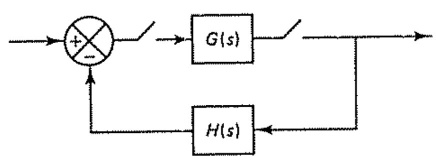
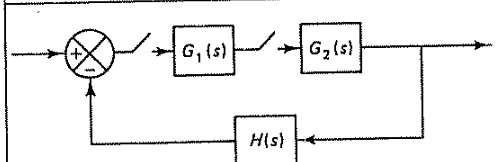
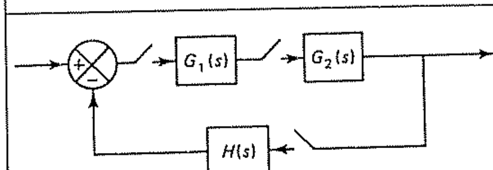
System	Steady-state errors in response to		
	Step input $r(t) = 1$	Ramp input $r(t) = t$	Acceleration input $r(t) = \frac{1}{2}t^2$
Type 0 system	$\frac{1}{1 + K_p}$	∞	∞
Type 1 system	0	$\frac{1}{K_v}$	∞
Type 2 system	0	0	$\frac{1}{K_a}$

Table 4–4 lists system types and the corresponding steady-state errors in response to step, ramp, and acceleration inputs for the discrete-time control system of the configuration shown in Figure 4–18.

The steady-state error analysis just presented applies to the closed-loop discrete-time control system shown in Figure 4–18. For a different closed-loop configuration, it is noted that if the closed-loop discrete-time control system has a closed-loop pulse transfer function, then the static error constants can be determined by an analysis similar to the one just presented. Table 4–5 lists the static error constants for typical closed-loop configurations of discrete-time control systems. If the closed-loop discrete-time control system does not have a closed-loop pulse transfer function, however, the static error constants cannot be defined, because the input signal cannot be separated from the system dynamics.

It is important to note that the terms "position error," "velocity error," and "acceleration error" mean steady-state deviations in the output position. A finite velocity error implies that after transients have died out the input and output move at the same velocity, but have a finite position difference.

TABLE 4–5 STATIC ERROR CONSTANTS FOR TYPICAL CLOSED-LOOP CONFIGURATIONS OF DISCRETE-TIME CONTROL SYSTEMS

Closed-loop configuration	Values of K_p, K_v, and K_a
	$K_p = \lim_{z\to1} GH(z)$ $K_v = \lim_{z\to1} \frac{(1-z^{-1})GH(z)}{T}$ $K_a = \lim_{z\to1} \frac{(1-z^{-1})^2GH(z)}{T^2}$
	$K_p = \lim_{z\to1} G(z)H(z)$ $K_v = \lim_{z\to1} \frac{(1-z^{-1})G(z)H(z)}{T}$ $K_a = \lim_{z\to1} \frac{(1-z^{-1})^2G(z)H(z)}{T^2}$
	$K_p = \lim_{z\to1} G_1(z)HG_2(z)$ $K_v = \lim_{z\to1} \frac{(1-z^{-1})G_1(z)HG_2(z)}{T}$ $K_a = \lim_{z\to1} \frac{(1-z^{-1})^2G_1(z)HG_2(z)}{T^2}$
	$K_p = \lim_{z\to1} G_1(z)G_2(z)H(z)$ $K_v = \lim_{z\to1} \frac{(1-z^{-1})G_1(z)G_2(z)H(z)}{T}$ $K_a = \lim_{z\to1} \frac{(1-z^{-1})^2G_1(z)G_2(z)H(z)}{T^2}$

Response to Disturbances. In examining transient response characteristics and steady-state errors, it is important to note that the effects of disturbances, in addition to those of reference inputs, must be explored.

For the system shown in Figure 4–19(a), let us assume that the reference input is zero, or $R(z) = 0$, but the system is subjected to disturbance $N(z)$. For this case the system block diagram can be redrawn as shown in Figure 4–19(b). Then the response $C(z)$ to disturbance $N(z)$ can be found from the closed-loop pulse transfer function

$$\frac{C(z)}{N(z)} = \frac{G(z)}{1 + G_D(z)G(z)}$$

If $|G_D(z)G(z)| \gg 1$, then we find

$$\frac{C(z)}{N(z)} \cong \frac{1}{G_D(z)}$$

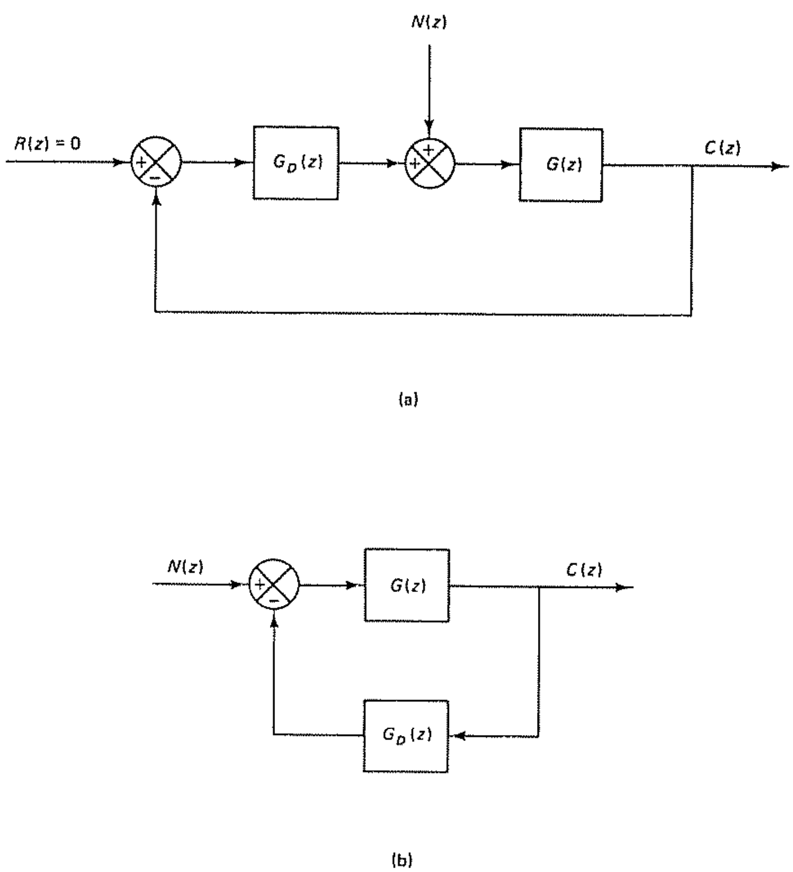


Figure 4–19 (a) Digital closed-loop control system subjected to reference input and disturbance input; (b) modified block diagram where the disturbance input is considered the input to the system

Since the system error is

$$E(z) = R(z) - C(z) = -C(z)$$

we find the error $E(z)$ due to the disturbance $N(z)$ to be

$$E(z) = -\frac{1}{G_D(z)}N(z)$$

Thus, the larger the gain of $G_D(z)$ is, the smaller the error $E(z)$. If $G_D(z)$ includes an integrator [which means that $G_D(z)$ has a pole at $z = 1$], then the steady-state error due to a constant disturbance is zero. This may be seen as follows. Since for a constant disturbance of magnitude N we have

$$N(z) = \frac{N}{1 - z^{-1}}$$

if $G_D(z)$ involves a pole at $z = 1$, then it may be written as

$$G_D(z) = \frac{\hat{G}_D(z)}{z - 1} = \frac{\hat{G}_D(z)z^{-1}}{1 - z^{-1}}$$

where $\hat{G}_D(z)$ does not involve any zeros at $z = 1$. Then the steady-state error can be given by

$$\begin{aligned} e_{ss} &= \lim_{z\to 1}\left[(1 - z^{-1})E(z)\right] = \lim_{z\to 1}\left[(1 - z^{-1})\frac{-N(z)}{G_D(z)}\right] \\ &= -\lim_{z\to 1}\left[(1 - z^{-1})\frac{N}{1 - z^{-1}}\frac{1}{G_D(z)}\right] = -\lim_{z\to 1}\frac{(1 - z^{-1})N}{\hat{G}_D(z)z^{-1}} = 0 \end{aligned}$$

If a linear system is subjected to both the reference input and a disturbance input, then the resulting error is the sum of the errors due to the reference input and the disturbance input. The total error must be kept within acceptable limits.

Note that the point where the disturbance enters the system is very important in adjusting the gain of $G_D(z)G(z)$. For example, consider the system shown in Figure 4–20(a). The closed-loop pulse transfer function for the disturbance is

$$\frac{C(z)}{N(z)} = -\frac{E(z)}{N(z)} = \frac{1}{1 + G_D(z)G(z)}$$

To minimize the effects of disturbance $N(z)$ on the system error $E(z)$, the gain of $G_D(z)G(z)$ must be made as large as possible. However, for the system shown in Figure 4–20(b), the closed-loop pulse transfer function for the disturbance is

$$\frac{C(z)}{N(z)} = -\frac{E(z)}{N(z)} = \frac{G_D(z)G(z)}{1 + G_D(z)G(z)}$$

and to minimize the effects of disturbance $N(z)$ on the system error $E(z)$, the gain of $G_D(z)G(z)$ must be made as small as possible.

Therefore, it is advantageous to obtain the expression for $E(z)/N(z)$ before concluding whether the gain of $G_D(z)G(z)$ should be large or small to minimize the error due to disturbances. It is important to remember, however, that the magnitude of the gain cannot be determined solely from the disturbance considerations. It must be determined by considering the responses to both reference and disturbance inputs. If the frequency regions for the reference input and disturbance input are sufficiently apart, a suitable filter may be inserted in the system. If the frequency

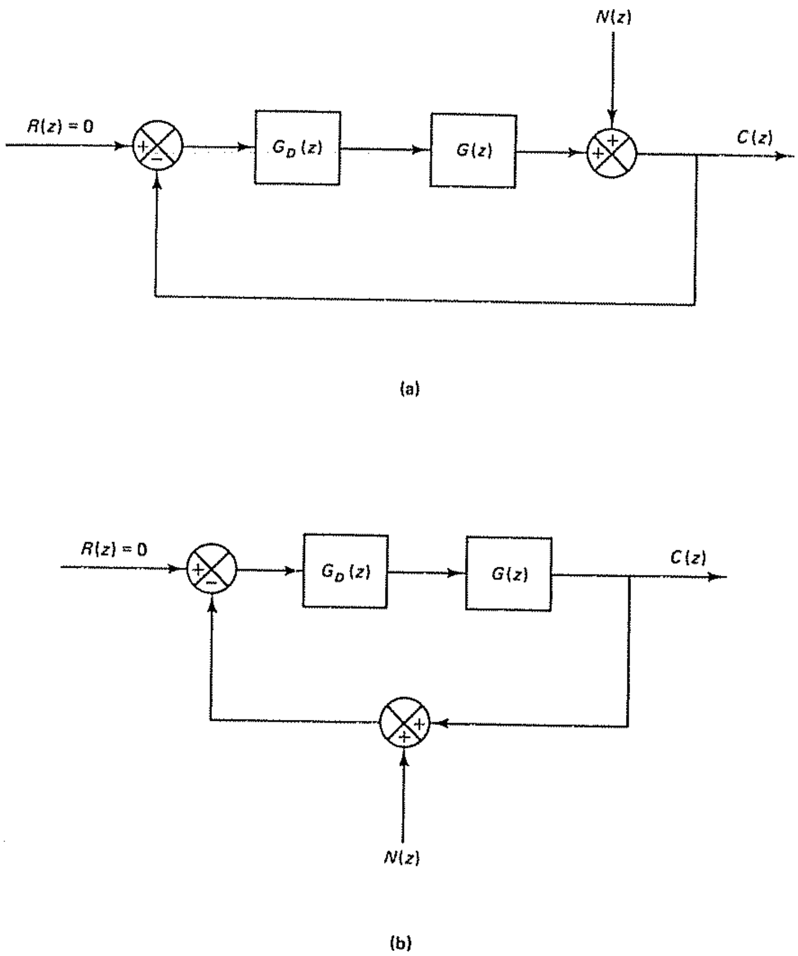


Figure 4–20 (a) Digital closed-loop control system subjected to reference input and disturbance input; (b) digital closed-loop control system where the disturbance enters the feedback loop

regions overlap, then modification of the block diagram configuration may become necessary to get acceptable responses to both reference and disturbance inputs.

4–5 DESIGN BASED ON THE ROOT-LOCUS METHOD

As discussed in Section 4–4, the relative stability of the discrete-time control system may be investigated with respect to the unit circle in the z plane. For example, if the closed-loop poles are complex conjugates and lie inside the unit circle, the unit-step response will be oscillatory.

In addition to the transient response characteristics of a given system, it is often necessary to investigate the effects of the system gain and/or sampling period on the absolute and relative stability of the closed-loop system. For such purposes the root-locus method proves to be very useful.

The root-locus method developed for continuous-time systems can be extended to discrete-time systems without modifications, except that the stability boundary is changed from the $j\omega$ axis in the s plane to the unit circle in the z plane. The reason the root-locus method can be extended to discrete-time systems is that the characteristic equation for the discrete-time system is of the same form as that for the continuous-time system in the s plane. For example, for the system shown in Figure 4–21 the characteristic equation is

$$1 + G(z)H(z) = 0$$

which is of exactly the same form as the equation for root-locus analysis in the s plane. However, the pole locations for closed-loop systems in the z plane must be interpreted differently from those in the s plane.

In this section we shall demonstrate the application of the root-locus method to the design of discrete-time or digital control systems.

Computer programs for calculating and tracing root loci are available for most computer systems. In particular, MATLAB provides a convenient means to plot root loci for both continuous-time and discrete-time closed-loop systems. Exact plotting of the root loci can be done on the computer and, therefore, we may not need graphical plotting procedures. However, skill in plotting root loci is an advantage, since it will enable the control engineer to make quick graphical plots for given problems to speed up preliminary stages of system design. In fact, the experienced control engineer frequently uses the root-locus approach to a preliminary design to locate the dominant closed-loop poles at desired positions in the z plane and then uses a digital simulation to improve the closed-loop performance.

Angle and Magnitude Conditions. In many linear time-invariant discrete-time control systems, the characteristic equation may have either of the following two forms:

$$1 + G(z)H(z) = 0$$

and

$$1 + GH(z) = 0$$

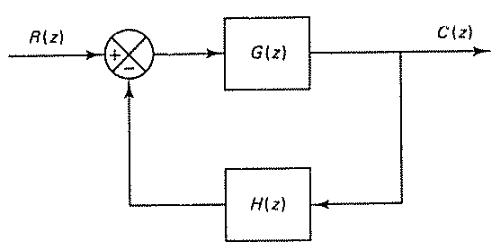


Figure 4–21 Closed-loop control system.

To combine these two forms into one, let us define the characteristic equation as

$$1 + F(z) = 0 \tag{4–20}$$

where

$$F(z) = G(z)H(z) \quad \text{or} \quad F(z) = GH(z)$$

Note that $F(z)$ is the open-loop pulse transfer function. The characteristic equation given by Equation (4–20) can then be written as

$$F(z) = -1$$

Since $F(z)$ is a complex quantity, this last equation can be split into two equations by equating first the angles and then the magnitudes of the two sides to obtain

ANGLE CONDITION:

$$\angle F(z) = \pm 180°(2k + 1), \qquad k = 0, 1, 2, \ldots$$

MAGNITUDE CONDITION:

$$|F(z)| = 1$$

The values of z that fulfill both the angle and the magnitude conditions are the roots of the characteristic equation, or the closed-loop poles.

A plot of the points in the complex plane satisfying the angle condition alone is the root locus. The roots of the characteristic equation (the closed-loop poles) corresponding to a given value of the gain can be located on the root loci by use of the magnitude condition. The details of applying the angle and magnitude conditions to obtain the closed-loop poles are presented in the following.

General Procedure for Constructing Root Loci. For a complicated system with many open-loop poles and zeros, constructing a root-locus plot may seem complicated, but actually it is not difficult if the established rules for constructing root loci are applied.

By locating particular points and asymptotes and by computing angles of departure from complex poles and angles of arrival at complex zeros, it is possible to construct root loci without difficulty. Note that while root loci may be conveniently drawn with a digital computer, if manual construction of the root locus plot is attempted, we essentially proceed on a trial-and-error basis. But the number of trials required can be greatly reduced if the established rules are used.

Because the open-loop conjugate complex poles and conjugate complex zeros, if any, are always located symmetrically about the real axis, the root loci are always symmetric with respect to the real axis. Hence, we need only construct the upper half of the root loci and draw the mirror image of the upper half in the lower half of the z plane. Remember that the angles of the complex quantities originating from the open-loop poles and open-loop zeros and drawn to a test point z are measured in the counterclockwise direction. We shall now present the general rules and procedures for constructing root loci.

General Rules for Constructing Root Loci

1. Obtain the characteristic equation

$$1 + F(z) = 0$$

and then rearrange this equation so that the parameter of interest, such as gain K, appears as the multiplying factor in the form

$$1 + \frac{K(z + z_1)(z + z_2) \cdots (z + z_m)}{(z + p_1)(z + p_2) \cdots (z + p_n)} = 0$$

In the present discussion, we assume that the parameter of interest is gain K, where $K > 0$. From the factored form of the open-loop pulse transfer function, locate the open-loop poles and zeros in the z plane. [Note that if $F(z) = G(z)H(z)$ then the open-loop zeros are zeros of $G(z)H(z)$, while the closed-loop zeros consist of the zeros of $G(z)$ and the poles of $H(z)$.]

2. Find the starting points and terminating points of the root loci. Find also the number of separate branches of the root loci. The points on the root loci corresponding to $K = 0$ are open-loop poles and those corresponding to $K = \infty$ are open-loop zeros. Hence, as K is increased from zero to infinity, a root locus starts from an open-loop pole and terminates at a finite open-loop zero or an open-loop zero at infinity. This means that a root-locus plot will have just as many branches as there are roots of the characteristic equation. [If the zeros at infinity are included in the count, $F(z)$ has the same number of zeros as poles.]

If the number n of closed-loop poles is the same as the number of open-loop poles, then the number of individual root locus branches terminating at finite open-loop zeros is equal to the number m of the open-loop zeros. The remaining $n - m$ branches terminate at infinity (at $n - m$ implicit zeros at infinity) along asymptotes.

3. Determine the root loci on the real axis. Root loci on the real axis are determined by open-loop poles and zeros lying on it. The conjugate complex poles and zeros of the open-loop pulse transfer function have no effect on the location of the root loci on the real axis because the angle contribution of a pair of conjugate complex poles or zeros is 360° on the real axis. Each portion of the root locus on the real axis extends over a range from a pole or zero to another pole or zero.

In constructing the root loci on the real axis, choose a test point on it. If the total number of real poles and real zeros to the right of this test point is odd, then this point lies on a root locus. The root locus and its complement form alternate segments along the real axis.

4. Determine the asymptotes of the root loci. If the test point z is located far from the origin, then the angles of all the complex quantities may be considered the same. One open-loop zero and one open-loop pole then each cancel the effects of the other.

Therefore, the root loci for very large values of z must be asymptotic to straight lines whose angles are given as follows:

$$\text{Angle of asymptote} = \frac{\pm 180°(2N + 1)}{n - m}, \qquad N = 0, 1, 2, \ldots$$

where

$$n = \text{number of finite poles of } F(z)$$

$$m = \text{number of finite zeros of } F(z)$$

Here, $N = 0$ corresponds to the asymptote with the smallest angle with the real axis. Although N assumes an infinite number of values, the angle repeats itself, as N is increased, and the number of distinct asymptotes is $n - m$.

All the asymptotes intersect on the real axis. The point at which they do so is obtained as follows. Since

$$F(z) = \frac{K[z^m + (z_1 + z_2 + \cdots + z_m)z^{m-1} + \cdots + z_1 z_2 \cdots z_m]}{z^n + (p_1 + p_2 + \cdots + p_n)z^{n-1} + \cdots + p_1 p_2 \cdots p_n}$$

$$= \frac{K}{z^{n-m} + [(p_1 + p_2 + \cdots + p_n) - (z_1 + z_2 + \cdots + z_m)]z^{n-m-1} + \cdots}$$

for a large value of z this last equation may be approximated as follows:

$$F(z) \doteqdot \frac{K}{\left[z + \dfrac{(p_1 + p_2 + \cdots + p_n) - (z_1 + z_2 + \cdots + z_m)}{n - m}\right]^{n-m}}$$

If the abscissa of the intersection of the asymptotes and the real axis is denoted by $-\sigma_a$, then

$$-\sigma_a = -\frac{(p_1 + p_2 + \cdots + p_n) - (z_1 + z_2 + \cdots + z_m)}{n - m} \tag{4-21}$$

Because all the complex poles and zeros occur in conjugate pairs, $-\sigma_a$ given by Equation (4–21) is always a real quantity.

Once the intersection of the asymptotes and the real axis is found, the asymptotes can be readily drawn in the complex z plane.

5. Find the breakaway and break-in points. Because of the conjugate symmetry of the root loci, the breakaway points and break-in points either lie on the real axis or occur in conjugate complex pairs.

If a root locus lies between two adjacent open-loop poles on the real axis, then there exists at least one breakaway point between the two poles. Similarly, if the root locus lies between two adjacent zeros (one zero may be located at $-\infty$) on the real axis, then there always exists at least one break-in point between the two zeros.

If the root locus lies between an open-loop pole and a zero (finite or infinite) on the real axis, then there may exist no breakaway or break-in points or there may exist both breakaway and break-in points.

If the characteristic equation

$$1 + F(z) = 0$$

is written as

$$1 + \frac{KB(z)}{A(z)} = 0$$

where $KB(z)/A(z) = F(z)$, then

$$K = -\frac{A(z)}{B(z)} \tag{4-22}$$

and the breakaway and break-in points (which correspond to multiple roots) can be determined from the roots of

$$\frac{dK}{dz} = -\frac{A'(z)B(z) - A(z)B'(z)}{B^2(z)} = 0 \tag{4-23}$$

where the prime indicates differentiation with respect to z. (See Problem A-4-5 for a proof.)

If the value of K corresponding to a root $z = z_0$ of $dK/dz = 0$ is positive, point $z = z_0$ is an actual breakaway or break-in point. Since K is assumed to be nonnegative, if the value of K thus obtained is negative, then point $z = z_0$ is neither a breakaway nor a break-in point.

Note that this approach can be used when there are complex poles and/or complex zeros.

6. Determine the angle of departure (or angle of arrival) of the root loci from the complex poles (or at the complex zeros). To sketch the root loci with reasonable accuracy, we must find the direction of the root loci near the complex poles and zeros. The angle of departure (or angle of arrival) of the root locus from a complex pole (or at a complex zero) can be found by subtracting from 180° the sum of all the angles of lines (complex quantities) from all other poles and zeros to the complex pole (or complex zero) in question, with appropriate signs included. The angle of departure is shown in Figure 4-22.

7. Find the points where the root loci cross the imaginary axis. The points where the root loci intersect the imaginary axis can be found by setting $z = j\nu$ in the characteristic equation (which involves undetermined gain K), equating both the real part and the imaginary part to zero, and solving for ν and K. The values of ν and

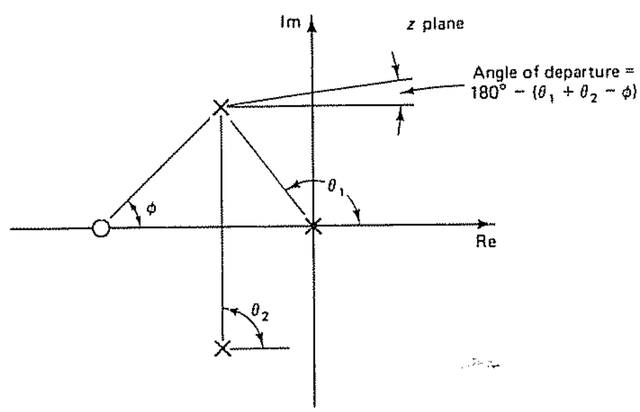


Figure 4-22 Diagram showing angle of departure

K thus found give the location at which the root loci cross the imaginary axis and the value of the corresponding gain K, respectively.

8. Any point on the root loci is a possible closed-loop pole. A particular point will be a closed-loop pole when the value of gain K satisfies the magnitude condition. Conversely, the magnitude condition enables us to determine the value of gain K at any specific root location on the locus. The magnitude condition is

$$|F(z)| = 1$$

or

$$\left| \frac{(z + z_1)(z + z_2)\cdots(z + z_m)}{(z + p_1)(z + p_2)\cdots(z + p_n)} \right| = \frac{1}{K} \tag{4–24}$$

If gain K of the open-loop pulse transfer function is given in the problem, then by applying the magnitude condition, Equation (4–24), it is possible to locate the closed-loop poles for a given K on each branch of the root loci by a trial-and-error method.

Cancellation of Poles of G(z) With Zeros of H(z). It is important to note that if $F(z) = G(z)H(z)$ and the denominator of $G(z)$ and the numerator of $H(z)$ involve common factors then the corresponding open-loop poles and zeros will cancel each other, reducing the degree of the characteristic equation by one or more. The root locus plot of $G(z)H(z)$ will not show all the roots of the characteristic equation, but only the roots of the reduced equation.

To obtain the complete set of closed-loop poles, we must add the canceled pole or poles of $G(z)H(z)$ to those closed-loop poles obtained from the root locus plot of $G(z)H(z)$. The important thing to remember is that a canceled pole of $G(z)H(z)$ is a closed-loop pole of the system.

As an example, consider the case where $G(z)$ and $H(z)$ of the system shown in Figure 4–21 are given by

$$G(z) = \frac{z + c}{(z + a)(z + b)}$$

and

$$H(z) = \frac{z + a}{z + d}$$

Then, clearly, the pole $z = -a$ of $G(z)$ and the zero $z = -a$ of $H(z)$ cancel each other, resulting in

$$G(z)H(z) = \frac{z + c}{(z + a)(z + b)} \frac{z + a}{z + d} = \frac{z + c}{(z + b)(z + d)}$$

However, the closed-loop pulse transfer function of the system is

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)H(z)} = \frac{(z + c)(z + d)}{(z + a)[(z + b)(z + d) + z + c]}$$

and we see that $z = -a$, the canceled pole of $G(z)H(z)$, is a closed-loop pole of the closed-loop system.

Note, however, that if pole–zero cancellation occurs in the feed-forward pulse transfer function, then the same pole–zero cancellation occurs in the closed-loop pulse transfer function. Consider again the system shown in Figure 4–21, where we assume

$$G(z) = G_D(z)G_1(z), \qquad H(z) = 1$$

Suppose pole–zero cancellation occurs in $G_D(z)G_1(z)$. For example, suppose

$$G_D(z)G_1(z) = \frac{z+b}{z+a}\,\frac{z+d}{(z+b)(z+c)} = \frac{z+d}{(z+a)(z+c)}$$

Then the closed-loop pulse transfer function becomes

$$\frac{C(z)}{R(z)} = \frac{G_D(z)G_1(z)}{1+G_D(z)G_1(z)} = \frac{(z+b)(z+d)}{(z+b)[(z+a)(z+c)+z+d]}$$

$$= \frac{z+d}{(z+a)(z+c)+z+d}$$

Because of the pole–zero cancellation, the third-order system becomes one of second order.

It is important to summarize that the effect of pole–zero cancellation in $G(z)$ and $H(z)$ is different from that of pole–zero cancellation in the feed-forward pulse transfer function (such as pole–zero cancellation in the digital controller and the plant). In the former, the canceled pole is still a pole of the closed-loop system, whereas in the latter the canceled pole does not appear as a pole of the closed-loop system (in the latter the order of the system is reduced by the number of canceled poles).

Root-Locus Diagrams of Digital Control Systems. In what follows we shall investigate the effects of gain K and sampling period T on the relative stability of the closed-loop control system. Consider the system shown in Figure 4–23. Assume that the digital controller is of the integral type, or that

$$G_D(z) = \frac{K}{1-z^{-1}} = K\frac{z}{z-1}$$

Let us draw root locus diagrams for the system for three values of the sampling period T: 0.5 sec, 1 sec, and 2 sec. Let us also determine the critical value of K for each case. And finally let us locate the closed-loop poles corresponding to $K = 2$ for each of the three cases.

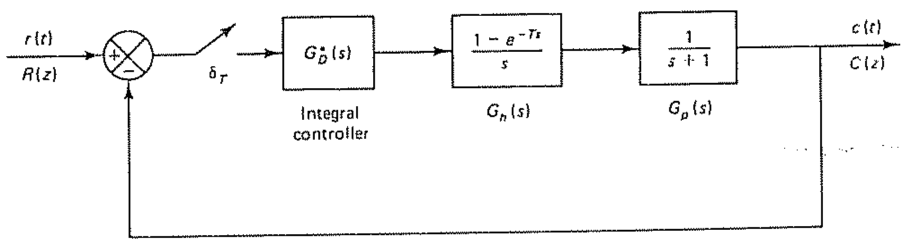


Figure 4–23 Digital control system

We shall first obtain the z transform of $G_h(s)G_p(s)$:

$$\begin{aligned}\mathcal{Z}[G_h(s)G_p(s)] &= \mathcal{Z}\left[\frac{1-e^{-Ts}}{s}\frac{1}{s+1}\right] \\ &= (1-z^{-1})\mathcal{Z}\left[\frac{1}{s(s+1)}\right] \\ &= (1-z^{-1})\mathcal{Z}\left[\frac{1}{s}-\frac{1}{s+1}\right] \\ &= \frac{z-1}{z}\left(\frac{z}{z-1}-\frac{z}{z-e^{-T}}\right) \\ &= \frac{1-e^{-T}}{z-e^{-T}}\end{aligned}$$

The feedforward pulse transfer function becomes

$$G(z) = G_D(z)\mathcal{Z}[G_h(s)G_p(s)] = \frac{Kz}{z-1}\frac{1-e^{-T}}{z-e^{-T}} \tag{4–25}$$

The characteristic equation is

$$1 + G(z) = 0$$

or

$$1 + \frac{Kz(1-e^{-T})}{(z-1)(z-e^{-T})} = 0 \tag{4–26}$$

1. *Sampling period $T = 0.5$ sec*: For this case, Equation (4–25) becomes

$$G(z) = \frac{0.3935Kz}{(z-1)(z-0.6065)}$$

Notice that $G(z)$ has poles at $z = 1$ and $z = 0.6065$ and a zero at $z = 0$.

To draw a root locus diagram, we first locate the poles and zero in the z plane and then find the breakaway point and break-in point. Notice that this open-loop pulse transfer function with two poles and one zero results in a circular root locus centered at the zero. The breakaway point and break-in point are determined by writing the characteristic equation in the form of Equation (4–22),

$$K = -\frac{(z-1)(z-0.6065)}{0.3935z} \tag{4–27}$$

and differentiating K with respect to z and equating the result to zero:

$$\frac{dK}{dz} = -\frac{z^2-0.6065}{0.3935z^2} = 0$$

Hence,

$$z^2 = 0.6065$$

or

$$z = 0.7788 \qquad \text{and} \qquad z = -0.7788$$

Notice that substitution of 0.7788 for z in Equation (4–27) yields $K = 0.1244$, while letting $z = -0.7788$ yields $K = 8.041$. Since both K values are positive, $z = 0.7788$ is the actual breakaway point and $z = -0.7788$ is the actual break-in point.

Figure 4–24(a) shows the root locus diagram when $T = 0.5$ sec. The critical value of gain K for this case is obtained by use of the magnitude condition, which can be obtained from Equation (4–26) as follows:

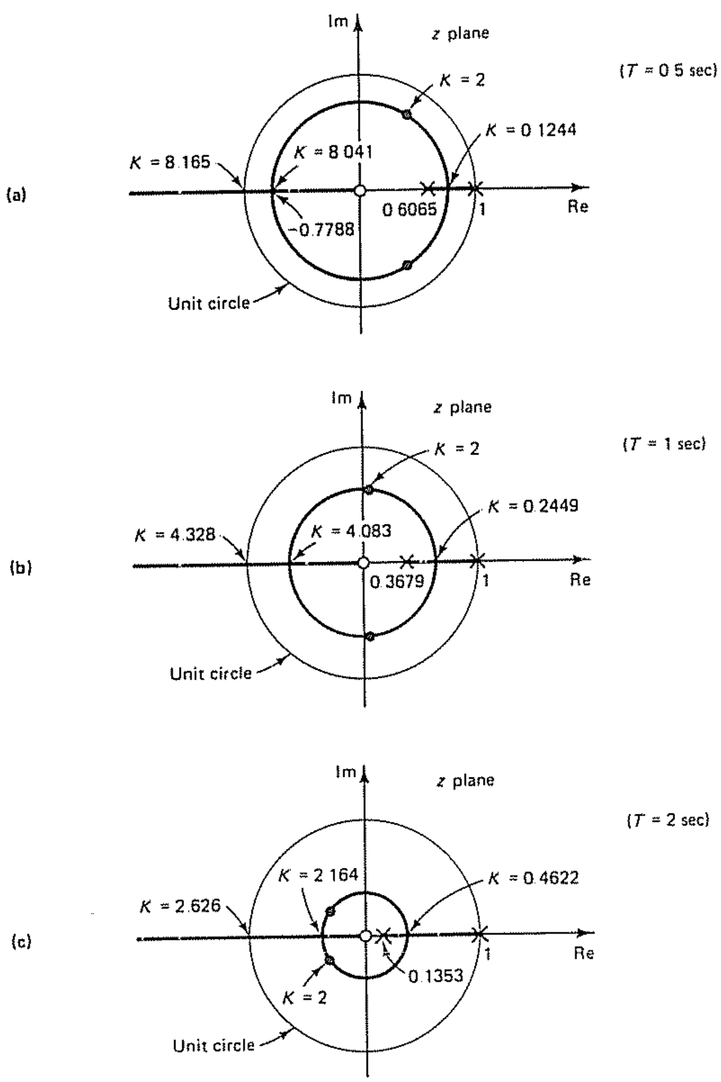


Figure 4–24 (a) Root locus diagram for the system shown in Figure 4–23 when $T = 0.5$ sec; (b) root locus diagram when $T = 1$ sec; (c) root locus diagram when $T = 2$ sec.

$$\left| \frac{z(1 - e^{-T})}{(z - 1)(z - e^{-T})} \right| = \frac{1}{K}$$

For the present case, $T = 0.5$ and this last equation becomes

$$\left| \frac{0.3935z}{(z - 1)(z - 0.6065)} \right| = \frac{1}{K} \qquad (4\text{–}28)$$

Since the critical gain K_c corresponds to point $z = -1$, we substitute -1 for z in Equation (4–28):

$$\left| \frac{0.3935(-1)}{(-2)(-1.6065)} \right| = \frac{1}{K}$$

or

$$K = 8.165$$

The critical gain K_c is thus 8.165.

The closed-loop poles corresponding to $K = 2$ can be found to be

$$z_1 = 0.4098 + j0.6623 \qquad \text{and} \qquad z_2 = 0.4098 - j0.6623$$

These closed-loop poles are indicated by dots in the root locus diagram.

2. *Sampling period $T = 1$ sec*: For this case, Equation (4–25) becomes as follows:

$$G(z) = \frac{0.6321Kz}{(z - 1)(z - 0.3679)}$$

Hence, $G(z)$ has poles at $z = 1$ and $z = 0.3679$ and a zero at $z = 0$.

The breakaway point and break-in point are found to be $z = 0.6065$ and $z = -0.6065$, respectively. The corresponding gain values are $K = 0.2449$ and $K = 4.083$, respectively.

Figure 4–24(b) shows the root locus diagram when $T = 1$ sec. The critical value of gain K is 4.328. The closed-loop poles corresponding to $K = 2$ are found to be

$$z_1 = 0.05185 + j0.6043 \qquad \text{and} \qquad z_2 = 0.05185 - j0.6043$$

and are shown in the root locus diagram by dots.

3. *Sampling period $T = 2$ sec*: For this case, Equation (4–25) becomes

$$G(z) = \frac{0.8647Kz}{(z - 1)(z - 0.1353)}$$

We see that $G(z)$ has poles at $z = 1$ and $z = 0.1353$ and a zero at $z = 0$.

The breakaway point and break-in point are found to be $z = 0.3678$ and $z = -0.3678$, with corresponding gain values $K = 0.4622$ and $K = 2.164$, respectively. The critical value of gain K for this case is 2.626.

Figure 4–24(c) shows the root locus diagram when $T = 2$ sec. The closed-loop poles corresponding to $K = 2$ are found to be

$$z_1 = -0.2971 + j0.2169 \qquad \text{and} \qquad z_2 = -0.2971 - j0.2169$$

These closed-loop poles are shown by dots in the root locus diagram.

Effects of Sampling Period T on Transient Response Characteristics. The transient response characteristics of the discrete-time control system depend on the sampling period T. A large sampling period has detrimental effects on the relative stability of the system. A rule of thumb is to sample eight to ten times during a cycle of the damped sinusoidal oscillations of the output of the closed-loop system, if it is underdamped. For overdamped systems, sample eight to ten times during the rise time in the step response.

As seen from the preceding analysis, for a given value of gain K, increasing the sampling period T will make the discrete-time control system less stable and eventually will make it unstable. Conversely, making the sampling period T shorter allows the critical value of gain K for stability to be larger. In fact, making the sampling period shorter and shorter tends to make the system behave more like the continuous-time system. (For the continuous-time second-order control system, the critical gain for stability is infinity, or $K = \infty$.)

For the system shown in Figure 4–23, the damping ratio ζ for the closed-loop poles for $K = 2$ for each of the preceding three cases can be found from Figure 4–25. Graphically, the damping ratios for the closed-loop poles corresponding to $T = 0.5$, $T = 1$, and $T = 2$ are determined approximately as $\zeta = 0.24$, $\zeta = 0.32$, and $\zeta = 0.37$, respectively.

The damping ratio ζ of a closed-loop pole can be analytically determined from the location of the closed-loop pole in the z plane. If the damping ratio of a closed-loop pole is ζ, then in the s plane the closed-loop pole location (in the upper half-plane) can be given by

$$s = -\zeta\omega_n + j\omega_n\sqrt{1 - \zeta^2}$$

Since $z = e^{Ts}$, the corresponding point in the z plane is

$$z = \exp[T(-\zeta\omega_n + j\omega_n\sqrt{1 - \zeta^2})]$$

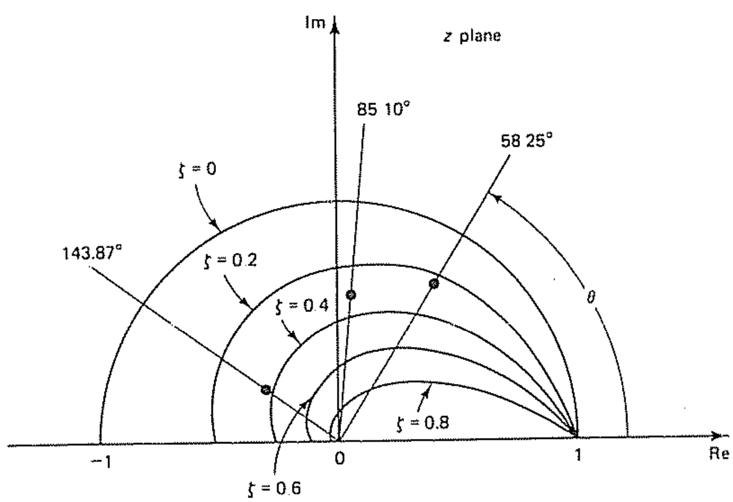


Figure 4–25 Closed-loop pole locations in the z plane shown with constant ζ loci.

from which we get

$$|z| = e^{-T\zeta\omega_n} \tag{4–29}$$

and

$$\underline{/z} = T\omega_n\sqrt{1-\zeta^2} = T\omega_d = \theta \text{ (rad)} \tag{4–30}$$

From Equations (4–29) and (4–30), the value of ζ can be calculated. For example, in the case where the sampling period T is 0.5 sec, we have the closed-loop pole for $K = 2$ at $z = 0.4098 + j0.6623$. Hence,

$$|z| = \sqrt{0.4098^2 + 0.6623^2} = 0.7788$$

By solving

$$|z| = e^{-T\zeta\omega_n} = 0.7788$$

for the exponent, we find

$$T\zeta\omega_n = 0.25 \tag{4–31}$$

Also,

$$\underline{/z} = \tan^{-1}\frac{0.6623}{0.4098} = 58.25° = 1.0167 \text{ rad}$$

Hence,

$$\underline{/z} = T\omega_n\sqrt{1-\zeta^2} = 1.0167 \text{ rad} \tag{4–32}$$

From Equations (4–31) and (4–32), we obtain

$$\frac{T\zeta\omega_n}{T\omega_n\sqrt{1-\zeta^2}} = \frac{0.25}{1.0167}$$

or

$$\frac{\zeta}{\sqrt{1-\zeta^2}} = 0.2459$$

which yields

$$\zeta = 0.2388$$

(From Figure 4–25 we graphically obtained 0.24 for ζ, which is very close to the actual ζ value of 0.2388.)

It is important to point out that in the second-order system the damping ratio ζ is indicative of the relative stability (for example, in respect to the maximum overshoot in the unit-step response) only if the sampling frequency is sufficiently high (so that there are eight to ten samplings in a cycle of oscillation). If the sampling frequency is not high enough, the maximum overshoot in the unit-step response will be much higher than would be predicted by the damping ratio ζ.

To compare the effects of different sampling periods T on the transient response, we shall compare the unit-step response sequences for the three values of T considered in the preceding analysis.

The closed-loop pulse transfer function for the system of Figure 4–23, whose feedforward pulse transfer function $G(z)$ is given by Equation (4–25), is

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)} = \frac{Kz(1 - e^{-T})}{(z - 1)(z - e^{-T}) + Kz(1 - e^{-T})}$$

For $T = 0.5$ sec and $K = 2$, the unit-step response can be given by

$$C(z) = \frac{0.3935 \times 2z}{(z - 1)(z - 0.6065) + 0.3935 \times 2z} R(z)$$
$$= \frac{0.7870z^{-1}}{1 - 0.8195z^{-1} + 0.6065z^{-2}} \frac{1}{1 - z^{-1}}$$

from which we obtain the unit-step response sequence $c(kT)$ versus kT shown in Figure 4–26(a).

From Figure 4–25 we see that the angle θ of the line connecting the origin and the dominant closed-loop pole at $z = 0.4098 + j0.6623$ (this line is a constant ω line in the s plane) is approximately 58.25°. The angle θ of the dominant closed-loop poles determines the number of samples per cycle of sinusoidal oscillation. Note that

$$\cos \theta k = \cos \theta\left(k + \frac{360°}{\theta}\right)$$

Hence, for $\theta = 58.25°$, we have $360°/\theta = 360°/58.25° = 6.18$ samples per cycle of damped oscillation, as seen from Figure 4–26(a).

Similarly, for $T = 1$ sec and $K = 2$, the unit-step response is given by

$$C(z) = \frac{1.2642z^{-1}}{1 - 0.1037z^{-1} + 0.3679z^{-2}} \frac{1}{1 - z^{-1}}$$

The unit-step response sequence $c(kT)$ versus kT is shown in Figure 4–26(b). Since the angle of the line connecting the origin and the closed-loop pole for the present case is 85.10°, as shown in Figure 4–25, we have approximately $360°/85.10° = 4.23$ samples per cycle, which is very much less than what we normally recommend. (We recommend eight or more samples per cycle of damped sinusoidal oscillation.)

Finally, for $T = 2$ sec and $K = 2$, the unit-step response is given by

$$C(z) = \frac{1.7294z^{-1}}{1 + 0.5941z^{-1} + 0.1353z^{-2}} \frac{1}{1 - z^{-1}}$$

The unit-step response sequence $c(kT)$ versus kT is shown in Figure 4–26(c). From Figure 4–25, the angle of the line connecting the origin and the closed-loop pole for the present case is 143.87°, and consequently we have $360°/143.87° = 2.50$ samples per cycle, as seen from Figure 4–26(c). (Note that a slow sampling frequency such as 2.50 samples per cycle is unacceptable.)

Figure 4–26 has shown three different plots of the unit-step response sequence $c(kT)$ versus kT. As can be seen from these plots, if the sampling period is small, a plot of $c(kT)$ versus kT will give a fairly accurate portrait of the response $c(t)$. However, if the sampling period is not sufficiently small, then the plot of $c(kT)$ versus kT will not portray an accurate result. It is very important to choose an adequate sampling period based on the satisfaction of the sampling theorem, system

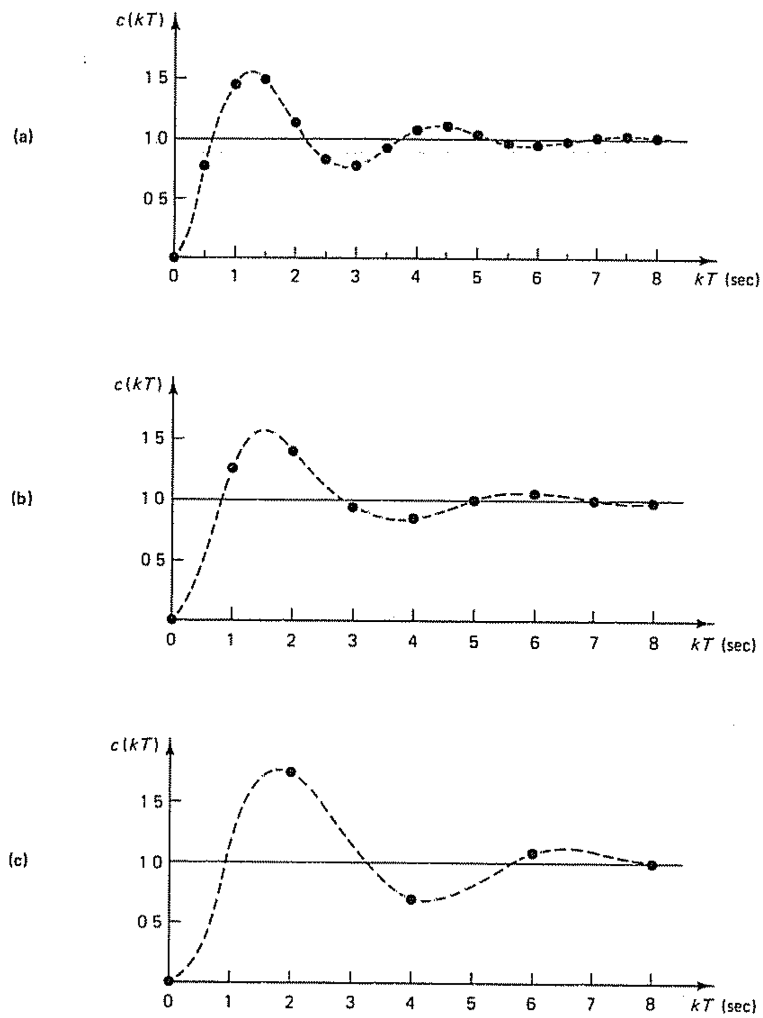


Figure 4–26 (a) Unit-step response sequence of the system shown in Figure 4–23 when $T = 0.5$ sec and $K = 2$; (b) unit-step response sequence when $T = 1$ sec and $K = 2$; (c) unit-step response sequence when $T = 2$ sec and $K = 2$

dynamics, and actual hardware considerations. Note that barely satisfying the sampling theorem is not sufficient. An acceptable rule of thumb is to have eight to ten samples per cycle (six samples per cycle is marginal) if the system is underdamped and exhibits oscillation in the response.

Next, let us investigate the effect of the sampling period T on the steady-state accuracy. We shall consider the unit-ramp response for each of the three cases.

For the case where the sampling period T is 0.5 sec and gain K is 2, the open-loop pulse transfer function is

$$G(z) = \frac{0.7870z}{(z-1)(z-0.6065)}$$

and the static velocity error constant K_v is given by

$$\begin{aligned} K_v &= \lim_{z \to 1} \frac{(1-z^{-1})G(z)}{T} \\ &= \lim_{z \to 1} \left[\frac{z-1}{0.5z} \frac{0.7870z}{(z-1)(z-0.6065)} \right] \\ &= 4 \end{aligned}$$

Thus, the steady-state error in response to a unit-ramp input is

$$e_{ss} = \frac{1}{K_v} = \frac{1}{4} = 0.25$$

Similarly, for the case where $T = 1$ sec and $K = 2$, the open-loop pulse transfer function is

$$G(z) = \frac{1.2642z}{(z-1)(z-0.3679)}$$

and the static velocity error constant K_v is given by

$$\begin{aligned} K_v &= \lim_{z \to 1} \frac{(1-z^{-1})G(z)}{T} \\ &= \lim_{z \to 1} \left[\frac{z-1}{z} \frac{1.2642z}{(z-1)(z-0.3679)} \right] \\ &= 2 \end{aligned}$$

and the steady-state error in response to a unit-ramp input is

$$e_{ss} = \frac{1}{K_v} = \frac{1}{2} = 0.5$$

Finally, for the case where $T = 2$ sec and $K = 2$, the open-loop pulse transfer function is

$$G(z) = \frac{1.7294z}{(z-1)(z-0.1353)}$$

and the static velocity error constant K_v and the steady-state error in response to a unit-ramp input are obtained, respectively, as

$$K_v = 1$$

and

$$e_{ss} = \frac{1}{K_v} = 1$$

Parts (a), (b), and (c) of Figure 4–27 show, respectively, the plots of the unit-ramp response sequence $c(kT)$ versus kT for the three cases considered.

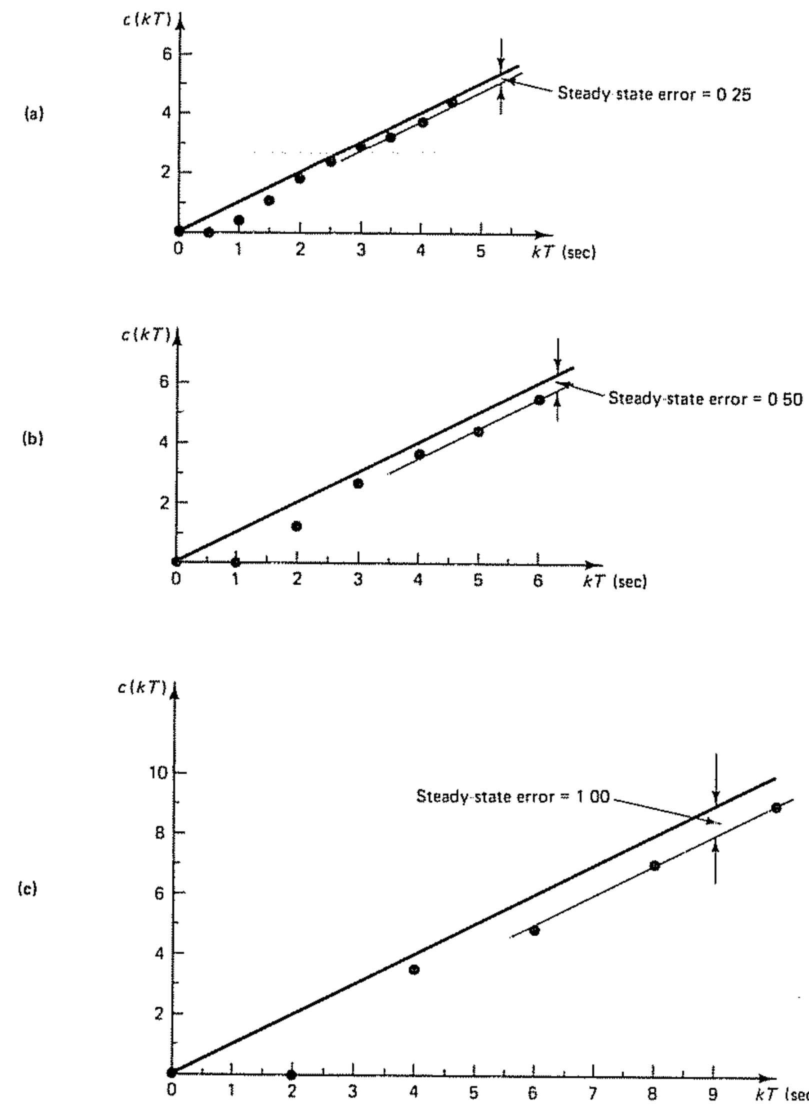


Figure 4–27 (a) Unit-ramp response sequence of the system shown in Figure 4–23 when $T = 0.5$ sec and $K = 2$; (b) unit-ramp response sequence when $T = 1$ sec and $K = 2$; (c) unit-ramp response sequence when $T = 2$ sec and $K = 2$

The three cases we have considered demonstrate that increasing the sampling period T adversely affects the system's relative stability. (It may even cause instability in some cases.) It is important to remember that the damping ratio ζ of the closed-loop poles of the digital control system is indicative of the relative stability only if the sampling frequency is sufficiently high (that is, eight or more samples per cycle of damped sinusoidal oscillation). If the sampling frequency is low (that is, less than six samples per cycle of damped sinusoidal oscillation), then predicting the relative stability from the damping ratio value is erroneous.

Example 4-9

Consider the digital control system shown in Figure 4-28. In the z plane, design a digital controller such that the dominant closed-loop poles have a damping ratio ζ of 0.5 and a settling time of 2 sec. The sampling period is assumed to be 0.2 sec, or $T = 0.2$. Obtain the response of the designed digital control system to a unit-step input. Also, obtain the static velocity error constant K_v of the system.

For the standard second-order system having a pair of dominant closed-loop poles, the settling time of 2 sec means that

$$\text{settling time} = \frac{4}{\zeta\omega_n} = \frac{4}{0.5\omega_n} = 2$$

which gives the undamped natural frequency ω_n of the dominant closed-loop poles as

$$\omega_n = 4$$

The damped natural frequency ω_d is determined to be

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 4\sqrt{1-0.5^2} = 3.464$$

Since the sampling period T is 0.2 sec, we have

$$\omega_s = \frac{2\pi}{T} = \frac{2\pi}{0.2} = 10\pi = 31.42$$

[Notice that there are approximately nine samples per cycle of damped oscillation (31.42/3.464 = 9.07). Thus, the sampling period of 0.2 sec is satisfactory.]

We shall first locate the desired dominant closed-loop poles in the z plane. Referring to Equations (4-29) and (4-30), for a constant-damping-ratio locus we have

$$|z| = e^{-T\zeta\omega_n} = \exp\left(-\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\frac{\omega_d}{\omega_s}\right)$$

and

$$\angle z = T\omega_d = 2\pi\frac{\omega_d}{\omega_s}$$

From the given specifications ($\zeta = 0.5$ and $\omega_d = 3.464$), the magnitude and angle of the dominant closed-loop pole in the upper half of the z plane are determined as follows:

$$|z| = \exp\left(-\frac{2\pi \times 0.5}{\sqrt{1-0.5^2}}\frac{3.464}{31.42}\right) = e^{-0.400} = 0.6703$$

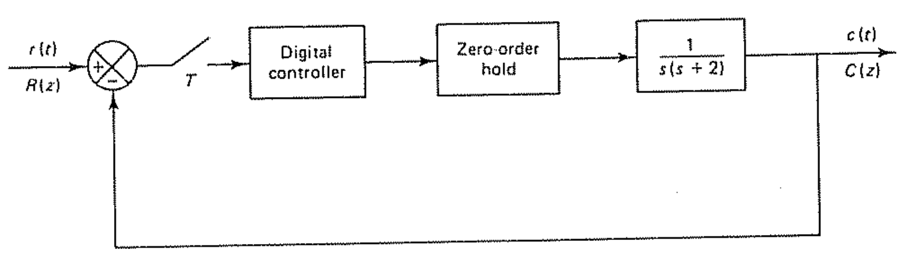


Figure 4-28 Digital control system for Example 4-9

and

$$\angle z = 2\pi \frac{3.464}{31.42} = 0.6927 \text{ rad} = 39.69°$$

We can now locate the desired dominant closed-loop pole in the upper half of the z plane, shown in Figure 4–29 as point P. Note that at point P

$$z = 0.6703 \angle 39.69° = 0.5158 + j0.4281$$

Noting that the sampling period T is 0.2 sec, the pulse transfer function $G(z)$ of the plant preceded by the zero-order hold can be obtained as follows:

$$G(z) = \mathcal{Z}\left[\frac{1 - e^{-0.2s}}{s} \frac{1}{s(s+2)}\right] = (1 - z^{-1})\mathcal{Z}\left[\frac{1}{s^2(s+2)}\right]$$

This last equation can be written as

$$G(z) = \frac{0.01758(z + 0.8760)}{(z - 1)(z - 0.6703)}$$

Next, we locate the poles ($z = 1$ and $z = 0.6703$) and zero ($z = -0.8760$) of $G(z)$ on the z plane, as shown in Figure 4–29. If point P is to be the location for the desired dominant closed-loop pole in the upper half of the z plane, then the sum of the angles at point P must be equal to $\pm 180°$. However, the sum of the angle contributions at point P is

$$17.10° - 138.52° - 109.84° = -231.26°$$

Hence, the angle deficiency is

$$-231.26° + 180° = -51.26°$$

The controller pulse transfer function must provide $+51.26°$. The pulse transfer function for the controller may be assumed to be

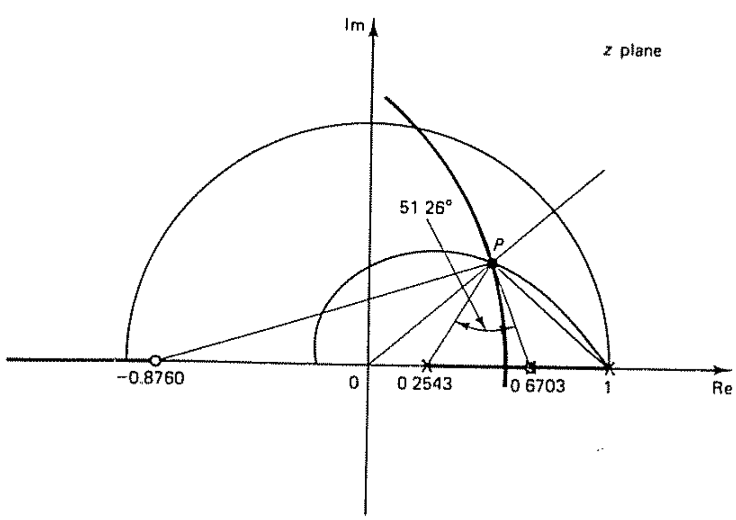


Figure 4–29 Root locus diagram of the system considered in Example 4–9

$$G_D(z) = K\frac{z + \alpha}{z + \beta}$$

where K is the gain constant of the controller.

If we decide to cancel the pole at $z = 0.6703$ by the zero of the controller at $z = -\alpha$, then the pole of the controller can be determined (from the condition that the controller must provide $+51.26°$) as a point at $z = 0.2543$ ($\beta = -0.2543$). Thus, the pulse transfer function for the controller may be determined as

$$G_D(z) = K\frac{z - 0.6703}{z - 0.2543}$$

The open-loop pulse transfer function now becomes

$$G_D(z)G(z) = K\frac{z - 0.6703}{z - 0.2543}\frac{0.01758(z + 0.8760)}{(z - 1)(z - 0.6703)}$$

$$= K\frac{0.01758(z + 0.8760)}{(z - 0.2543)(z - 1)}$$

The gain constant K can be determined from the following magnitude condition:

$$|G_D(z)G(z)|_{z = 0.5158 + j0.4281} = 1$$

Hence,

$$K\left|\frac{0.01758(z + 0.8760)}{(z - 0.2543)(z - 1)}\right|_{z=0.5158+j0.4281} = 1$$

which gives

$$K = 12.67$$

The designed digital controller is

$$G_D(z) = 12.67\frac{z - 0.6703}{z - 0.2543} \tag{4–33}$$

The open-loop pulse transfer function for the present system is

$$G_D(z)G(z) = \frac{12.67 \times 0.01758(z + 0.8760)}{(z - 0.2543)(z - 1)} = \frac{0.2227(z + 0.8760)}{(z - 0.2543)(z - 1)}$$

Hence, the closed-loop pulse transfer function is

$$\frac{C(z)}{R(z)} = \frac{G_D(z)G(z)}{1 + G_D(z)G(z)} = \frac{0.2227z + 0.1951}{z^2 - 1.0316z + 0.4494}$$

The response to the unit-step input $R(z) = 1/(1 - z^{-1})$ can be obtained from

$$C(z) = \frac{0.2227z + 0.1951}{z^2 - 1.0316z + 0.4494}\frac{1}{1 - z^{-1}}$$

$$= \frac{0.2227z^{-1} + 0.1951z^{-2}}{1 - 1.0316z^{-1} + 0.4494z^{-2}}\frac{1}{1 - z^{-1}}$$

Figure 4–30 shows the unit-step response sequence $c(kT)$ versus kT. The plot shows that the maximum overshoot is approximately 16% (which means that the damping ratio is approximately 0.5) and the settling time is approximately 2 sec. The digital controller just designed satisfies the given specifications and is satisfactory.

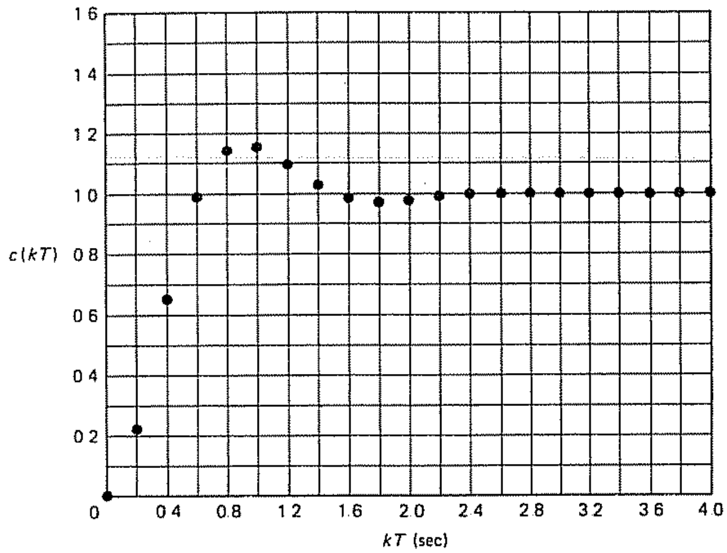


Figure 4–30 Unit-step response sequence of the system designed in Example 4–9.

The static velocity error constant K_v of the system is given by

$$K_v = \lim_{z\to 1}\left[\frac{1-z^{-1}}{T}G_D(z)G(z)\right]$$

$$= \lim_{z\to 1}\left[\frac{z-1}{0.2z}\,\frac{0.2227(z+0.8760)}{(z-0.2543)(z-1)}\right]$$

$$= 2.801$$

If it is required to have a large value of K_v, then we may include a lag compensator. For example, adding a zero at $z = 0.94$ and a pole at $z = 0.98$ would raise the K_v value three times, since $(1 - 0.94)/(1 - 0.98) = 3$. (It is important that the pole and zero of the lag compensator lie on a finite number of allocable discrete points.) A lag compensator, which has a pole and a zero very close to each other, does not significantly change the root locus near the dominant closed-loop poles. The effect of a lag compensator on the transient response is to introduce a small but slowly decreasing transient component. Such a small but slow transient, however, is not desirable from the viewpoint of disturbance or noise attenuation, since the response to disturbances would not attenuate promptly.

Finally, it is noted that although the designed system is of the third order, it acts as a second-order system, since one pole of the plant has been canceled by the zero of the controller. Because of this, the present system has only two closed-loop poles. The dominant closed-loop poles are the only closed-loop poles in this case. If a pole and a zero do not cancel each other, then the system will be of the third order.

Comments. It is important to note that the poles of the closed-loop pulse transfer function determine the natural modes of the system. The transient response and frequency response behaviors, however, are strongly influenced by the zeros of the closed-loop pulse transfer function.

Familiarity with the relationship between the z plane pole and zero locations and the time response characteristics is useful in designing discrete-time control systems. It is important to note that in the s plane adding a zero on the negative real axis near the origin increases the maximum overshoot in response to a step input. Such a zero in the s plane is mapped to a zero on the positive real axis in the z plane between 0 and 1. Therefore, in the z plane, adding a zero on the positive real axis between 0 and 1 increases the maximum overshoot. In fact, moving a zero toward point $z = 1$ will greatly increase the maximum overshoot.

Similarly, in the s plane a closed-loop pole on the negative real axis near the origin increases the settling time. In the z plane, such a closed-loop pole is mapped to a closed-loop pole on the positive real axis between 0 and 1. Thus, a closed-loop pole in the z plane between 0 and 1 (in particular, near point $z = 1$) increases the settling time. The presence of a closed-loop pole or zero on the negative real axis between 0 and -1 in the z plane, however, affects the transient response only slightly.

4-6 DESIGN BASED ON THE FREQUENCY-RESPONSE METHOD

The frequency-response concept plays the same powerful role in digital control systems as it does in continuous-time control systems. As stated earlier, it is assumed in this book that the reader is familiar with conventional frequency-response design techniques for continuous-time control systems. In fact, familiarity with Bode diagrams (logarithmic plots) is necessary in the extension of the conventional frequency-response techniques to the analysis and design of discrete-time control systems.

Frequency-response methods have very frequently been used in the compensator design. The basic reason is the simplicity of the methods. In performing frequency-response tests on a discrete-time system, it is important that the system have a low-pass filter before the sampler so that sidebands are filtered out. Then the response of the linear time-invariant system to a sinusoidal input preserves the frequency and modifies only the amplitude and phase of the input signal. Thus, the amplitude and phase are the only two quantities that must be dealt with.

In the following, we shall analyze the response of the linear time-invariant discrete-time system to a sinusoidal input; this analysis will be followed by the definition of the sinusoidal pulse transfer function. Then we discuss the design of a discrete-time control system in the w plane by use of a Bode diagram.

Response of a Linear Time-Invariant Discrete-Time System to a Sinusoidal Input. Earlier in this book we stated that the frequency response of $G(z)$ can be obtained by substituting $z = e^{j\omega T}$ into $G(z)$. In what follows we shall show that this is indeed true.

Consider the stable linear time-invariant discrete-time system shown in Figure 4-31. The input to the system $G(z)$ before sampling is

$$u(t) = \sin \omega t$$

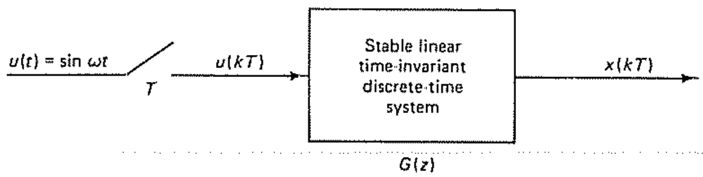


Figure 4–31 Stable linear time-invariant discrete-time system

The sampled signal $u(kT)$ is

$$u(kT) = \sin k\omega T$$

The z transform of the sampled input is

$$U(z) = \mathcal{Z}[\sin k\omega T] = \frac{z \sin \omega T}{(z - e^{j\omega T})(z - e^{-j\omega T})}$$

The response of the system is given by

$$X(z) = G(z)U(z) = G(z)\frac{z \sin \omega T}{(z - e^{j\omega T})(z - e^{-j\omega T})}$$

$$= \frac{az}{z - e^{j\omega T}} + \frac{\bar{a}z}{z - e^{-j\omega T}} + [\text{terms due to poles of } G(z)] \qquad (4\text{–}34)$$

Multiplying both sides of Equation (4–34) by $(z - e^{j\omega T})/z$, we obtain

$$G(z)\frac{\sin \omega T}{z - e^{-j\omega T}} = a + \frac{\bar{a}(z - e^{j\omega T})}{z - e^{-j\omega T}} + \frac{z - e^{j\omega T}}{z}[\text{terms due to poles of } G(z)]$$

The second term on the right-hand side of this last equation approaches zero as z approaches $e^{j\omega T}$. Since the system considered here is stable, the third term on the right-hand side also approaches zero as z approaches $e^{j\omega T}$. Hence, by letting z approach $e^{j\omega T}$, we have

$$a = G(z)\frac{\sin \omega T}{z - e^{-j\omega T}}\bigg|_{z=e^{j\omega T}} = \frac{G(e^{j\omega T})}{2j}$$

The coefficient $\bar{a}$, the complex conjugate of a, is then obtained as follows:

$$\bar{a} = -\frac{G(e^{-j\omega T})}{2j}$$

Let us define

$$G(e^{j\omega T}) = Me^{j\theta}$$

Then

$$G(e^{-j\omega T}) = Me^{-j\theta}$$

Equation (4–34) can now be written as

$$X(z) = \frac{Me^{j\theta}}{2j}\frac{z}{z - e^{j\omega T}} - \frac{Me^{-j\theta}}{2j}\frac{z}{z - e^{-j\omega T}} + [\text{terms due to poles of } G(z)]$$

or

$$X(z) = \frac{M}{2j}\left(\frac{e^{j\theta} z}{z - e^{j\omega T}} - \frac{e^{-j\theta} z}{z - e^{-j\omega T}}\right) + [\text{terms due to poles of } G(z)]$$

The inverse z transform of this last equation is

$$x(kT) = \frac{M}{2j}(e^{jk\omega T} e^{j\theta} - e^{-jk\omega T} e^{-j\theta}) + \mathcal{Z}^{-1}[\text{terms due to poles of } G(z)] \qquad (4\text{–}35)$$

The last term on the right-hand side of Equation (4–35) represents the transient response. Since the system $G(z)$ has been assumed to be stable, all transient response terms will disappear at steady state and we will get the following steady-state response $x_{ss}(kT)$:

$$x_{ss}(kT) = \frac{M}{2j}[e^{j(k\omega T+\theta)} - e^{-j(k\omega T+\theta)}] = M \sin(k\omega T + \theta) \qquad (4\text{–}36)$$

where M, the gain of the discrete-time system when subjected to a sinusoidal input, is given by

$$M = M(\omega) = |G(e^{j\omega T})|$$

and θ, the phase angle, is given by

$$\theta = \theta(\omega) = \underline{/G(e^{j\omega T})}$$

In terms of $G(e^{j\omega T})$, Equation (4–36) can be written as follows:

$$x_{ss}(kT) = |G(e^{j\omega T})| \sin(k\omega T + \underline{/G(e^{j\omega T})})$$

We have shown that $G(e^{j\omega T})$ indeed gives the magnitude and phase of the frequency response of $G(z)$. Thus, to obtain the frequency response of $G(z)$, we need only to substitute $e^{j\omega T}$ for z in $G(z)$. The function $G(e^{j\omega T})$ is commonly called the *sinusoidal pulse transfer function*. Noting that

$$e^{j(\omega+(2\pi/T))T} = e^{j\omega T} e^{j2\pi} = e^{j\omega T}$$

we find that the sinusoidal pulse transfer function $G(e^{j\omega T})$ is periodic, with the period equal to T.

Example 4–10

Consider the system defined by

$$x(kT) = u(kT) + ax((k-1)T), \qquad 0 < a < 1$$

where $u(kT)$ is the input and $x(kT)$ the output. Obtain the steady-state output $x(kT)$ when the input $u(kT)$ is the sampled sinusoid, or $u(kT) = A \sin k\omega T$.

The z transform of the system equation is

$$X(z) = U(z) + az^{-1} X(z)$$

By defining $G(z) = X(z)/U(z)$, we have

$$G(z) = \frac{X(z)}{U(z)} = \frac{1}{1 - az^{-1}}$$

Let us substitute $e^{j\omega T}$ for z in $G(z)$. Then the sinusoidal pulse transfer function $G(e^{j\omega T})$ can be obtained as

$$G(e^{j\omega T}) = \frac{1}{1 - ae^{-j\omega T}} = \frac{1}{1 - a\cos\omega T + ja\sin\omega T}$$

The amplitude of $G(e^{j\omega T})$ is

$$|G(e^{j\omega T})| = M = \frac{1}{\sqrt{1 + a^2 - 2a\cos\omega T}}$$

and the phase angle of $G(e^{j\omega T})$ is

$$\underline{/G(e^{j\omega T})} = \theta = -\tan^{-1}\frac{a\sin\omega T}{1 - a\cos\omega T}$$

Then the steady-state output $x_{ss}(kT)$ can be written as follows:

$$\begin{aligned} x_{ss}(kT) &= AM\sin(k\omega T + \theta) \\ &= \frac{A}{\sqrt{1 + a^2 - 2a\cos\omega T}}\sin\left(k\omega T - \tan^{-1}\frac{a\sin\omega T}{1 - a\cos\omega T}\right) \end{aligned}$$

Bilinear Transformation and the w Plane. Before we can advantageously apply our well-developed frequency-response methods to the analysis and design of discrete-time control systems, certain modifications in the z plane approach are necessary. Since in the z plane the frequency appears as $z = e^{j\omega T}$, if we treat frequency response in the z plane, the simplicity of the logarithmic plots will be completely lost. Thus, the direct application of frequency-response methods is not worthy of consideration. In fact, since the z transformation maps the primary and complementary strips of the left half of the s plane into the unit circle in the z plane, conventional frequency-response methods, which deal with the entire left half plane, do not apply to the z plane.

The difficulty, however, can be overcome by transforming the pulse transfer function in the z plane into that in the w plane. The transformation, commonly called the w transformation, a bilinear transformation, is defined by

$$z = \frac{1 + (T/2)w}{1 - (T/2)w} \tag{4–37}$$

where T is the sampling period involved in the discrete-time control system under consideration. By converting a given pulse transfer function in the z plane into a rational function of w, the frequency-response methods can be extended to discrete-time control systems. By solving Equation (4–37) for w, we obtain the inverse relationship

$$w = \frac{2}{T}\frac{z - 1}{z + 1} \tag{4–38}$$

Through the z transformation and the w transformation, the primary strip of the left half of the s plane is first mapped into the inside of the unit circle in the z plane and then mapped into the entire left half of the w plane. The two mapping processes are depicted in Figure 4–32. (Note that in the s plane we consider only the primary strip.) Notice that the origin of the z plane is mapped into the point $w = -2/T$ in the w plane. Notice also that, as s varies from 0 to $j\omega_s/2$ along the $j\omega$

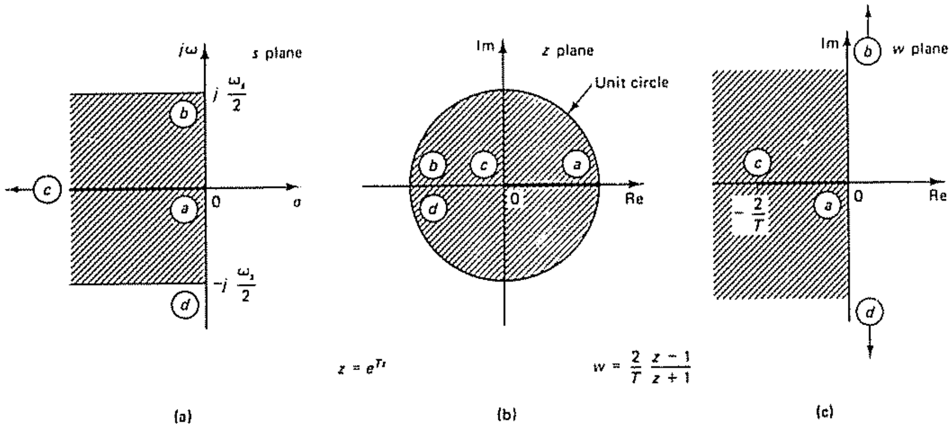


Figure 4–32 Diagrams showing mappings from the s plane to the z plane and from the z plane to the w plane. (a) Primary strip in the left half of the s plane; (b) z plane mapping of the primary strip in the s plane; (c) w plane mapping of the unit circle in the z plane

axis in the s plane, z varies from 1 to -1 along the unit circle in the z plane and w varies from 0 to ∞ along the imaginary axis in the w plane.

Although the left half of the w plane corresponds to the left half of the s plane and the imaginary axis of the w plane corresponds to the imaginary axis of the s plane, there are differences between the two planes. The chief difference is that the behavior in the s plane over the frequency range $-\frac{1}{2}\omega_s \le \omega \le \frac{1}{2}\omega_s$ maps to the range $-\infty < \nu < \infty$, where ν is the fictitious frequency in the w plane. This means that, although the frequency response characteristics of the analog filter will be reproduced in the discrete or digital filter, the frequency scale on which the response occurs will be compressed from an infinite interval in the analog filter to a finite interval in the digital filter.

Once the pulse transfer function $G(z)$ is transformed into $G(w)$ by means of the w transformation, it may be treated as a conventional transfer function in w. Conventional frequency-response techniques can then be used in the w plane, and so the well-established frequency-response design techniques can be applied to the design of discrete-time control systems.

As noted earlier, ν represents the fictitious frequency. By replacing w by $j\nu$, conventional frequency-response techniques may be used to draw the Bode diagram for the transfer function in w. (In the brief review of the Bode diagrams in this section, we shall use the fictitious frequency ν as the variable.)

Although the w plane resembles the s plane geometrically, the frequency axis in the w plane is distorted. The fictitious frequency ν and the actual frequency ω are related as follows:

$$w\Big|_{w=j\nu} = j\nu = \frac{2}{T}\frac{z-1}{z+1}\Big|_{z=e^{j\omega T}} = \frac{2}{T}\frac{e^{j\omega T}-1}{e^{j\omega T}+1}$$

$$= \frac{2}{T}\frac{e^{j(1/2)\omega T} - e^{-j(1/2)\omega T}}{e^{j(1/2)\omega T} + e^{j(1/2)\omega T}} = \frac{2}{T} j \tan\frac{\omega T}{2}$$

or

$$\nu = \frac{2}{T} \tan \frac{\omega T}{2} \tag{4-39}$$

Equation (4–39) gives the relationship between the actual frequency ω and the fictitious frequency ν. Note that as the actual frequency ω moves from $-\frac{1}{2}\omega_s$ to 0 the fictitious frequency ν moves from $-\infty$ to 0, and as ω moves from 0 to $\frac{1}{2}\omega_s$, ν moves from 0 to ∞.

Referring to Equation (4–39), the actual frequency ω can be translated into the fictitious frequency ν. For example, if the bandwidth is specified as ω_b, then the corresponding bandwidth in the w plane is $(2/T)\tan(\omega_b T/2)$. Similarly, $G(j\nu_1)$ corresponds to $G(j\omega_1)$, where $\omega_1 = (2/T)\tan^{-1}(\nu_1 T/2)$. Figure 4–33 shows the relationship between the fictitious frequency ν times $\frac{1}{2}T$ and the actual frequency ω for the frequency range between 0 and $\frac{1}{2}\omega_s$.

Notice that in Equation (4–39) if ωT is small then

$$\nu \doteq \omega$$

This means that for small ωT the transfer functions $G(s)$ and $G(w)$ resemble each other. Note that this is the direct result of the inclusion of the scale factor $2/T$ in Equation (4–38). The presence of this scale factor in the transformation enables us to maintain the same error constants before and after the w transformation. (This means that the transfer function in the w plane will approach that in the s plane as T approaches zero. See Example 4–11, which follows.)

Example 4–11

Consider the transfer-function system shown in Figure 4–34. The sampling period T is assumed to be 0.1 sec. Obtain $G(w)$.

The z transform of $G(s)$ is

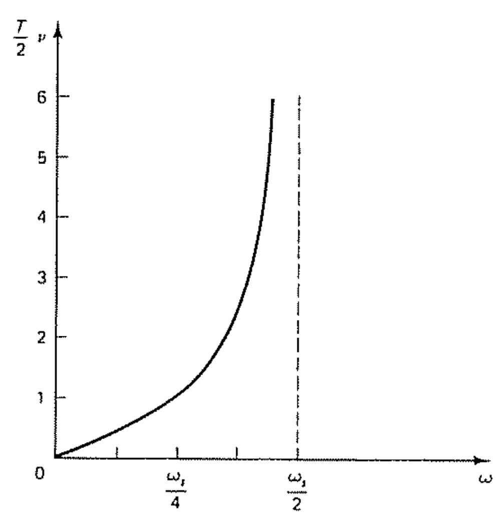


Figure 4–33 Relationship between the fictitious frequency ν times $\frac{1}{2}T$ and the actual frequency ω for the frequency range between 0 and $\frac{1}{2}\omega_s$

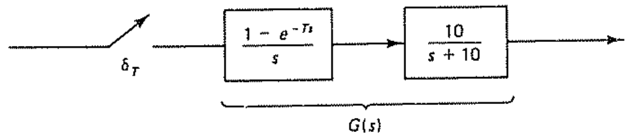


Figure 4-34 Transfer-function system of Example 4-11

$$G(z) = \mathcal{Z}\left[\frac{1-e^{-Ts}}{s}\frac{10}{s+10}\right]$$
$$= (1 - z^{-1})\mathcal{Z}\left[\frac{10}{s(s+10)}\right]$$
$$= \frac{0.6321}{z - 0.3679}$$

By use of the bilinear transformation given by Equation (4-37), or

$$z = \frac{1 + (T/2)w}{1 - (T/2)w} = \frac{1 + 0.05w}{1 - 0.05w}$$

$G(z)$ can be transformed into $G(w)$ as follows:

$$G(w) = \frac{0.6321}{\dfrac{1 + 0.05w}{1 - 0.05w} - 0.3679} = \frac{0.6321(1 - 0.05w)}{0.6321 + 0.06840w}$$
$$= 9.241\frac{1 - 0.05w}{w + 9.241}$$

Notice that the location of the pole of the plant is $s = -10$ and that of the pole in the w plane is $w = -9.241$. The gain value in the s plane is 10 and that in the w plane is 9.241. (Thus, both the pole locations and the gain values are similar in the s plane and the w plane.) However, $G(w)$ has a zero at $w = 2/T = 20$, although the plant does not have any zero. As the sampling period T becomes smaller, the w plane zero at $w = 2/T$ approaches infinity in the right half of the w plane. Note that we have

$$\lim_{w \to 0} G(w) = \lim_{s \to 0}\frac{10}{s + 10}$$

This fact is very useful in checking the numerical calculations in transforming $G(s)$ into $G(w)$.

To summarize, the w transformation, a bilinear transformation, maps the inside of the unit circle of the z plane into the left half of the w plane. The overall result due to the transformations from the s plane into the z plane and from the z plane into the w plane is that the w plane and the s plane are similar over the region of interest in the s plane. This is because some of the distortions caused by the transformation from the s plane into the z plane are partly compensated for by the transformation from the z plane into the w plane.

Note that if

$$G(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^n + a_1 z^{n-1} + \cdots + a_n}, \qquad m \le n$$

where the a_i's and b_i's are constants, is transformed into the w plane by the transformation

$$z = \frac{1 + (T/2)w}{1 - (T/2)w}$$

then $G(w)$ takes the form

$$G(w) = \frac{\beta_0 w^n + \beta_1 w^{n-1} + \cdots + \beta_n}{\alpha_0 w^n + \alpha_1 w^{n-1} + \cdots + \alpha_n}$$

where the α_i's and the β_i's are constants (some of them may be zero). Thus, $G(w)$ is a ratio of polynomials in w, where the degrees of the numerator and denominator may or may not be equal. Since $G(j\nu)$ is a rational function of ν, the Nyquist stability criterion can be applied to $G(j\nu)$. In terms of the Bode diagram, the conventional straight-line approximation to the magnitude curve as well as the concept of the phase margin and gain margin apply to $G(j\nu)$.

Bode Diagrams. Design by means of Bode diagrams has been widely used in dealing with single-input–single-output continuous-time control systems. In particular, if the transfer function is in a factored form, the simplicity and ease with which the asymptotic Bode diagram can be drawn and reshaped are well known.

As stated earlier, the conventional frequency-response methods apply to the transfer functions in the w plane. Recall that the Bode diagram consists of two separate plots, the logarithmic magnitude $|G(j\nu)|$ versus log ν and the phase angle $\angle G(j\nu)$ versus log ν. Sketching of the logarithmic magnitude is based on the factoring of $G(j\nu)$, so that it works on the principle of adding the individual factored terms instead of multiplying individual terms. Familiar asymptotic plotting techniques can be applied, and therefore the magnitude curve can be quickly drawn by using straight-line asymptotes. Using the Bode diagram, a digital compensator or digital controller may be designed with conventional design techniques.

It is important to note that there may be a difference in the high-frequency magnitudes for $G(j\omega)$ and $G(j\nu)$. The high-frequency asymptote of the logarithmic magnitude curve for $G(j\nu)$ may be a constant-decibel line (that is, a horizontal line). On the other hand, if $\lim_{s\to\infty} G(s) = 0$, then the magnitude of $G(j\omega)$ always approaches zero ($-\infty$ dB) as ω approaches infinity. For example, referring to Example 4–11, we obtained $G(w)$ for $G(s)$ as follows:

$$G(w) = 9.241\left(\frac{1 - 0.05w}{w + 9.241}\right)$$

The high-frequency magnitude of $G(j\nu)$ is

$$\lim_{\nu\to\infty} |G(j\nu)| = \lim_{\nu\to\infty} \left| 9.241\left(\frac{1 - 0.05j\nu}{j\nu + 9.241}\right)\right| = 0.4621$$

while the high-frequency magnitude of the plant is

$$\lim_{\omega\to\infty} \left|\frac{10}{j\omega + 10}\right| = 0$$

The difference in the Bode diagrams at the high-frequency end can be explained as follows. First, recall that we are interested only in the frequency range $0 \le \omega \le \frac{1}{2}\omega_s$, which corresponds to $0 \le \nu \le \infty$. Then, noting that $\nu = \infty$ in the w plane corresponds to $\omega = \frac{1}{2}\omega_s$ in the s plane, it can be said that $\lim_{\nu\to\infty}|G(j\nu)|$ corresponds to $\lim_{\omega\to\omega_s/2}|10/(j\omega + 10)|$, which is a constant. (It is important to note that these two values are generally not equal to each other.) From the pole–zero point of view, it can be said that when $|G(j\nu)|$ is a nonzero constant at $\nu = \infty$ it is implied that $G(w)$ contains the same number of poles and zeros.

In general, one or more zeros of $G(w)$ lie in the right half of the w plane. The presence of a zero in the right half of the w plane means that $G(w)$ is a nonminimum phase transfer function. Therefore, we must be careful in drawing the phase angle curve in the Bode diagram.

Advantages of the Bode Diagram Approach to the Design. The Bode diagram approach to the analysis and design of control systems is particularly useful for the following reasons:

1. In the Bode diagram the low-frequency asymptote of the magnitude curve is indicative of one of the static error constants K_p, K_v, or K_a.
2. Specifications of the transient response can be translated into those of the frequency response in terms of the phase margin, gain margin, bandwidth, and so forth. These specifications can easily be handled in the Bode diagram. In particular, the phase and gain margins can be read directly from the Bode diagram.
3. The design of a digital compensator (or digital controller) to satisfy the given specifications (in terms of the phase margin and gain margin) can be carried out in the Bode diagram in a simple and straightforward manner.

Phase Lead, Phase Lag, and Phase Lag–Lead Compensation. Before we discuss design procedures in the w plane, let us review the phase lead, phase lag, and phase lag–lead compensation techniques.

Phase lead compensation is commonly used for improving stability margins. The phase lead compensation increases the system bandwidth. Thus, the system has a faster speed to respond. However, such a system using phase lead compensation may be subjected to high-frequency noise problems due to its increased high-frequency gains.

Phase lag compensation reduces the system gain at higher frequencies without reducing the system gain at lower frequencies. The system bandwidth is reduced and thus the system has a slower speed of response. Because of the reduced high-frequency gain, the total system gain can be increased, and thereby low-frequency gain can be increased and the steady-state accuracy can be improved. Also, any high-frequency noises involved in the system can be attenuated.

In some applications, a phase lag compensator is cascaded with a phase lead compensator. The cascaded compensator is known as a *phase lag–lead* compensator. By use of the lag–lead compensator, the low-frequency gain can be increased (which means an improvement in steady-state accuracy), while at the same time the system bandwith and stability margins can be increased.

Note that the PID controller is a special case of a phase lag–lead controller. The PD control action, which affects the high-frequency region, increases the phase lead angle and improves system stability, as well as increasing the system bandwidth (and thus increasing the speed of response). That is, the PD controller behaves in much the same way as a phase lead compensator. The PI control action affects the low-frequency portion and, in fact, increases the low-frequency gain and improves steady-state accuracy. Therefore, the PI controller acts as a phase lag compensator. The PID control action is a combination of the PI and PD control actions. The design techniques for PID controllers basically follow those of phase lag–lead compensators. (In industrial control systems, however, each PID control action in the PID controller may be adjusted experimentally.)

Some Remarks on the Coefficient Quantization Problem. From the viewpoint of microprocessor implementation of the phase lead, phase lag, and phase lag–lead compensators, phase lead compensators present no coefficient quantization problem, because the locations of poles and zeros are widely separated. However, in the case of phase lag compensators and phase lag–lead compensators, the phase lag network presents a coefficient quantization problem because the locations of poles and zeros are close to each other. (They are near the point $z = 1$.)

Since the filter coefficients must be realized by binary words that use limited numbers of bits, if the number of bits employed is insufficient, the pole and zero locations of the filter may not be realized exactly as desired and the resulting filter will not behave as expected.

Since small deviations in the pole and zero locations from the desired locations can have significant effects on the frequency-response characteristics of the compensator, the digital version of the compensator may not perform as expected. To minimize the effect of the coefficient quantization problem, it is necessary to structure the filter so that it is least subject to coefficient inaccuracies due to quantization.

Because the sensitivity of the roots of polynomials to the parameter variations becomes severe as the order of the polynomial increases, direct realization of a higher-order filter is not desirable. It is preferable to place lower-order elements in cascade or in parallel, as discussed in Section 3–6. As a matter of course, from the outset if we choose poles and zeros of the digital compensator from allowable discrete points, then the coefficient quantization problem can be avoided.

In the analog compensator, the poles and zeros of the compensator can be placed with an arbitrary accuracy. In converting an analog compensator to a digital compensator, the digital version of the lag compensator may involve considerable inaccuracies in the locations of poles and zeros. (The important thing to remember is that the poles and zeros of the filter in the z plane must lie on a finite number of allowable discrete points.)

Design Procedure in the w Plane. Referring to the digital control system shown in Figure 4–35, the design procedure in the w plane may be stated as follows:

1. First, obtain $G(z)$, the z transform of the plant preceded by a hold. Then transform $G(z)$ into a transfer function $G(w)$ through the bilinear transformation given by Equation (4–37):

$$z = \frac{1 + (T/2)w}{1 - (T/2)w}$$

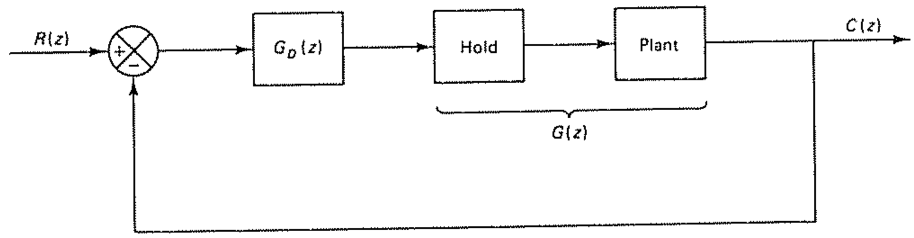


Figure 4–35 Digital control system

That is,

$$G(w) = G(z)|_{z=[1+(T/2)w]/[1-(T/2)w]}$$

It is important that the sampling period T be chosen properly. A rule of thumb is to sample at the frequency 10 times that of the bandwidth of the closed-loop system. (Although digital controls and signal processing use similar approaches in sampling continuous-time signals, the sampling frequencies involved are very different. In the field of signal processing, sampling frequencies are generally very high, while in the field of digital control systems, the sampling frequencies used are generally low. Such a difference in the sampling frequencies is mainly due to the different dynamics involved and the different trade-offs in these two fields.)

2. Substitute $w = j\nu$ into $G(w)$ and plot the Bode diagram for $G(j\nu)$.
3. Read from the Bode diagram the static error constants, the phase margin, and the gain margin.
4. By assuming that the low-frequency gain of the discrete-time controller (or digital controller) transfer function $G_D(w)$ is unity, determine the system gain by satisfying the requirement for a given static error constant. Then, by using conventional design techniques for continuous-time control systems, determine the pole(s) and zero(s) of the digital controller transfer function. [$G_D(w)$ is a ratio of two polynomials in w.] Then the open-loop transfer function of the designed system is given by $G_D(w)G(w)$.
5. Transform the controller transfer function $G_D(w)$ into $G_D(z)$ through the bilinear transformation given by Equation (4–38):

$$w = \frac{2}{T}\frac{z-1}{z+1}$$

Then

$$G_D(z) = G_D(w)|_{w=(2/T)(z-1)/(z+1)}$$

is the pulse transfer function of the digital controller.

6. Realize the pulse transfer function $G_D(z)$ by a computational algorithm.

In following the design procedure just given, it is important to note the following:

1. The transfer function $G(w)$ is a nonminimum phase transfer function. Hence, the phase angle curve is different from that for the more typical minimum phase transfer function. It is necessary to make sure that the phase angle curve is drawn correctly by taking into consideration the nonminimum phase term.
2. The frequency axis in the w plane is distorted. The relationship between the fictitious frequency ν and the actual frequency ω is

$$\nu = \frac{2}{T} \tan \frac{\omega T}{2}$$

If, for example, a bandwidth ω_b is specified, we need to design the system for a bandwidth ν_b, where

$$\nu_b = \frac{2}{T} \tan \frac{\omega_b T}{2}$$

Example 4–12

Consider the digital control system shown in Figure 4–36. Design a digital controller in the w plane such that the phase margin is 50°, the gain margin is at least 10 dB, and the static velocity error constant K_v is 2 sec^{-1}. Assume that the sampling period is 0.2 sec, or $T = 0.2$.

First, we obtain the pulse transfer function $G(z)$ of the plant that is preceded by the zero-order hold:

$$\begin{aligned} G(z) &= \mathcal{Z}\left[\frac{1 - e^{-0.2s}}{s} \frac{K}{s(s + 1)}\right] \\ &= (1 - z^{-1})\mathcal{Z}\left[\frac{K}{s^2(s + 1)}\right] \\ &= 0.01873\left[\frac{K(z + 0.9356)}{(z - 1)(z - 0.8187)}\right] \\ &= \frac{K(0.01873z + 0.01752)}{z^2 - 1.8187z + 0.8187} \end{aligned}$$

Next, we transform the pulse transfer function $G(z)$ into a transfer function $G(w)$ by means of the bilinear transformation given by Equation (4–37):

$$z = \frac{1 + (T/2)w}{1 - (T/2)w} = \frac{1 + 0.1w}{1 - 0.1w}$$

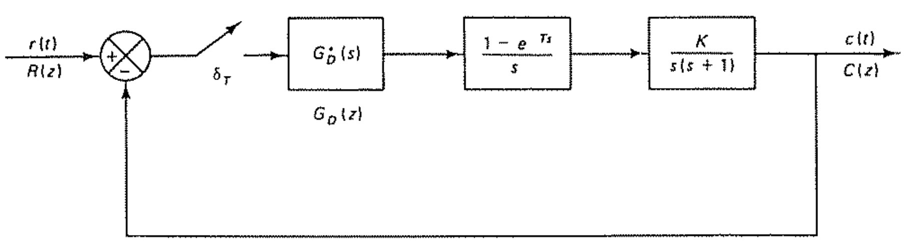


Figure 4–36 Digital control system of Example 4–12

Thus,

$$G(w) = \frac{K\left[0.01873\left(\dfrac{1+0.1w}{1-0.1w}\right) + 0.01752\right]}{\left(\dfrac{1+0.1w}{1-0.1w}\right)^2 - 1.8187\left(\dfrac{1+0.1w}{1-0.1w}\right) + 0.8187}$$

$$= \frac{K(-0.000333w^2 - 0.09633w + 0.9966)}{w^2 + 0.9969w}$$

$$\doteqdot \frac{K\left(1 + \dfrac{w}{300}\right)\left(1 - \dfrac{w}{10}\right)}{w(w+1)}$$

A simple *phase-lead compensator* will probably satisfy all requirements. Therefore, we shall try lead compensation. (If lead compensation does not satisfy all requirements, we need to use a different type of compensation.)

Now let us assume that the transfer function of the digital controller $G_D(w)$ has unity gain for the low-frequency range and has the following form:

$$G_D(w) = \frac{1 + \tau w}{1 + \alpha \tau w}, \qquad 0 < \alpha < 1$$

(This is a phase-lead compensator.) It is one of the simplest forms of the digital controller transfer function. (Other forms may be assumed as well for this problem.) The open-loop transfer function is

$$G_D(w)G(w) = \frac{1 + \tau w}{1 + \alpha \tau w} \frac{K(-0.000333w^2 - 0.09633w + 0.9966)}{w^2 + 0.9969w}$$

The static velocity error constant K_v is specified as 2 $\sec^{-1}$. Hence,

$$K_v = \lim_{w \to 0} wG_D(w)G(w) \doteqdot K = 2$$

The gain K is thus determined to be 2.

By setting $K = 2$, we plot the Bode diagram of $G(w)$:

$$G(w) = \frac{2(-0.000333w^2 - 0.09633w + 0.9966)}{w^2 + 0.9969w}$$

$$\doteqdot \frac{2\left(1 + \dfrac{w}{300}\right)\left(1 - \dfrac{w}{10}\right)}{w(w+1)}$$

Figure 4-37 shows the Bode diagram for the system. For the magnitude curves we have used straight-line asymptotes. The magnitude and phase angle of $G(j\nu)$ are shown by dashed curves. (Note that the zero at $\nu = 10$, which lies in the right half of the w plane, gives phase lag.) The phase margin can be read from the Bode diagram (dashed curves) as 30° and the gain margin as 14.5 dB.

The given specifications require, in addition to $K_v = 2$, the phase margin of 50° and a gain margin of at least 10 dB. Let us design a digital controller to satisfy these specifications.

Design of lead compensator. Since the specification calls for a phase margin of 50°, the additional phase-lead angle necessary to satisfy this requirement is 20°. To achieve a phase margin of 50° without decreasing the value of K, the lead compensator must contribute the required phase-lead angle.

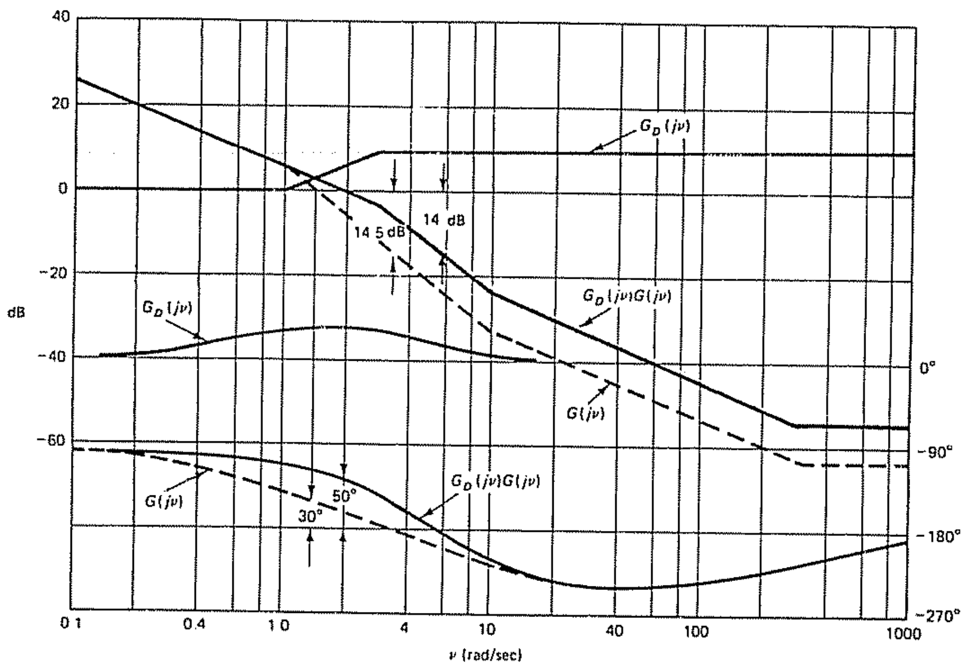


Figure 4–37 Bode diagram for the system designed in Example 4–12.

Noting that the addition of a lead compensator modifies the magnitude curve in the Bode diagram, the gain crossover frequency will be shifted to the right. Considering the shift of the gain crossover frequency, we may assume that ϕ_m, the maximum phase-lead angle required, is approximately 28°. (This means that 8° has been added to compensate for the shift in the gain crossover frequency.) Since

$$\sin \phi_m = \frac{1-\alpha}{1+\alpha}$$

$\phi_m = 28°$ corresponds to $\alpha = 0.361$.

Once the attenuation factor α has been determined on the basis of the required phase-lead angle, the next step is to determine the corner frequencies $\nu = 1/\tau$ and $\nu = 1/(\alpha\tau)$ of the lead compensator. To do so, we first note that the maximum phase-lead angle ϕ_m occurs at the geometric mean of the two corner frequencies, or $\nu = 1/(\sqrt{\alpha}\tau)$. The amount of the modification in the magnitude curve at $\nu = 1/(\sqrt{\alpha}\tau)$ due to the inclusion of the term $(1 + \tau j\nu)/(1 + \alpha\tau j\nu)$ is

$$\left|\frac{1+\tau j\nu}{1+\alpha\tau j\nu}\right|_{\nu=1/(\sqrt{\alpha}\tau)} = \frac{1}{\sqrt{\alpha}}$$

Next, we find the frequency point where the magnitude of the uncompensated system is equal to $-20 \log(1/\sqrt{\alpha})$. Note that

$$-20 \log \frac{1}{\sqrt{0.361}} = -20 \log 1.6643 = -4.425 \text{ dB}$$

To find the frequency point where the magnitude is −4.425 dB, we substitute $w = j\nu$ in $G(w)$ and find the magnitude of $G(j\nu)$:

$$|G(j\nu)| = \frac{2\sqrt{1 + \left(\frac{\nu}{300}\right)^2}\sqrt{1 + \left(\frac{\nu}{10}\right)^2}}{\nu\sqrt{1 + \nu^2}}$$

By trial and error, we find that at $\nu = 1.7$ the magnitude becomes approximately −4.4 dB. We select this frequency to be the new gain crossover frequency ν_c. Noting that this frequency corresponds to $1/(\sqrt{\alpha\tau})$, or

$$\nu_c = \frac{1}{\sqrt{\alpha\tau}} = 1.7$$

we obtain

$$\tau = \frac{1}{1.7\sqrt{\alpha}} = 0.9790$$

and

$$\alpha\tau = 0.3534$$

The lead compensator thus determined is

$$G_D(w) = \frac{1 + \tau w}{1 + \alpha\tau w} = \frac{1 + 0.9790w}{1 + 0.3534w} \tag{4-40}$$

The magnitude and phase angle curves for $G_D(j\nu)$ and the magnitude and phase angle curves of the compensated open-loop transfer function $G_D(j\nu)G(j\nu)$ are shown by solid curves in Figure 4-37. From the Bode diagram we see that the phase margin is 50° and the gain margin is 14 dB.

The controller transfer function given by Equation (4-40) will now be transformed back to the z plane by the bilinear transformation given by Equation (4-38):

$$w = \frac{2}{T}\frac{z - 1}{z + 1} = \frac{2}{0.2}\frac{z - 1}{z + 1} = 10\frac{z - 1}{z + 1}$$

Thus,

$$G_D(z) = \frac{1 + 0.9790\left(10\frac{z - 1}{z + 1}\right)}{1 + 0.3534\left(10\frac{z - 1}{z + 1}\right)} = \frac{2.3798z - 1.9387}{z - 0.5589}$$

The open-loop pulse transfer function of the compensated system is

$$G_D(z)G(z) = \frac{2.3798z - 1.9387}{z - 0.5589}\,\frac{0.03746(z + 0.9356)}{(z - 1)(z - 0.8187)} = \frac{0.0891z^2 + 0.0108z - 0.0679}{z^3 - 2.3776z^2 + 1.8352z - 0.4576}$$

The closed-loop pulse transfer function of the designed system is

$$\frac{C(z)}{R(z)} = \frac{0.0891z^2 + 0.0108z - 0.0679}{z^3 - 2.2885z^2 + 1.8460z - 0.5255}$$

$$= \frac{0.0891(z + 0.9357)(z - 0.8145)}{(z - 0.8126)(z - 0.7379 - j0.3196)(z - 0.7379 + j0.3196)}$$

Notice that the closed-loop pulse transfer function involves two zeros located at $z = -0.9357$ and $z = 0.8145$. The zero at $z = 0.8145$ almost cancels with the closed-loop pole at $z = 0.8126$. The effect of another zero at $z = -0.9357$ on the transient and frequency responses is very small, since it is located on the negative real axis of the z plane between 0 and -1 and is close to point $z = -1$. The pair of complex conjugate poles acts as dominant closed-loop poles. (The system behaves as if it is a second-order system.)

To check the transient response of the designed system, we shall obtain the unit-step response of this system using MATLAB. MATLAB Program 4–2 produces the unit-step response curve as shown in Figure 4–38. The plot of the unit-step response exhibits a maximum overshoot of approximately 20% and a settling time of approximately 4 sec. From this curve we see that the number of samples per cycle of sinusoidal oscillation is approximately 15. This means that the sampling frequency ω_s is 15 times the damped natural frequency ω_d. Thus, the sampling period of 0.2 sec is satisfactory under normal operation of this system.

MATLAB Program 4–2

```
% ---------- Unit-step response of designed system ----------

num = [0  0.0891  0.0108  -0.0679];
den = [1  -2.2885  1.8460  -0.5255];
r = ones(1,41);
v = [0  40  0  1.6];
axis(v);
k = 0:40;
c = filter(num,den,r);
plot(k,c,'o')
grid
title('Unit-Step Response of Designed System')
xlabel('k    (Sampling period T = 0.2 sec)')
ylabel('Output c(k)')
```

Comments. The advantage of the w transform method is that the conventional frequency-response method using Bode diagrams can be used for the design of discrete-time control systems. In applying this method, we must carefully choose a reasonable sampling frequency. Before we conclude this section, we summarize the important facts about design in the w plane.

1. The magnitude and phase angle of $G(j\nu)$ are the magnitude and phase angle of $G(z)$ as z moves on the unit circle from $z = 1$ to $z = -1$. Since $z = e^{j\omega T}$,

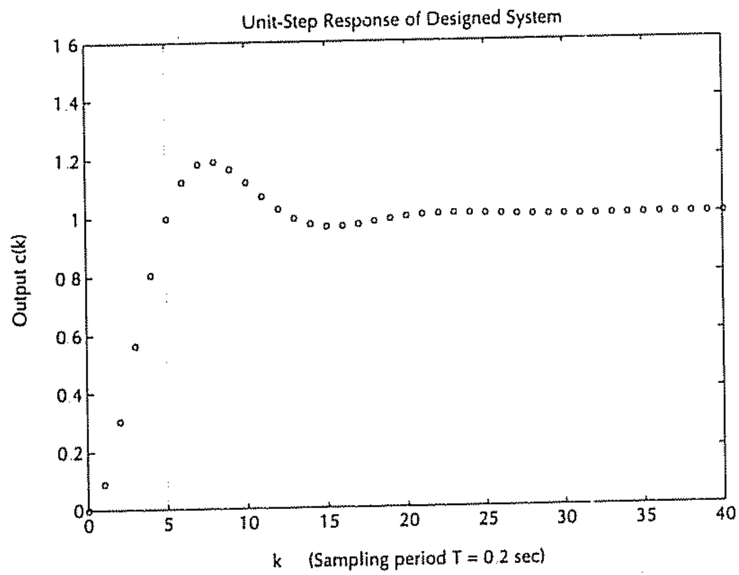


Figure 4–38 Plot of unit-step response of the designed system.

the ω value varies from 0 to $\frac{1}{2}\omega_s$. The fictitious frequency ν varies from 0 to ∞, since $\nu = (2/T)\tan(\omega T/2)$. Thus, the frequency response of the digital control system for $0 \le \omega \le \frac{1}{2}\omega_s$ is similar to the frequency response of the corresponding analog control system for $0 \le \nu \le \infty$.

2. Since $G(j\nu)$ is a rational function of ν, it is basically the same as $G(j\omega)$. In determining possible unstable zeros of the characteristic equation, the Nyquist stability criterion can be applied. Therefore, both the conventional straight-line approximation to the magnitude curve in the Bode diagram and the concept of phase margin and gain margin apply to $G(j\nu)$.
3. Compare transfer functions $G(w)$ and $G(s)$. As we mentioned earlier, because of the presence of the scale factor $2/T$ in the w transformation, the corresponding static error constants for $G(w)$ and $G(s)$ become identical. (Without the scale factor $2/T$, this will not be true.)
4. The w transformation may generate one or more right half-plane zeros in $G(w)$. If one or more right half-plane zeros exist, then $G(w)$ is a nonminimum phase transfer function. Because the zeros in the right half-plane are generated by the sample-and-hold operation, the locations of these zeros depend on the sampling period T. The effects of these zeros in the right half-plane on the response become smaller as the sampling period T becomes smaller.

   In what follows, let us consider the effects on the response of the zero in the right half-plane at $w = 2/T$. The zero at $w = 2/T$ causes distortion in the frequency response as ν approaches $2/T$. Since

$$\nu = \frac{2}{T}\tan\frac{\omega T}{2}$$

then, as ν approaches $2/T$, $\tan(\omega T/2)$ approaches 1, or

$$\frac{\omega T}{2} = \frac{\pi}{4}$$

and thus

$$\omega = \frac{\pi}{2T}$$

As we stated earlier, $\omega = \frac{1}{2}\omega_s = \pi/T$ is the highest frequency that we consider in the response of the discrete-time or digital control system. Therefore, $\omega = \omega_s/4 = \pi/2T$, which is one-half the highest frequency considered, is well within the frequency range of interest. Thus, the zero at $w = 2/T$, which is in the right half of the w plane, will seriously affect the response.

5. It should be noted that the Bode diagram method in the w plane is frequently used in practice, and many successful digital control systems have been designed by this approach.

4-7 ANALYTICAL DESIGN METHOD

The main reason why the control actions of analog controllers are limited is that there are physical limitations in pneumatic, hydraulic, and electronic components. Such limitations may be completely ignored in designing digital controllers. Thus, many control schemes that have been impossible with analog controls are possible with digital controls. In fact, optimal control schemes that are not possible with analog controllers are made possible by digital control schemes.

In this section we specifically present an analytical design method for digital controllers that will force the error sequence, when subjected to a specific type of time-domain input, to become zero after a finite number of sampling periods and, in fact, to become zero and stay zero after the minimum possible number of sampling periods.

If the response of a closed-loop control system to a step input exhibits the minimum possible settling time (that is, the output reaches the final value in the minimum time and stays there), no steady-state error, and no ripples between the sampling instants, then this type of response is commonly called a *deadbeat response*. The deadbeat response will be discussed in this section. (We shall treat the deadbeat response again in Chapter 6, where we discuss the pole placement technique and the design of state observers.)

The discussions that follow are limited to the determination of the control algorithms or pulse transfer functions of digital controllers for single-input–single-output systems, given desired optimal response characteristics. For optimal control of multiple-input–multiple-output systems, see Chapter 8, where the state-space approach is used.

Design of Digital Controllers for Minimum Settling Time with Zero Steady-State Error. Consider the digital control system shown in Figure 4–39(a). The error signal $e(t)$, which is the difference between the input $r(t)$ and the output $c(t)$, is

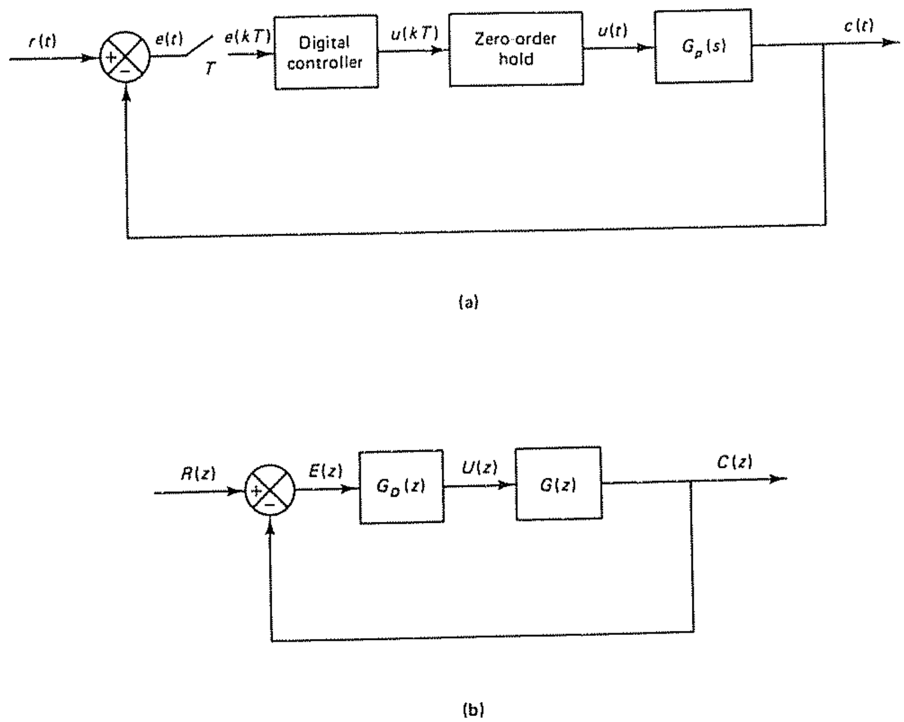


Figure 4–39 (a) A digital control system; (b) diagram showing equivalent digital control system.

sampled every time interval T. The input to the digital controller is the error signal $e(kT)$. The output of the digital controller is the control signal $u(kT)$. The control signal $u(kT)$ is fed to the zero-order hold, and the output of the hold, $u(t)$, which is a piecewise continuous-time signal, is fed to the plant. [Although the sampler at the input of the zero-order hold is not shown, the signal $u(kT)$ is first sampled and fed to the zero-order hold. As mentioned earlier, the zero-order hold shown in the diagram is a sample-and-hold device.] It is desired to design a digital controller $G_D(z)$ such that the closed-loop control system will exhibit the minimum possible settling time with zero steady-state error in response to a step, a ramp, or an acceleration input. It is required that the output not exhibit intersampling ripples after the steady state is reached. The system must satisfy any other specifications, if required, such as a specification for the static velocity error constant.

Let us define the z transform of the plant that is preceded by the zero-order hold as $G(z)$, or

$$G(z) = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s} G_p(s)\right]$$

Then the open-loop pulse transfer function becomes $G_D(z)G(z)$, as shown in Figure 4–39(b). Next, define the desired closed-loop pulse transfer function as $F(z)$:

$$\frac{C(z)}{R(z)} = \frac{G_D(z)G(z)}{1 + G_D(z)G(z)} = F(z) \tag{4-41}$$

Since it is required that the system exhibit a finite settling time with zero steady-state error, the system must exhibit a finite impulse response. Hence, the desired closed-loop pulse transfer function must be of the following form:

$$F(z) = \frac{a_0 z^N + a_1 z^{N-1} + \cdots + a_N}{z^N}$$

or

$$F(z) = a_0 + a_1 z^{-1} + \cdots + a_N z^{-N} \tag{4-42}$$

where $N \geq n$ and n is the order of the system. [Note that $F(z)$ must not contain any terms with positive powers in z, since such terms in the series expansion of $F(z)$ imply that the output precedes the input, which is not possible for a physically realizable system.] In our design approach, we solve the closed-loop pulse transfer function for the digital controller pulse transfer function $G_D(z)$. That is, we find the pulse transfer function $G_D(z)$ that will satisfy Equation (4–41). Solving Equation (4–41) for $G_D(z)$, we obtain

$$G_D(z) = \frac{F(z)}{G(z)[1 - F(z)]} \tag{4-43}$$

The designed system must be physically realizable. The conditions for physical realizability place certain constraints on the closed-loop pulse transfer function $F(z)$ and the digital controller pulse transfer function $G_D(z)$. The conditions for physical realizability may be stated as follows:

1. The order of the numerator of $G_D(z)$ must be equal to or lower than the order of the denominator. (Otherwise, the controller requires future input data to produce the current output.)
2. If the plant $G_p(s)$ involves a transportation lag e^{-Ls}, then the designed closed-loop system must involve at least the same magnitude of the transportation lag. (Otherwise, the closed-loop system would have to respond before an input was given, which is impossible for a physically realizable system.)
3. If $G(z)$ is expanded into a series in z^{-1}, the lowest-power term of the series expansion of $F(z)$ in z^{-1} must be at least as large as that of $G(z)$. For example, if an expansion of $G(z)$ into a series in z^{-1} begins with the z^{-1} term, then the first term of $F(z)$ given by Equation (4–42) must be zero, or a_0 must equal 0; that is, the expansion has to be of the form

   $$F(z) = a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}$$

   where $N \geq n$ and n is the order of the system. This means that the plant cannot respond instantaneously when a control signal of finite magnitude is applied: the response comes at a delay of at least one sampling period if the series expansion of $G(z)$ begins with a term in z^{-1}.

In addition to the physical realizability conditions, we must pay attention to the stability aspects of the system. Specifically, we must avoid canceling an unstable pole of the plant by a zero of the digital controller. If such a cancellation is attempted, any error in the pole–zero cancellation will diverge as time elapses and the system will become unstable. Similarly, the digital controller pulse transfer function should not involve unstable poles to cancel plant zeros that lie outside the unit circle.

Next, let us investigate what will happen to the closed-loop pulse transfer function $F(z)$ if $G(z)$ involves an unstable (or critically stable) pole, that is, a pole $z = \alpha$ outside (or on) the unit circle. [Note that the following argument applies equally, if $G(z)$ involves two or more unstable—or critically stable—poles.] Let us define

$$G(z) = \frac{G_1(z)}{z - \alpha}$$

where $G_1(z)$ does not include a term that cancels with $z - \alpha$. Then the closed-loop pulse transfer function becomes

$$\frac{C(z)}{R(z)} = \frac{G_D(z)G(z)}{1 + G_D(z)G(z)} = \frac{G_D(z)\dfrac{G_1(z)}{z-\alpha}}{1 + G_D(z)\dfrac{G_1(z)}{z-\alpha}} = F(z) \tag{4-44}$$

Since we require that no zero of $G_D(z)$ cancel the unstable pole of $G(z)$ at $z = \alpha$, we must have

$$1 - F(z) = \frac{1}{1 + G_D(z)\dfrac{G_1(z)}{z-\alpha}} = \frac{z - \alpha}{z - \alpha + G_D(z)G_1(z)}$$

That is, $1 - F(z)$ must have $z = \alpha$ as a zero. Also, notice that from Equation (4–44) if zeros of $G(z)$ do not cancel poles of $G_D(z)$, the zeros of $G(z)$ become zeros of $F(z)$. [$F(z)$ may involve additional zeros.]

Let us summarize what we have stated concerning stability.

1. Since the digital controller $G_D(z)$ should not cancel unstable (or critically stable) poles of $G(z)$, all unstable (or critically stable) poles of $G(z)$ must be included in $1 - F(z)$ as zeros.
2. Zeros of $G(z)$ that lie inside the unit circle may be canceled with poles of $G_D(z)$. However, zeros of $G(z)$ that lie on or outside the unit circle must not be canceled with poles of $G_D(z)$. Hence, all zeros of $G(z)$ that lie on or outside the unit circle must be included in $F(z)$ as zeros.

Now we shall proceed with the design. Since $e(kT) = r(kT) - c(kT)$, referring to Equation (4–41) we have

$$E(z) = R(z) - C(z) = R(z)[1 - F(z)] \tag{4-45}$$

Note that for a unit-step input $r(t) = 1(t)$

$$R(z) = \frac{1}{1 - z^{-1}}$$

For a unit-ramp input $r(t) = t1(t)$,

$$R(z) = \frac{Tz^{-1}}{(1 - z^{-1})^2}$$

And for a unit-acceleration input $r(t) = \frac{1}{2}t^2 1(t)$,

$$R(z) = \frac{T^2 z^{-1}(1 + z^{-1})}{2(1 - z^{-1})^3}$$

Thus, in general, z transforms of such time-domain polynomial inputs may be written as

$$R(z) = \frac{P(z)}{(1 - z^{-1})^{q+1}} \tag{4–46}$$

where $P(z)$ is a polynomial in z^{-1}. Notice that for a unit-step input $P(z) = 1$ and $q = 0$; for a unit-ramp input, $P(z) = Tz^{-1}$ and $q = 1$; and for a unit-acceleration input, $P(z) = \frac{1}{2}T^2 z^{-1}(1 + z^{-1})$ and $q = 2$.

By substituting Equation (4–46) into Equation (4–45), we obtain

$$E(z) = \frac{P(z)[1 - F(z)]}{(1 - z^{-1})^{q+1}} \tag{4–47}$$

To ensure that the system reaches steady state in a finite number of sampling periods and maintains zero steady-state error, $E(z)$ must be a polynomial in z^{-1} with a finite number of terms. Then, by referring to Equation (4–47), we choose the function $1 - F(z)$ to be of the form

$$1 - F(z) = (1 - z^{-1})^{q+1} N(z) \tag{4–48}$$

where $N(z)$ is a polynomial in z^{-1} with a finite number of terms. Then

$$E(z) = P(z)N(z) \tag{4–49}$$

which is a polynomial in z^{-1} with a finite number of terms. This means that the error signal becomes zero in a finite number of sampling periods.

From the preceding analysis, the pulse transfer function of the digital controller can be determined as follows. By first letting $F(z)$ satisfy the physical realizability and stability conditions and then substituting Equation (4–48) into Equation (4–43), we obtain

$$G_D(z) = \frac{F(z)}{G(z)(1 - z^{-1})^{q+1} N(z)} \tag{4–50}$$

Equation (4–50) gives the pulse transfer function of the digital controller that will produce zero steady-state error after a finite number of sampling periods.

For a stable plant $G_p(s)$, the condition that the output not exhibit intersampling ripples after the settling time is reached may be written as follows:

$$c(t \geq nT) = \text{constant}, \qquad \text{for step inputs}$$
$$\dot{c}(t \geq nT) = \text{constant}, \qquad \text{for ramp inputs}$$
$$\ddot{c}(t \geq nT) = \text{constant}, \qquad \text{for acceleration inputs}$$

The applicable condition must be satisfied when the system is designed. In designing the system, the condition on $c(t)$, $\dot{c}(t)$, or $\ddot{c}(t)$ must be interpreted in terms of $u(t)$. Note that the plant is continuous time and the input to the plant is $u(t)$, a continuous-time function; therefore, to have no ripples in the output $c(t)$, the control signal $u(t)$ at steady state must be either constant or monotonically increasing (or monotonically decreasing) for step, ramp, or acceleration inputs.

Comments

1. Since the closed-loop pulse transfer function $F(z)$ is a polynomial in z^{-1}, all the closed-loop poles are at the origin or at $z = 0$. The multiple closed-loop pole at the origin is very sensitive to system parameter variations.
2. Although a digital control system designed to exhibit minimum settling time with zero steady-state error in response to a specific type of input has excellent transient response characteristics for the input it is designed for, it may exhibit inferior or sometimes unacceptable transient response characteristics for other types of input. (This is always true in optimal control systems. An optimal control system will exhibit the best response characteristics for the type of input it is designed for, but will not exhibit optimal response characteristics for other types of input.)
3. In the case in which an analog controller is discretized, an increase in the sampling period changes the system dynamics and may lead to system instability. On the other hand, the behavior of the digital control system we are designing in this section does not depend on the choice of the sampling period. Since the inputs $r(t)$ considered here are time-domain inputs (such as step inputs, ramp inputs, and acceleration inputs), the sampling period T can be chosen arbitrarily. For a smaller sampling period, the response time (which is an integral multiple of the sampling period T) becomes smaller. However, for a very small sampling period T, the magnitude of the control signal will become excessively large, with the result that saturation phenomena will take place in the system, and the design method presented in this section will no longer apply. Hence, the sampling period T should not be too small. On the other hand, if the sampling period T is chosen too large, the system may behave unsatisfactorily or may even become unstable when it is subjected to sufficiently time varying inputs (such as frequency-domain inputs). Thus, a compromise is necessary. A rule of thumb is to choose the smallest sampling period T such that no saturation phenomena occur in the control signal.

Example 4–13

Consider the digital control system shown in Figure 4–39(a), where the plant transfer function $G_p(s)$ is given by

$$G_p(s) = \frac{1}{s(s+1)}$$

Design a digital controller $G_D(z)$ such that the closed-loop system will exhibit a deadbeat response to a unit-step input. (In a deadbeat response the system should not exhibit intersampling ripples in the output after the settling time is reached.) The sampling period T is assumed to be 1 sec. Then, using the digital controller $G_D(z)$ so designed, investigate the response of this system to a unit-ramp input.

The first step in the design is to determine the z transform of the plant that is preceded by the zero-order hold:

$$\begin{aligned} G(z) &= \mathcal{Z}\left[\frac{1-e^{-Ts}}{s}\frac{1}{s(s+1)}\right] \\ &= (1-z^{-1})\mathcal{Z}\left[\frac{1}{s^2(s+1)}\right] \\ &= (1-z^{-1})\left[\frac{z^{-1}}{(1-z^{-1})^2} - \frac{1}{1-z^{-1}} + \frac{1}{1-0.3679z^{-1}}\right] \\ &= \frac{0.3679(1+0.7181z^{-1})z^{-1}}{(1-z^{-1})(1-0.3679z^{-1})} \end{aligned} \tag{4-51}$$

Now redraw the block diagram of the system as shown in Figure 4–39(b). Define the closed-loop pulse transfer function as $F(z)$, or

$$\frac{C(z)}{R(z)} = \frac{G_D(z)G(z)}{1+G_D(z)G(z)} = F(z)$$

Notice that if $G(z)$ is expanded into a series in z^{-1} then the first term will be $0.3679z^{-1}$. Hence, $F(z)$ must begin with a term in z^{-1}.

Referring to Equation (4–42) and noting that the system is of the second order ($n = 2$), we assume $F(z)$ to be of the following form:

$$F(z) = a_1 z^{-1} + a_2 z^{-2} \tag{4-52}$$

Since the input is a step function, from Equation (4–48) we require that

$$1 - F(z) = (1 - z^{-1})N(z) \tag{4-53}$$

Since $G(z)$ has a critically stable pole at $z = 1$, the stability requirement states that $1 - F(z)$ must have a zero at $z = 1$. However, the function $1 - F(z)$ already has a term $1 - z^{-1}$ and therefore satisfies the requirement.

Since the system should not exhibit intersampling ripples and the input is a step function, we require $c(t \geq 2T)$ to be constant. Noting that $u(t)$, the output of the zero-order hold, is a continuous-time function, a constant $c(t \geq 2T)$ requires that $u(t)$ also be constant for $t \geq 2T$. In terms of the z transform, $U(z)$ must be of the following type of series in z^{-1}:

$$U(z) = b_0 + b_1 z^{-1} + b(z^{-2} + z^{-3} + z^{-4} + \cdots)$$

where b is a constant. Because the plant transfer function $G_p(s)$ involves an integrator, b must be zero. (Otherwise, the output cannot stay constant.) Consequently, we have

$$U(z) = b_0 + b_1 z^{-1}$$

From Figure 4–39(b), $U(z)$ can be given as follows:

$$U(z) = \frac{C(z)}{G(z)} = \frac{C(z)}{R(z)}\frac{R(z)}{G(z)} = F(z)\frac{R(z)}{G(z)}$$

$$= F(z)\frac{1}{1 - z^{-1}}\frac{(1 - z^{-1})(1 - 0.3679z^{-1})}{0.3679(1 + 0.7181z^{-1})z^{-1}}$$

$$= F(z)\frac{1 - 0.3679z^{-1}}{0.3679(1 + 0.7181z^{-1})z^{-1}}$$

For $U(z)$ to be a series in z^{-1} with only two terms, $F(z)$ must be of the following form:

$$F(z) = (1 + 0.7181z^{-1})z^{-1}F_1 \qquad (4\text{–}54)$$

where F_1 is a constant. Then $U(z)$ can be written as follows:

$$U(z) = 2.7181(1 - 0.3679z^{-1})F_1 \qquad (4\text{–}55)$$

Equation (4–55) gives $U(z)$ in terms of F_1. Once constant F_1 is determined, $U(z)$ can be given as a series in z^{-1} with only two terms.

Now we shall determine $N(z)$, $F(z)$, and F_1. By substituting Equation (4–52) into Equation (4–53), we obtain

$$1 - a_1 z^{-1} - a_2 z^{-2} = (1 - z^{-1})N(z)$$

The left-hand side of this last equation must be divisible by $1 - z^{-1}$. If we divide the left-hand side by $1 - z^{-1}$, the quotient is $1 + (1 - a_1)z^{-1}$ and the remainder is $(1 - a_1 - a_2)z^{-2}$. Hence, $N(z)$ is determined as

$$N(z) = 1 + (1 - a_1)z^{-1} \qquad (4\text{–}56)$$

and the remainder must be zero. This requires that

$$1 - a_1 - a_2 = 0 \qquad (4\text{–}57)$$

Also, from Equations (4–52) and (4–54) we have

$$F(z) = a_1 z^{-1} + a_2 z^{-2} = (1 + 0.7181z^{-1})z^{-1}F_1$$

Hence,

$$a_1 + a_2 z^{-1} = (1 + 0.7181z^{-1})F_1$$

Division of the left-hand side of this last equation by $1 + 0.7181z^{-1}$ yields the quotient a_1 and the remainder $(a_2 - 0.7181a_1)z^{-1}$. By equating the quotient with F_1 and the remainder with zero, we obtain

$$F_1 = a_1$$

and

$$a_2 - 0.7181a_1 = 0 \qquad (4\text{–}58)$$

Solving Equations (4–57) and (4–58) for a_1 and a_2 gives

$$a_1 = 0.5820, \qquad a_2 = 0.4180$$

Thus, $F(z)$ is determined as

$$F(z) = 0.5820z^{-1} + 0.4180z^{-2} \qquad (4\text{–}59)$$

and

$$F_1 = 0.5820$$

Equation (4–56) gives

$$N(z) = 1 + 0.4180z^{-1} \qquad (4\text{–}60)$$

The digital controller pulse transfer function $G_D(z)$ is then determined from Equation (4–50), as follows. By referring to Equations (4–51), (4–54), and (4–60),

$$G_D(z) = \frac{F(z)}{G(z)(1 - z^{-1})N(z)}$$
$$= \frac{(1 + 0.7181z^{-1})z^{-1}(0.5820)}{\dfrac{0.3679(1 + 0.7181z^{-1})z^{-1}}{(1 - z^{-1})(1 - 0.3679z^{-1})}(1 - z^{-1})(1 + 0.4180z^{-1})}$$
$$= \frac{1.5820 - 0.5820z^{-1}}{1 + 0.4180z^{-1}}$$

With the digital controller thus designed, the closed-loop pulse transfer function becomes as follows:

$$\frac{C(z)}{R(z)} = F(z) = 0.5820z^{-1} + 0.4180z^{-2}$$
$$= \frac{0.5820(z + 0.7181)}{z^2}$$

The system output in response to a unit-step input $r(t) = 1$ can be obtained as follows:

$$C(z) = F(z)R(z)$$
$$= (0.5820z^{-1} + 0.4180z^{-2})\frac{1}{1 - z^{-1}}$$
$$= 0.5820z^{-1} + z^{-2} + z^{-3} + z^{-4} + \cdots$$

Hence,

$$c(0) = 0$$
$$c(1) = 0.5820$$
$$c(k) = 1, \qquad k = 2, 3, 4, \ldots$$

Notice that substitution of 0.5820 for F_1 in Equation (4–55) yields

$$U(z) = 2.7181(1 - 0.3679z^{-1})(0.5820)$$
$$= 1.5820 - 0.5820z^{-1}$$

Thus, the control signal $u(k)$ becomes zero for $k \geq 2$, as required. There is no intersampling ripple in the output after the settling time is reached. Figure 4–40(a) shows plots of $c(k)$ versus k, $u(k)$ versus k, and $u(t)$ versus t in the unit-step response.

Next, let us investigate the response of this system to a unit-ramp input:

$$C(z) = F(z)R(z)$$
$$= (0.5820z^{-1} + 0.4180z^{-2})\frac{z^{-1}}{(1 - z^{-1})^2}$$
$$= 0.5820z^{-2} + 1.5820z^{-3} + 2.5820z^{-4} + 3.5820z^{-5} + \cdots$$

For the unit-ramp response, the control signal $U(z)$ is obtained as follows. Referring to Equations (4–51) and (4–59),

$$U(z) = \frac{C(z)}{G(z)} = \frac{F(z)}{G(z)}R(z) = \frac{F(z)}{G(z)}\frac{z^{-1}}{(1 - z^{-1})^2}$$

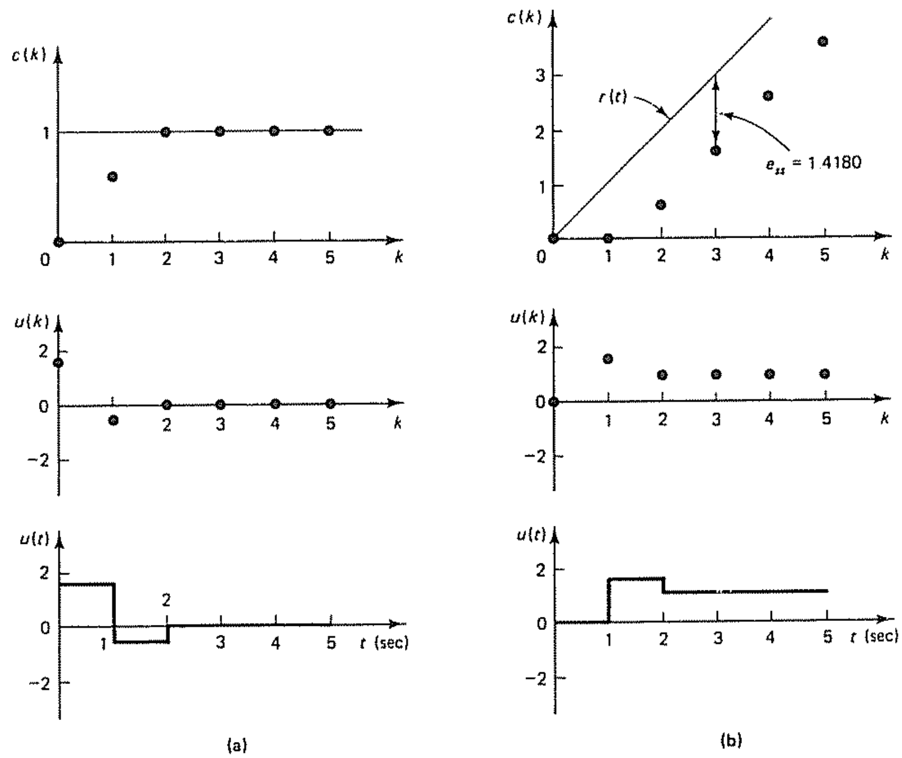


Figure 4-40 Responses of the system designed in Example 4-13 (a) Plots of $c(k)$ versus k, $u(k)$ versus k, and $u(t)$ versus t in the unit-step response; (b) plots of $c(k)$ versus k, $u(k)$ versus k, and $u(t)$ versus t in the unit-ramp response.

$$= (1.5820 - 0.5820z^{-1})\frac{z^{-1}}{1 - z^{-1}}$$
$$= 1.5820z^{-1} + z^{-2} + z^{-3} + z^{-4} + \cdots$$

The signal $u(k)$ becomes constant ($b = 1$) for $k \geq 2$. Hence, the system output will not exhibit intersampling ripples. Figure 4-40(b) shows plots of $c(k)$ versus k, $u(k)$ versus k, and $u(t)$ versus t in the unit-ramp response.

Note that the static velocity error constant K_v for the present system is

$$K_v = \lim_{z \to 1}\left[\frac{1 - z^{-1}}{T} G_D(z)G(z)\right]$$
$$= \lim_{z \to 1}\left[(1 - z^{-1})\frac{F(z)}{(1 - z^{-1})N(z)}\right]$$
$$= \lim_{z \to 1}\frac{0.5820z^{-1} + 0.4180z^{-2}}{1 + 0.4180z^{-1}} = 0.7052$$

Thus, the steady-state error in the unit-ramp response is

$$e_{ss} = \frac{1}{K_v} = 1.4180$$

which is indicated in Figure 4–40(b).

In the present design problem, we have required that in response to a step input the system exhibit the minimum settling time with no steady-state error and no ripples in the output after the settling time is reached. If one or more additional constraints are present in the design problem (for example, if the value of the static velocity error constant K_v is arbitrarily specified), then the number of sampling periods required before reaching the steady-state must be increased. For example, the second-order system may require three or more sampling periods before the steady state is reached, depending on the additional constraints imposed. See Example 4–14, which follows.

Example 4–14

Consider a design problem the same as that of Example 4–13 except that the static velocity error constant K_v is specified. (Because of this additional constraint, the settling time will be longer than 2 sec.) The block diagram of the digital control system is shown in Figure 4–39(a). The plant transfer function $G_p(s)$ under consideration is

$$G_p(s) = \frac{1}{s(s+1)}$$

The design specifications are (1) that the closed-loop system is to exhibit a finite settling time with zero steady-state error in the unit-step response, (2) that the output is not to exhibit intersampling ripples after the settling time is reached, (3) that the static velocity error constant K_v is to be 4 sec^{-1}, and (4) that the settling time is to be the minimum possible that will satisfy all these specifications. The sampling period T is assumed to be 1 sec. Design a digital controller $G_D(z)$ that satisfies the given specifications. After the controller is designed, investigate the response of the system to a unit-ramp input.

The z transform of the plant that is preceded by the zero-order hold was obtained in Example 4–13 as

$$G(z) = \mathcal{Z}\left[\frac{1-e^{-Ts}}{s}\frac{1}{s(s+1)}\right]$$
$$= \frac{0.3679(1+0.7181z^{-1})z^{-1}}{(1-z^{-1})(1-0.3679z^{-1})}$$

Define the closed-loop pulse transfer function as $F(z)$:

$$\frac{C(z)}{R(z)} = \frac{G_D(z)G(z)}{1+G_D(z)G(z)} = F(z)$$

Since the first term in the expansion of $G(z)$ is $0.3679z^{-1}$, $F(z)$ must begin with a term in z^{-1}:

$$F(z) = a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}$$

where $N \geq n$ and n is the order of the system (that is, $n = 2$ in the present case) Because of the added constraint, we may assume $N > 2$. We shall try $N = 3$. Thus, we assume

$$F(z) = a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} \qquad (4\text{–}61)$$

(If a satisfactory result is not obtained, we must assume $N > 3$.) Since the input is a step function, from Equation (4–48) we require that

$$1 - F(z) = (1 - z^{-1})N(z) \tag{4-62}$$

Note that the presence of a critically stable pole at $z = 1$ in the plant pulse transfer function $G(z)$ requires $1 - F(z)$ to have a zero at $z = 1$. However, the function $1 - F(z)$ already has a term $1 - z^{-1}$ and therefore satisfies the stability requirement.

The requirement that the static velocity error constant be 4 sec^{-1} can be written as follows:

$$K_v = \lim_{z \to 1}\left[\frac{1 - z^{-1}}{T} G_D(z)G(z)\right]$$
$$= \lim_{z \to 1}\left[(1 - z^{-1})\frac{F(z)}{(1 - z^{-1})N(z)}\right]$$
$$= \frac{F(1)}{N(1)} = 4$$

where we used Equation (4–50) with $q = 0$. Notice that from Equation (4–62) we have $F(1) = 1$. Hence, K_v can be written as follows:

$$K_v = \frac{1}{N(1)} = 4 \tag{4-63}$$

Since the system output should not exhibit intersampling ripples after the settling time is reached, we require $U(z)$ to be of the following form:

$$U(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + b(z^{-3} + z^{-4} + z^{-5} + \cdots)$$

Because the plant transfer function $G_p(s)$ involves an integrator, b must be zero. Consequently, we have

$$U(z) = b_0 + b_1 z^{-1} + b_2 z^{-2}$$

Also, from Figure 4-39(b), $U(z)$ can be given by

$$U(z) = \frac{C(z)}{G(z)} = \frac{C(z)}{R(z)}\frac{R(z)}{G(z)} = F(z)\frac{R(z)}{G(z)}$$
$$= F(z)\frac{1 - 0.3679z^{-1}}{0.3679(1 + 0.7181z^{-1})z^{-1}}$$

For $U(z)$ to be a series in z^{-1} with three terms, $F(z)$ must be of the following form:

$$F(z) = (1 + 0.7181z^{-1})z^{-1}F_1(z) \tag{4-64}$$

where $F_1(z)$ is a first-degree polynomial in z^{-1}. Then $U(z)$ can be written as follows:

$$U(z) = 2.7181(1 - 0.3679z^{-1})F_1(z) \tag{4-65}$$

From Equations (4–61) and (4–62), we have

$$1 - F(z) = 1 - a_1 z^{-1} - a_2 z^{-2} - a_3 z^{-3} = (1 - z^{-1})N(z)$$

If we divide $1 - a_1 z^{-1} - a_2 z^{-2} - a_3 z^{-3}$ by $1 - z^{-1}$, the quotient is $1 + (1 - a_1)z^{-1} + (1 - a_1 - a_2)z^{-2}$ and the remainder is $(1 - a_1 - a_2 - a_3)z^{-3}$. Hence, $N(z)$ is determined as

$$N(z) = 1 + (1 - a_1)z^{-1} + (1 - a_1 - a_2)z^{-2} \tag{4-66}$$

and the remainder must be zero, so that

$$1 - a_1 - a_2 - a_3 = 0 \tag{4–67}$$

Note that from Equation (4–63) we require $N(1) = \frac{1}{4}$. Therefore, by substituting $z^{-1} = 1$ into Equation (4–66), we obtain

$$2a_1 + a_2 = 2.75 \tag{4–68}$$

Also, Equation (4–64) can be rewritten as

$$F(z) = a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} = (1 + 0.7181z^{-1})z^{-1} F_1(z)$$

Hence,

$$a_1 + a_2 z^{-1} + a_3 z^{-2} = (1 + 0.7181z^{-1})F_1(z)$$

Division of the left-hand side of this last equation by $1 + 0.7181z^{-1}$ yields the quotient $[a_1 + (a_2 - 0.7181a_1)z^{-1}]$ and the remainder $[a_3 - 0.7181(a_2 - 0.7181a_1)]z^{-2}$. By equating the quotient with $F_1(z)$ and the remainder with zero, we obtain

$$F_1(z) = a_1 + (a_2 - 0.7181a_1)z^{-1}$$

and

$$a_3 - 0.7181(a_2 - 0.7181a_1) = 0 \tag{4–69}$$

Solving Equations (4–67), (4–68), and (4–69) for a_1, a_2, and a_3 gives

$$a_1 = 1.26184, \qquad a_2 = 0.22633, \qquad a_3 = -0.48816$$

Thus, $F(z)$ is determined as

$$F(z) = 1.26184z^{-1} + 0.22633z^{-2} - 0.48816z^{-3}$$

and

$$F_1(z) = 1.26184 - 0.67979z^{-1}$$

Equation (4–66) gives

$$N(z) = 1 - 0.26184z^{-1} - 0.48817z^{-2}$$

The digital controller pulse transfer function $G_D(z)$ is then determined from Equation (4–50) as follows:

$$\begin{aligned}
G_D(z) &= \frac{F(z)}{G(z)(1 - z^{-1})N(z)} \\
&= \frac{(1 + 0.7181z^{-1})z^{-1}(1.26184 - 0.67980z^{-1})}{\dfrac{0.3679(1 + 0.7181z^{-1})z^{-1}}{(1 - z^{-1})(1 - 0.3679z^{-1})}(1 - z^{-1})(1 - 0.26184z^{-1} - 0.48817z^{-2})} \\
&= 3.4298\frac{(1 - 0.5387z^{-1})(1 - 0.3679z^{-1})}{(1 - 0.8418z^{-1})(1 + 0.5799z^{-1})}
\end{aligned}$$

With the digital controller thus designed, the system output in response to a unit-step input $r(t) = 1$ is obtained as follows:

$$\begin{aligned}
C(z) &= F(z)R(z) \\
&= (1.26184z^{-1} + 0.22633z^{-2} - 0.48816z^{-3})\frac{1}{1 - z^{-1}} \\
&= 1.2618z^{-1} + 1.4882z^{-2} + z^{-3} + z^{-4} + \cdots
\end{aligned}$$

Hence,

$$c(0) = 0$$
$$c(1) = 1.2618$$
$$c(2) = 1.4882$$
$$c(k) = 1, \qquad k = 3, 4, 5, \ldots$$

The unit-step response sequence has a maximum overshoot of approximately 50%. The settling time is 3 sec.

Notice that from Equation (4–65) we have

$$U(z) = 2.7181(1 - 0.3679z^{-1})(1.26184 - 0.67979z^{-1})$$
$$= 3.4298 - 3.1096z^{-1} + 0.6798z^{-2}$$

Thus, the control signal $u(k)$ becomes zero for $k \geq 3$. Consequently, there are no intersampling ripples in the response. Figure 4–41 shows plots of $c(k)$ versus k, $u(k)$ versus k, and $u(t)$ versus t in the unit-step response. Notice that the assumption of $N = 3$, that is, the assumption of $F(z)$ as given by Equation (4–61), is satisfactory.

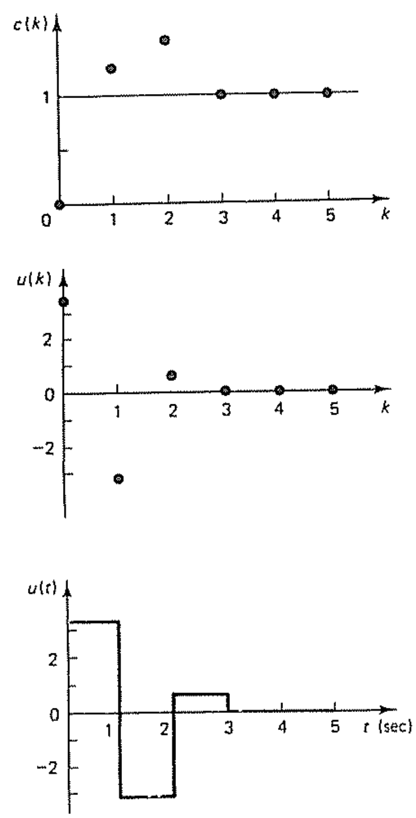


Figure 4–41 Plots of $c(k)$ versus k, $u(k)$ versus k, and $u(t)$ versus t in the unit-step response of the system designed in Example 4–14.

Next, let us investigate the response of this system to a unit-ramp input:

$$
\begin{aligned}
C(z) &= F(z)R(z) \\
&= (1.26184z^{-1} + 0.22633z^{-2} - 0.48816z^{-3})\frac{z^{-1}}{(1 - z^{-1})^2} \\
&= 1.2618z^{-2} + 2.7500z^{-3} + 3.7500z^{-4} + \cdots
\end{aligned}
$$

In the unit-ramp response, the control signal $U(z)$ is obtained as follows:

$$
\begin{aligned}
U(z) &= \frac{C(z)}{G(z)} = \frac{F(z)}{G(z)}R(z) = \frac{F(z)}{G(z)}\frac{1}{1 - z^{-1}}\frac{z^{-1}}{1 - z^{-1}} \\
&= (3.4298 - 3.1096z^{-1} + 0.6798z^{-2})\frac{z^{-1}}{1 - z^{-1}} \\
&= 3.4298z^{-1} + 0.3202z^{-2} + z^{-3} + z^{-4} + z^{-5} + \cdots
\end{aligned}
$$

The signal $u(k)$ becomes constant $(b = 1)$ for $k \geq 3$. Hence, the system output will not exhibit intersampling ripples. Figure 4–42 shows plots of $c(k)$ versus k, $u(k)$ versus k, and $u(t)$ versus t in the unit-ramp response. Notice that the steady-state error in the unit-ramp response is $e_{ss} = 1/K_v = \frac{1}{4}$, as indicated in Figure 4–42.

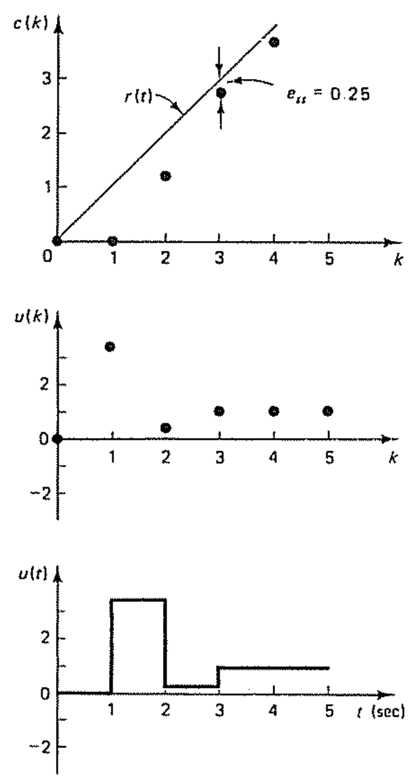


Figure 4–42 Plots of $c(k)$ versus k, $u(k)$ versus k, and $u(t)$ versus t in the unit-ramp response of the system designed in Example 4–14.

Comparing the digital control systems designed in Examples 4–13 and 4–14, we note that the latter improves the ramp response characteristics at the expense of the settling time. (The latter system requires one extra sampling period to reach the steady state.) Note also that the former has better step response characteristics, that is, a shorter settling time and no overshoot. Depending on the objectives of the system, we may choose one over the other. If good ramp response characteristics are required, then the system should be designed using the ramp input as the reference input, rather than the step input. (See Problem A–4–14.)

EXAMPLE PROBLEMS AND SOLUTIONS

Problem A–4–1

Show that geometrically the patterns of the poles near $z = 1$ in the z plane are similar to the patterns of poles in the s plane near the origin.

Solution Note that

$$z = e^{Ts}$$

Near the origin of the s plane,

$$z = e^{Ts} = 1 + Ts + \tfrac{1}{2}T^2s^2 + \cdots$$

or

$$z - 1 \doteqdot Ts$$

Thus, geometrical patterns of the poles near $z = 1$ in the z plane are similar to the patterns of poles in the s plane near the origin.

Problem A–4–2

Consider the system described by

$$y(k) - 0.6y(k-1) - 0.81y(k-2) + 0.67y(k-3) - 0.12y(k-4) = x(k)$$

where $x(k)$ is the input and $y(k)$ is the output of the system. Determine the stability of the system.

Solution The pulse transfer function for the system is

$$\frac{Y(z)}{X(z)} = \frac{1}{1 - 0.6z^{-1} - 0.81z^{-2} + 0.67z^{-3} - 0.12z^{-4}}$$

$$= \frac{z^4}{z^4 - 0.6z^3 - 0.81z^2 + 0.67z - 0.12}$$

Define

$$P(z) = z^4 - 0.6z^3 - 0.81z^2 + 0.67z - 0.12$$

$$= a_0z^4 + a_1z^3 + a_2z^2 + a_3z + a_4, \qquad a_0 > 0$$

Then we have

$$a_0 = 1$$

$$a_1 = -0.6$$

$$a_2 = -0.81$$
$$a_3 = 0.67$$
$$a_4 = -0.12$$

The Jury stability conditions are:

1. $|a_4| < a_0$. This condition is clearly satisfied.
2. $P(1) > 0$. Since
$$P(1) = 1 - 0.6 - 0.81 + 0.67 - 0.12 = 0.14 > 0$$
the condition is satisfied.
3. $P(-1) > 0$. Since
$$P(-1) = 1 + 0.6 - 0.81 - 0.67 - 0.12 = 0$$
the condition is not satisfied. $P(-1) = 0$ implies that there is one root at $z = -1$.
4. $|b_3| > |b_0|$. Since
$$b_3 = \begin{vmatrix} a_4 & a_0 \\ a_0 & a_4 \end{vmatrix} = \begin{vmatrix} -0.12 & 1 \\ 1 & -0.12 \end{vmatrix} = -0.9856$$
$$b_0 = \begin{vmatrix} a_4 & a_3 \\ a_0 & a_1 \end{vmatrix} = \begin{vmatrix} -0.12 & 0.67 \\ 1 & -0.6 \end{vmatrix} = -0.5980$$
the condition is satisfied.
5. $|c_2| > |c_0|$. Since
$$c_2 = \begin{vmatrix} b_3 & b_0 \\ b_0 & b_3 \end{vmatrix} = \begin{vmatrix} -0.9856 & -0.5980 \\ -0.5980 & -0.9856 \end{vmatrix} = 0.6138$$
$$c_0 = \begin{vmatrix} b_3 & b_2 \\ b_0 & b_1 \end{vmatrix} = \begin{vmatrix} -0.9856 & 0.5196 \\ -0.5980 & 0.9072 \end{vmatrix} = -0.5834$$
the condition is satisfied.

From the preceding analysis, we conclude that the characteristic equation $P(z) = 0$ involves a root at $z = -1$ and the other three roots are in the unit circle centered at the origin of the z plane. The system is critically stable.

Problem A-4-3

Consider the following characteristic equation:

$$P(z) = z^3 - 1.3z^2 - 0.08z + 0.24 = 0 \qquad (4\text{-}70)$$

Determine whether or not any of the roots of the characteristic equation lie outside the unit circle in the z plane. Use the bilinear transformation and the Routh stability criterion.

Solution Let us substitute $(w + 1)/(w - 1)$ for z in the given characteristic equation, resulting in

$$\left(\frac{w+1}{w-1}\right)^3 - 1.3\left(\frac{w+1}{w-1}\right)^2 - 0.08\frac{w+1}{w-1} + 0.24 = 0$$

Clearing the fractions by multiplying both sides of this last equation by $(w - 1)^3$, we get

$$-0.14w^3 + 1.06w^2 + 5.10w + 1.98 = 0$$

By dividing both sides of this last equation by -0.14, we obtain

$$w^3 - 7.571w^2 - 36.43w - 14.14 = 0 \tag{4-71}$$

The Routh array for Equation (4–71) becomes as follows:

one sign change	w^3	1	-36.43
	w^2	-7.571	-14.14
	w^1	-38.30	0
	w^0	-14.14	

Routh stability criterion states that the number of roots with positive real parts is equal to the number of changes in sign of the coefficients of the first column of the array. Since there is one sign change for the coefficients in the first column, there is one root in the right half of the w plane. This means that the original characteristic equation given by Equation (4–70) has one root outside the unit circle in the z plane. The system is unstable. (Compare the amount of computation needed in the present method and that needed in the Jury stability test. See in particular Example 4–6.)

Problem A–4–4

Consider the system defined by

$$\frac{Y(z)}{U(z)} = \frac{0.7870z^{-1}}{1 - 0.8195z^{-1} + 0.6065z^{-2}} = \frac{0.7870z}{z^2 - 0.8195z + 0.6065}$$

The sampling period T is 0.5 sec. Using MATLAB, plot the unit-ramp response up to $k = 20$.

Solution The unit-ramp input u may be written as

$$u = kT, \qquad k = 0, 1, 2, \ldots$$

In the MATLAB program, this input can be given as

```
k = 0:N;     u = [T*k];
```

where N is the end of the process considered.

A MATLAB program for plotting the unit-ramp response of the system considered is given in MATLAB Program 4–3. The resulting plot is shown in Figure 4–43

Problem A–4–5

Show that if the characteristic equation for a closed-loop system is written as

$$1 + \frac{KB(z)}{A(z)} = 0$$

where $A(z)$ and $B(z)$ do not contain K, then the breakaway and break-in points can be determined from the roots of

$$\frac{dK}{dz} = -\frac{A'(z)B(z) - A(z)B'(z)}{B^2(z)} = 0$$

where the primes indicate differentiation with respect to z.

MATLAB Program 4–3

```
% --------- Unit-ramp response ---------

% ***** Enter the numerator and denominator of the system *****

num = [0  0.7870  0];
den = [1  -0.8195  0.6065];

% ***** Enter k, unit-ramp input, filter command and plot
% command *****

k = 0:20;
u = [0.5*k];
y = filter(num,den,u);
plot(k,y,'o',k,y,'-',k,0.5*k,'--')

% ***** Add grid, title, xlabel, and ylabel *****

grid
title('Unit-Ramp Response')
xlabel('k')
ylabel('y(k)')
```

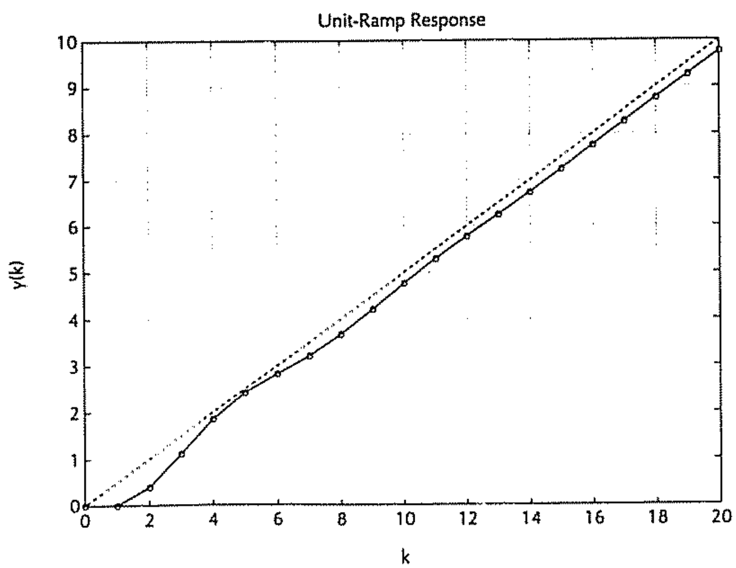


Figure 4–43 Unit-ramp response of the system considered in Problem A–4–4

Solution Let us write the characteristic equation as

$$f(z) = A(z) + KB(z) = 0 \tag{4-72}$$

Suppose that $f(z) = 0$ has a multiple root of order r. Then $f(z)$ may be written as

$$f(z) = (z - z_1)^r(z - z_2) \cdots (z - z_p)$$

If we differentiate this equation with respect to z and set $z = z_1$, we get

$$\left.\frac{df(z)}{dz}\right|_{z=z_1} = 0$$

This means that multiple roots of $f(z)$ will satisfy the following equation:

$$\frac{df(z)}{dz} = 0$$

or

$$\frac{df(z)}{dz} = A'(z) + KB'(z) = 0 \tag{4-73}$$

where

$$A'(z) = \frac{dA(z)}{dz}, \qquad B'(z) = \frac{dB(z)}{dz}$$

Solving Equation (4–73) for K, we obtain

$$K = -\frac{A'(z)}{B'(z)}$$

This particular value of K will yield multiple roots of the characteristic equation. If we substitute this value of K into Equation (4–72), we obtain

$$f(z) = A(z) - \frac{A'(z)}{B'(z)}B(z) = 0$$

or

$$B'(z)A(z) - A'(z)B(z) = 0 \tag{4-74}$$

If this last equation is solved for z, the points where multiple roots occur can be obtained. On the other hand, from Equation (4–72) we have

$$K = -\frac{A(z)}{B(z)}$$

and

$$\frac{dK}{dz} = -\frac{A'(z)B(z) - A(z)B'(z)}{B^2(z)}$$

If dK/dz is set equal to zero, we get the same equation as Equation (4–74). Therefore, the breakaway or break-in points can be simply determined from the roots of

$$\frac{dK}{dz} = 0$$

It should be noted that not all the solutions of Equation (4–74) or of $dK/dz = 0$ correspond to actual breakaway or break-in points. Such a point for which $dK/dz = 0$ is an actual breakaway or break-in point if and only if the value of K at this point is a real, positive value.

Problem A–4–6

Discuss the procedure for designing lead compensators for digital control systems by the root-locus method.

Solution We shall consider the system shown in Figure 4–44 to discuss the procedure for designing lead compensators. Lead compensation is useful when the system is either unstable for all values of gain or is stable but has undesirable transient response characteristics. To design lead compensators, we may use the following procedure:

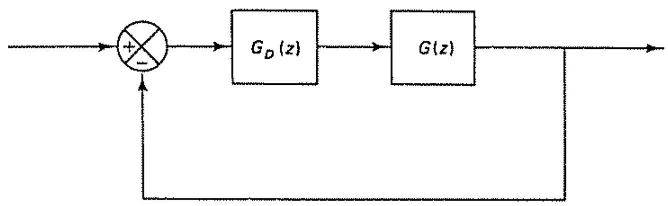


Figure 4–44 Digital control system.

1. From the performance specifications, determine the desired location for the dominant closed-loop poles.
2. By drawing the root-locus plot, ascertain whether or not the gain adjustment alone can yield the desired closed-loop poles If not, calculate the angle deficiency ϕ. This angle must be contributed by the lead compensator if the new root locus is to pass through the desired locations for the dominant closed-loop poles.
3. Assume the lead compensator $G_D(z)$ to be

$$G_D(z) = K_D \alpha \frac{1 + \alpha z}{1 + \alpha \tau z}, \qquad 0 < \alpha < 1$$

4. If static error constants are not specified, determine the location of the pole and zero of the lead compensator so that the lead compensator will contribute the necessary angle ϕ. If no other requirements are imposed on the system, try to make the value of α as large as possible. A larger value of α generally results in a larger value of K_v, which is desirable. (If a particular static error constant is specified, it is generally simpler to use the frequency-response approach.)
5. Determine the open-loop gain of the compensated system from the magnitude condition.

Once a compensator has been designed, check to see whether or not all performance specifications have been met. If the compensated system does not meet the performance specifications, then repeat the design procedure by adjusting the compensator pole and zero until all such specifications are met. If a large static error constant is required, cascade a lag network or alter the lead compensator to a lag–lead compensator.

Problem A–4–7

Draw root locus diagrams in the z plane for the system shown in Figure 4–45 for the following three sampling periods: $T = 1$ sec, $T = 2$ sec, and $T = 4$ sec.

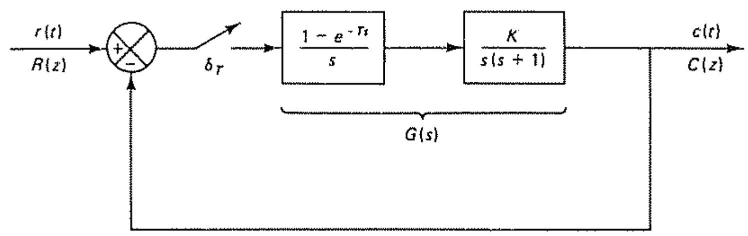


Figure 4–45 Digital control system.

Solution We first obtain the z transform of $G(s)$. Referring to Example 3–5, we get

$$G(z) = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s}\frac{K}{s(s+1)}\right]$$
$$= (1 - z^{-1})\mathcal{Z}\left[\frac{K}{s^2(s+1)}\right]$$
$$= \frac{K[(T - 1 + e^{-T})z^{-1} + (1 - e^{-T} - Te^{-T})z^{-2}]}{(1 - z^{-1})(1 - e^{-T}z^{-1})} \qquad (4\text{–}75)$$

Next we construct root locus diagrams for the three cases considered.

1. *Sampling period $T = 1$:* For $T = 1$, Equation (4–75) becomes

$$G(z) = \frac{K[(1 - 1 + e^{-1})z^{-1} + (1 - e^{-1} - e^{-1})z^{-2}]}{(1 - z^{-1})(1 - e^{-1}z^{-1})}$$
$$= \frac{0.3679K(z + 0.7181)}{(z - 1)(z - 0.3679)}$$

Notice that $G(z)$ possesses a zero at $z = -0.7181$ and poles at $z = 1$ and $z = 0.3679$. The breakaway point is at $z = 0.6479$, and the break-in point is at $z = -2.0841$. The root-locus diagram for this case is shown in Figure 4–46(a). The value of gain K of any point on the root loci can be determined from the magnitude condition

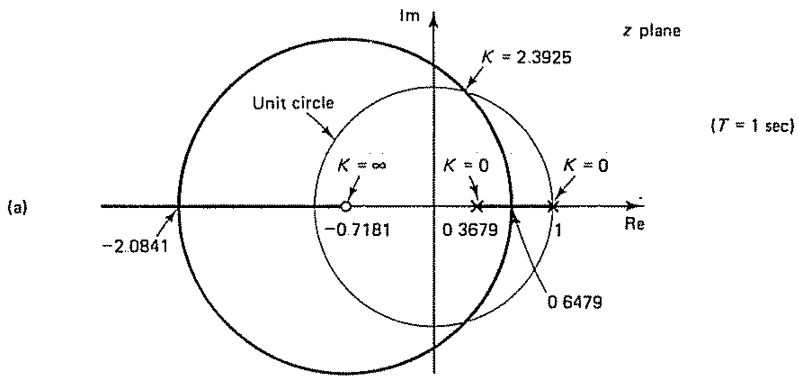
$$K = \left|\frac{(z - 1)(z - 0.3679)}{0.3679(z + 0.7181)}\right|$$

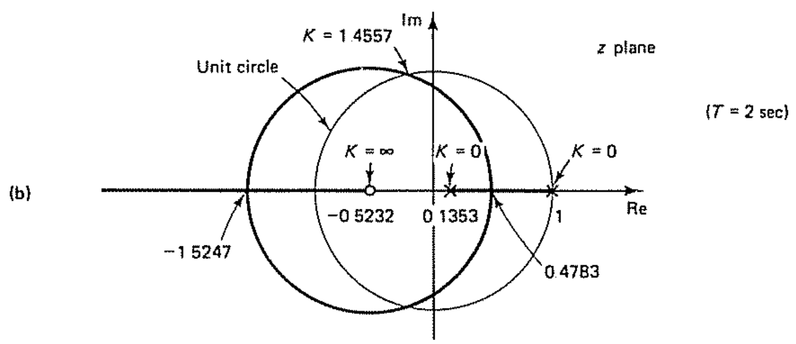
If we choose a point z on the root loci, the value of K at that point can be calculated by substituting the value of z into this last equation. (This means that with this value of K that particular point becomes a closed-loop pole.) The critical gain is found to be $K = 2.3925$.

2. *Sampling period $T = 2$:* For the sampling period $T = 2$, we have from Equation (4–75)

$$G(z) = \frac{1.1353K(z + 0.5232)}{(z - 1)(z - 0.1353)}$$

The pulse transfer function $G(z)$ in this case possesses a zero at $z = -0.5232$ and poles at $z = 1$ and $z = 0.1353$. The breakaway point is at $z = 0.4783$, and the break-in point is at $z = -1.5247$. The root-locus diagram for this case is shown in Figure 4–46(b). The critical gain K for stability is $K = 1.4557$.





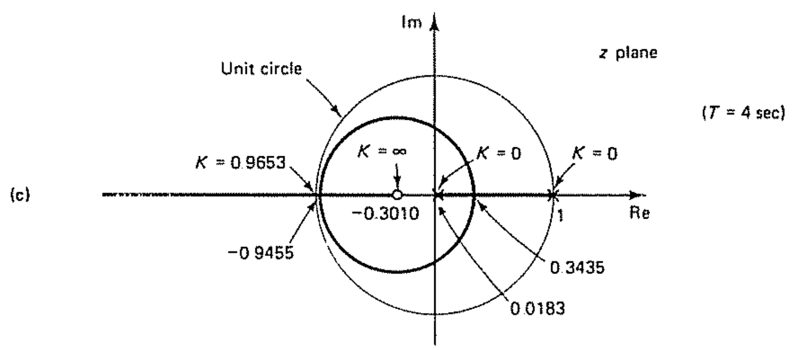


Figure 4–46 Root-locus diagrams for the system shown in Figure 4–45 when (a) $T = 1$ sec, (b) $T = 2$ sec, and (c) $T = 4$ sec

3. *Sampling period $T = 4$* For the case of $T = 4$, Equation (4–75) gives

$$G(z) = \frac{3.0183K(z + 0.3010)}{(z - 1)(z - 0.0183)}$$

The breakaway point is at $z = 0.3435$, and the break-in point is at $z = -0.9455$.

The root-locus diagram is shown in Figure 4–46(c). The critical gain for stability is $K = 0.9653$.

From the three cases considered, notice that the smaller the sampling period is, the larger the critical gain K for stability.

Problem A–4–8

Consider the digital control system shown in Figure 4–47, where the plant is of the first order and has a dead time of 2 sec. The sampling period is assumed to be 1 sec, or $T = 1$.

Design a digital PI controller such that the dominant closed-loop poles have a damping ratio ζ of 0.5 and the number of samples per cycle of damped sinusoidal oscillation is 10. Obtain the response of the system to a unit-step input. Also, obtain the static velocity error constant K_v and find the steady-state error in the response to a unit-ramp input.

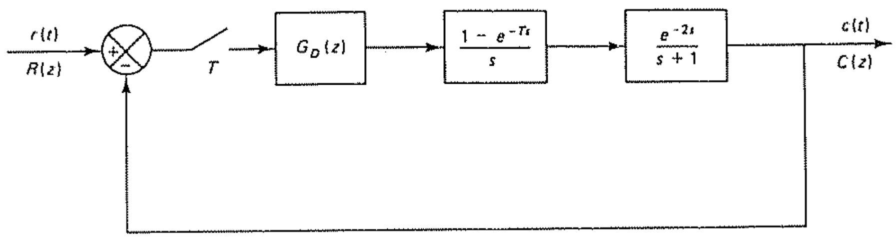


Figure 4–47 Digital control system

Solution The pulse transfer function of the plant that is preceded by a zero-order hold is

$$\begin{aligned} G(z) &= \mathcal{Z}\left[\frac{1-e^{-Ts}}{s}\frac{e^{-2s}}{s+1}\right] \\ &= (1-z^{-1})z^{-2}\mathcal{Z}\left[\frac{1}{s(s+1)}\right] \\ &= (1-z^{-1})z^{-2}\frac{(1-e^{-1})z^{-1}}{(1-z^{-1})(1-e^{-1}z^{-1})} \\ &= \frac{0.6321z^{-3}}{1-0.3679z^{-1}} = \frac{0.6321}{z^2(z-0.3679)} \end{aligned}$$

The digital PI controller has the following pulse transfer function:

$$\begin{aligned} G_D(z) &= K_p + K_i\frac{1}{1-z^{-1}} \\ &= (K_p+K_i)\frac{z - \dfrac{K_p}{k_p+K_i}}{z-1} \end{aligned}$$

The open-loop pulse transfer function becomes

$$G_D(z)G(z) = \frac{(K_p+K_i)\left(z-\dfrac{K_p}{K_p+K_i}\right)}{z-1}\frac{0.6321}{z^2(z-0.3679)}$$

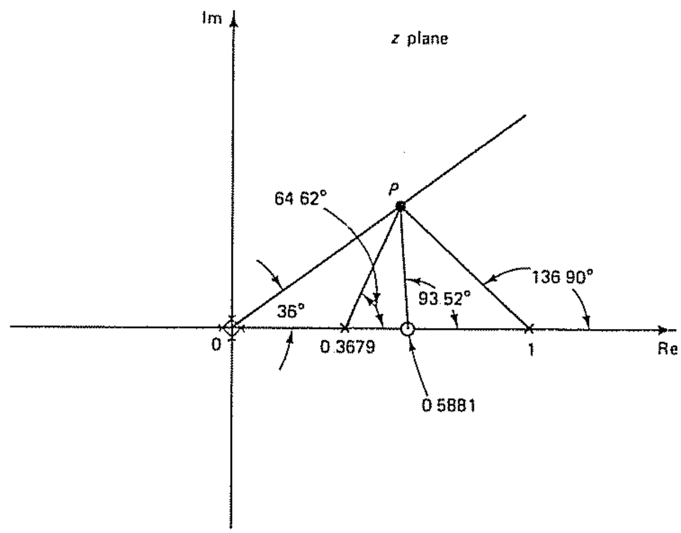


Figure 4–48 Pole and zero locations in the z plane of the system considered in Problem A–4–8.

We locate the open-loop poles in the z plane as shown in Figure 4–48. There is one open-loop zero involved in this case, but its location is unknown at this point.

Since it is required to have 10 samples per cycle of damped sinusoidal oscillation, the dominant closed-loop pole in the upper half of the z plane must lie on a line from the origin having an angle of $360°/10 = 36°$. From Equations (4–1) and (4–2), rewritten as

$$|z| = \exp\left(-\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\frac{\omega_d}{\omega_s}\right)$$

$$\angle z = 2\pi\frac{\omega_d}{\omega_s}$$

the desired closed-loop pole location can be determined as follows. Noting that $\angle z = 36°$, we have

$$2\pi\frac{\omega_d}{\omega_s} = \frac{2\pi}{10}$$

or $\omega_d/\omega_s = 0.1$. Since ζ is specified as 0.5, we have

$$|z| = \exp\left(-\frac{2\pi \times 0.5}{\sqrt{1-0.5^2}}\frac{1}{10}\right) = e^{-0.3628} = 0.6958$$

The closed-loop pole is located at point P in Figure 4–48, where (at point P)

$$z = 0.6958 \angle 36^\circ$$
$$= 0.5629 + j0.4090$$

(Note that this point is the intersection of the $\zeta = 0.5$ locus and the line from the origin having an angle of 36°.)

If point P is to be the closed-loop pole location in the upper half of the z plane, then the angle deficiency at point P is

$$-36^\circ - 36^\circ - 136.90^\circ - 64.62^\circ + 180^\circ = -93.52^\circ$$

The controller zero must contribute $+93.52^\circ$. This means that the zero of the digital controller must be located at $z = 0.5881$. Therefore,

$$\frac{K_p}{K_p + K_i} = 0.5881 \tag{4-76}$$

Hence, the PI controller is determined as follows:

$$G_D(z) = K\frac{z - 0.5881}{z - 1}$$

where $K = K_p + K_i$. Gain constant K is determined from the magnitude condition:

$$K\left|\frac{z - 0.5881}{z - 1}\,\frac{0.6321}{z^2(z - 0.3679)}\right|_{z=0.5629+j0.4090} = 1$$

or

$$K = 0.5070$$

Thus,

$$K_p + K_i = 0.5070 \tag{4-77}$$

From Equations (4–76) and (4–77), we find that

$$K_p = 0.2982 \qquad \text{and} \qquad K_i = 0.2088$$

Hence, the PI controller just designed can be given by

$$G_D(z) = 0.5070\,\frac{1 - 0.5881z^{-1}}{1 - z^{-1}}$$

Finally, the open-loop pulse transfer function becomes

$$G_D(z)G(z) = 0.5070\left(\frac{1 - 0.5881z^{-1}}{1 - z^{-1}}\,\frac{0.6321z^{-3}}{1 - 0.3679z^{-1}}\right)$$
$$= \frac{0.3205(1 - 0.5881z^{-1})z^{-3}}{(1 - z^{-1})(1 - 0.3679z^{-1})}$$

The closed-loop pulse transfer function becomes

$$\frac{C(z)}{R(z)} = \frac{0.3205z^{-3} - 0.1885z^{-4}}{1 - 1.3679z^{-1} + 0.3679z^{-2} + 0.3205z^{-3} - 0.1885z^{-4}}$$

The response $c(kT)$ to the unit-step input can be obtained easily by use of MATLAB. A MATLAB program for plotting the unit-step response is shown in MATLAB Program 4–4. The resulting plot is shown in Figure 4–49.

MATLAB Program 4–4

```
% ---------- Unit-step response ----------

num = [0  0  0  0.3205  -0.1885];
den = [1  -1.3679  0.3679  0.3205  -0.1885];
r = ones(1,51);
v = [0  50  0  1.6];
axis(v);
k = 0:50;
c = filter(num,den,r);
plot(k,c,'o')
grid
title('Unit-Step Response')
xlabel('k')
ylabel('c(k)')
```

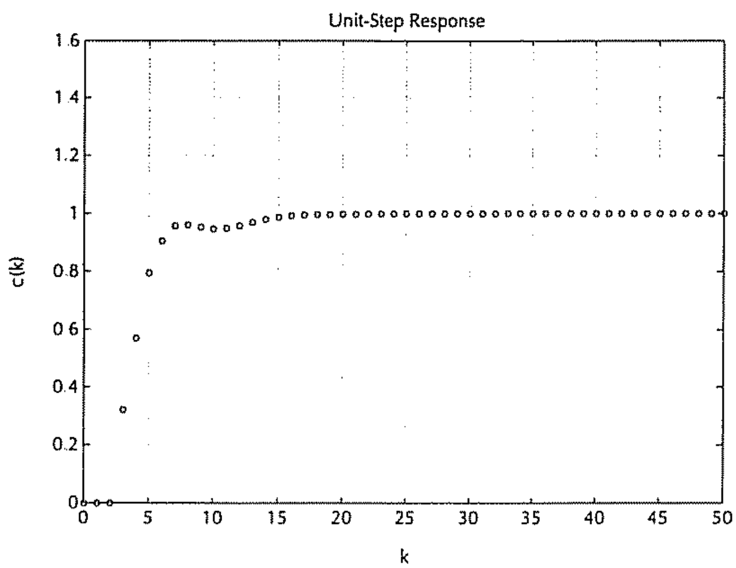


Figure 4–49 Plot of $c(kT)$ versus kT for the system designed in Problem A–4–8. (Sampling period $T = 1$ sec.)

Problem A–4–9

Consider the system shown in Figure 4–50. We wish to design a digital controller such that the dominant closed-loop poles of the system will have a damping ratio ζ of 0.5. We also want the number of samples per cycle of damped sinusoidal oscillation to be 8. Assume that the sampling period T is 0.2 sec.

Using the root-locus method in the z plane, determine the pulse transfer function of the digital controller. Obtain the response of the designed system to a unit-step input. Also obtain the static velocity error constant K_v.

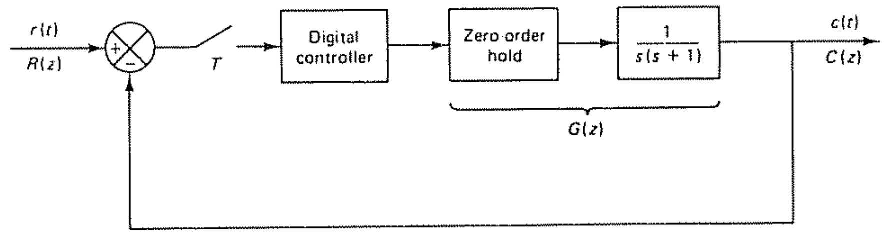


Figure 4–50 Digital control system

Solution We shall first locate the desired closed-loop poles in the z plane. Referring to Equations (4–1) and (4–2), for a constant-damping-ratio locus we have

$$|z| = e^{-\zeta T\omega_n} = \exp\left(-\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\frac{\omega_d}{\omega_s}\right) \tag{4–78}$$

and

$$\angle z = T\omega_d = 2\pi\frac{\omega_d}{\omega_s} = \theta$$

Since we require eight samples per cycle of damped sinusoidal oscillation, the dominant closed-loop pole in the upper half of the z plane must be located on a line having an angle of 45° and passing through the origin as shown in Figure 4–51. (Note that the number of samples per cycle is $360°/\theta$. Hence, eight samples per cycle requires $\theta = 360°/8 = 45°$.) Thus,

$$\angle z = 45° = \frac{\pi}{4} = 2\pi\frac{\omega_d}{\omega_s}$$

which gives

$$\frac{\omega_d}{\omega_s} = \frac{1}{8} \tag{4–79}$$

Since the sampling period T is specified as 0.2 sec, we have

$$\omega_s = \frac{2\pi}{T} = \frac{2\pi}{0.2} = 10\pi$$

Therefore,

$$\omega_d = \frac{1}{8}\omega_s = \frac{10\pi}{8} = 3.9270$$

By letting $\zeta = 0.5$ and substituting Equation (4–79) into Equation (4–78), we obtain

$$|z| = e^{-0.4535} = 0.6354$$

Hence, we can locate the desired closed-loop pole in the upper half of the z plane, as shown by point P in Figure 4–51. Note that at point P

$$|z|\angle z = 0.6354\angle 45° = 0.4493 + j0.4493$$

Next, we obtain the pulse transfer function $G(z)$ of the plant that is preceded by a zero-order hold:

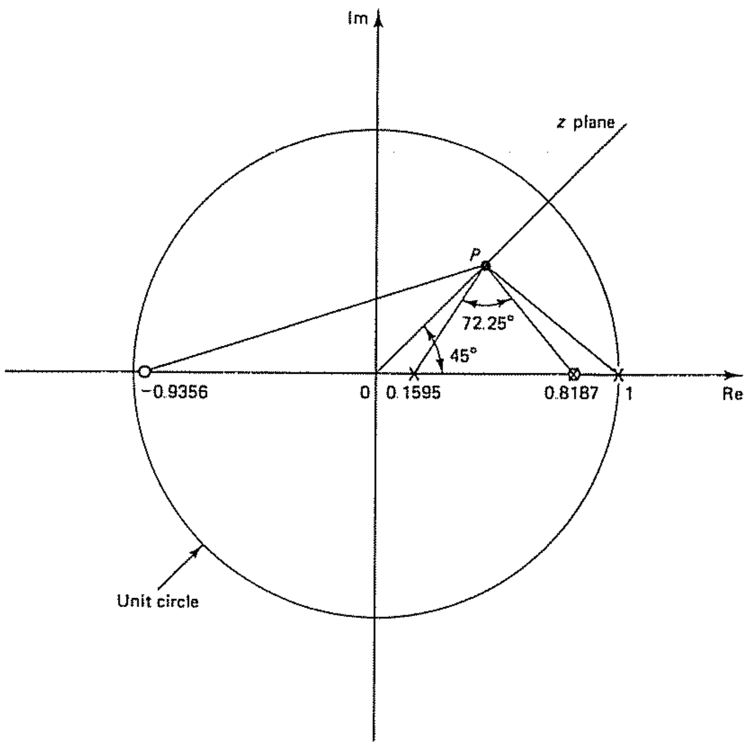


Figure 4–51 Pole and zero locations for the system considered in Problem A–4–9.

$$
\begin{aligned}
G(z) &= \mathcal{Z}\left[\frac{1-e^{-Ts}}{s}\frac{1}{s(s+1)}\right] \\
&= (1-z^{-1})\,\mathcal{Z}\left[\frac{1}{s^2(s+1)}\right] \\
&= (1-z^{-1})\left[\frac{0.2z^{-1}}{(1-z^{-1})^2} - \frac{1}{1-z^{-1}} + \frac{1}{1-e^{-0.2}z^{-1}}\right] \\
&= \frac{0.01873(1+0.9356z^{-1})z^{-1}}{(1-z^{-1})(1-0.8187z^{-1})} = \frac{0.01873(z+0.9356)}{(z-1)(z-0.8187)}
\end{aligned}
$$

We can now locate the open-loop poles and a zero on the z plane as shown in Figure 4–51. Since point P is the location of the desired closed-loop pole, the angle deficiency at point P can be calculated easily as follows:

$$-140.79^\circ - 129.43^\circ + 17.97^\circ + 180^\circ = -72.25^\circ$$

The controller pulse transfer function must contribute 72.25°

Let us choose the controller pulse transfer function to be

$$G_D(z) = K\frac{z+\alpha}{z+\beta}$$

and choose the zero of the controller to cancel the pole at $z = 0.8187$. Then the pole of the controller can be determined easily from the angle condition as $z = 0.1595$. Thus, we have

$$G_D(z) = K\frac{1 - 0.8187z^{-1}}{1 - 0.1595z^{-1}}$$

The open-loop pulse transfer function of the system is therefore obtained as follows:

$$\begin{aligned} G_D(z)G(z) &= K\frac{1 - 0.8187z^{-1}}{1 - 0.1595z^{-1}}\frac{0.01873(1 + 0.9356z^{-1})z^{-1}}{(1 - z^{-1})(1 - 0.8187z^{-1})} \\ &= K\frac{0.01873(1 + 0.9356z^{-1})z^{-1}}{(1 - 0.1595z^{-1})(1 - z^{-1})} \end{aligned}$$

The gain constant K can be determined from the magnitude condition:

$$K\left|\frac{0.01873(z + 0.9356)}{(z - 0.1595)(z - 1)}\right|_{z=0.4493+j0.4493} = 1$$

or

$$K = 13.934$$

Hence, we have determined the pulse transfer function of the digital controller to be

$$G_D(z) = 13.934\left(\frac{1 - 0.8187z^{-1}}{1 - 0.1595z^{-1}}\right)$$

The open-loop pulse transfer function is

$$G_D(z)G(z) = \frac{0.2610(1 + 0.9356z^{-1})z^{-1}}{(1 - 0.1595z^{-1})(1 - z^{-1})}$$

The closed-loop pulse transfer function is

$$\begin{aligned} \frac{C(z)}{R(z)} &= \frac{G_D(z)G(z)}{1 + G_D(z)G(z)} \\ &= \frac{0.2610z^{-1} + 0.2442z^{-2}}{1 - 0.8985z^{-1} + 0.4037z^{-2}} \end{aligned}$$

Because of the cancellation of a pole of the plant and the zero of the controller, the order of the system is reduced from third to second. The system has only a pair of conjugate complex closed-loop poles.

Figure 4–52 shows the unit-step response sequence $c(kT)$ versus kT. The plot shows the maximum overshoot to be approximately 16.5%.

Finally, the static velocity error constant K_v is determined as follows:

$$\begin{aligned} K_v &= \lim_{z\to 1}\left[\frac{1 - z^{-1}}{T}G_D(z)G(z)\right] \\ &= \lim_{z\to 1}\left[\frac{1 - z^{-1}}{0.2}\frac{0.2610(1 + 0.9356z^{-1})z^{-1}}{(1 - 0.1595z^{-1})(1 - z^{-1})}\right] \\ &= 3.005 \end{aligned}$$

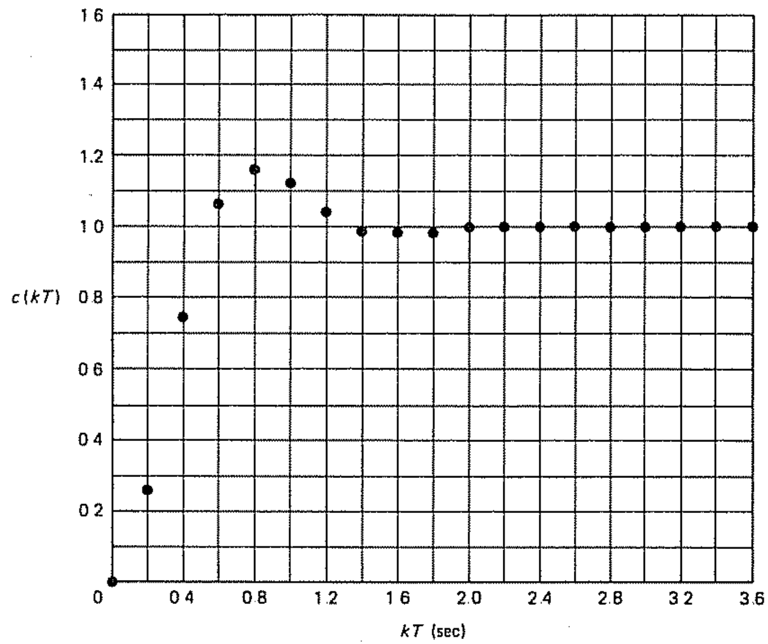


Figure 4–52 Plot of the unit-step response sequence $c(kT)$ versus kT for the system designed in Problem A–4–9.

Problem A–4–10

Consider the system shown in Figure 4–53. Assume that the performance specifications are given in terms of phase margin, gain margin, static velocity error constants, and the like. State procedures for designing lead compensators and lag compensators by the frequency-response approach.

Solution The procedures for designing lead compensators and lag compensators may be stated as follows:

LEAD COMPENSATOR

1. Assume the following form for the lead compensator:

$$G_D(w) = K_D \frac{1 + \tau w}{1 + \alpha \tau w}, \qquad 0 < \alpha < 1$$

The open-loop transfer function of the compensated system may be written as

$$G_D(w)G(w) = K_D \frac{1 + \tau w}{1 + \alpha \tau w} G(w)$$

$$= \frac{1 + \tau w}{1 + \alpha \tau w} G_1(w)$$

where $G_1(w) = K_D G(w)$ Determine gain K_D to satisfy the requirement on the given static velocity error constant.

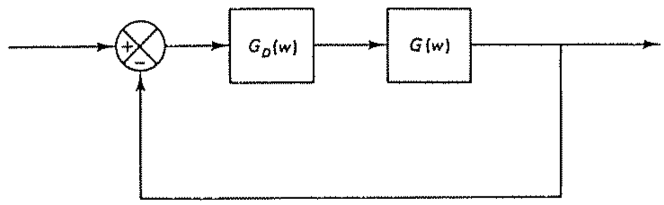


Figure 4–53 Digital control system in the w plane

2. Using the gain K_D thus determined, draw a Bode diagram of $G_1(w)$, the gain-adjusted but uncompensated system. Evaluate the phase margin.
3. Determine the necessary phase lead angle ϕ to be added to the system.
4. Add $5° \sim 12°$ to ϕ to compensate for the shift of the gain crossover frequency. Define this added ϕ as ϕ_m. Determine the attenuation factor α from the following equation:

$$\sin \phi_m = \frac{1 - \alpha}{1 + \alpha}$$

5. Determine the frequency point where the magnitude of the uncompensated system $G_1(j\nu)$ is equal to $-20 \log (1/\sqrt{\alpha})$. Select this frequency as the new gain crossover frequency. This frequency corresponds to $\nu_m = 1/(\sqrt{\alpha}\tau)$, and the maximum phase shift ϕ_m occurs at this frequency.
6. Determine the corner frequencies of the lead compensator as follows:

$$\text{Zero of lead compensator:} \quad \nu = \frac{1}{\tau}$$

$$\text{Pole of lead compensator:} \quad \nu = \frac{1}{\alpha\tau}$$

7. Check the gain margin to be sure it is satisfactory. If not, repeat the design process by modifying the pole–zero location of the compensator until a satisfactory result is obtained.

The primary function of a lag compensator is to provide attenuation in the high-frequency range to give a system sufficient phase margin. The phase lag characteristic is of no consequence in lag compensation.

LAG COMPENSATOR

1. Assume the following form for the lag compensator:

$$G_D(w) = K_D \frac{1 + \tau w}{1 + \beta\tau w} \qquad (\beta > 1)$$

The open-loop transfer function of the compensated system may be written as

$$G_D(w)G(w) = K_D \frac{1 + \tau w}{1 + \beta\tau w} G(w)$$

$$= \frac{1 + \tau w}{1 + \beta\tau w} G_1(w)$$

where $G_1(w) = K_D G(w)$. Determine gain K_D to satisfy the requirement on the given static velocity error constant.

2. If the uncompensated system $G_1(w)$ does not satisfy the specifications on the phase and gain margins, then find the frequency point where the phase angle of the open-loop transfer function is equal to $-180°$ plus the required phase margin. The required phase margin is the specified phase margin plus 5° to 12°. (The addition of 5° to 12° compensates for the phase lag of the lag compensator.) Choose this frequency as the new gain crossover frequency.
3. To prevent detrimental effects of phase lag due to the lag compensator, the pole and zero of the lag compensator must be located substantially lower than the new gain crossover frequency. Therefore, choose the corner frequency $\nu = 1/\tau$ (corresponding to the zero of the lag compensator) one decade below the new gain crossover frequency.
4. Determine the attenuation necessary to bring the magnitude curve down to 0 dB at the new gain crossover frequency. Noting that this attenuation is $-20 \log \beta$, determine the value of β. Then the other corner frequency (corresponding to the pole of the lag compensator) is determined from $\nu = 1/(\beta\tau)$.

Caution. Once the lag compensator is designed in the w plane, $G_D(w)$ must be transformed to the z plane lag compensator, $G_D(z)$. Note that the locations of the pole and zero of the lag compensator in the z plane are close to each other. (They are near point $z = 1$.) Since the filter coefficients must be realized by binary words that use limited number of bits, if the number of bits employed is insufficient, the pole and zero locations of the filter may not be realized exactly as desired, and the resulting compensator may not behave as expected. It is important that the pole and zero of the lag compensator lie on a finite number of allocable discrete points.

Problem A–4–11

Design a digital controller for the system shown in Figure 4–54. Use the Bode diagram approach in the w plane. The design specifications are that the phase margin be 55°, the gain margin be at least 10 dB, and the static velocity error constant be 5 sec^{-1}. The sampling period is specified as 0.1 sec, or $T = 0.1$. After the controller is designed, draw a root-locus diagram. Locate the closed-loop poles on the diagram and find the number of samples per cycle of damped sinusoidal oscillation.

Solution The z transform of the plant that is preceded by a zero-order hold is

$$\begin{aligned} G(z) &= \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s}\,\frac{1}{s(s + 2)}\right] \\ &= (1 - z^{-1})\mathcal{Z}\left[\frac{1}{s^2(s + 2)}\right] \\ &= 0.004683z^{-1}\frac{1 + 0.9355z^{-1}}{(1 - z^{-1})(1 - 0.8187z^{-1})} \\ &= (0.004683)\frac{z + 0.9355}{(z - 1)(z - 0.8187)} \end{aligned}$$

Let us transform $G(z)$ into $G(w)$ by using the following bilinear transformation:

$$z = \frac{1 + (Tw/2)}{1 - (Tw/2)} = \frac{1 + 0.05w}{1 - 0.05w}$$

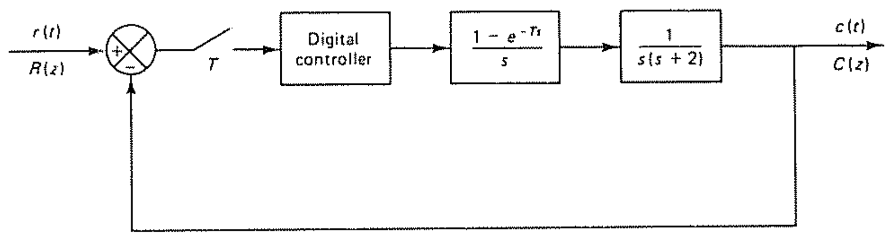


Figure 4–54 Digital control system.

By substituting this last equation into $G(z)$, we obtain

$$G(w) = \frac{0.004683\left(\dfrac{1 + 0.05w}{1 - 0.05w} + 0.9355\right)}{\left(\dfrac{1 + 0.05w}{1 - 0.05w} - 1\right)\left(\dfrac{1 + 0.05w}{1 - 0.05w} - 0.8187\right)}$$

$$= \frac{0.5(1 + 0.001666w)(1 - 0.05w)}{w(1 + 0.5016w)}$$

The Bode diagram of $G(j\nu)$ is shown in Figure 4–55.

We shall now choose the controller transfer function to be of the form

$$G_D(w) = K_D\frac{1 + \tau w}{1 + \alpha\tau w} = K_D\frac{1 + \dfrac{w}{a}}{1 + \dfrac{w}{b}}$$

where $a = 1/\tau$ and $b = 1/(\alpha\tau)$. The open-loop transfer function is

$$G_D(w)G(w) = K_D\frac{1 + (w/a)}{1 + (w/b)}\,\frac{0.5(1 + 0.001666w)(1 - 0.05w)}{w(1 + 0.5016w)}$$

The required static velocity error constant K_v is 5 $\sec^{-1}$. Hence,

$$K_v = \lim_{w\to 0}[wG_D(w)G(w)] = 0.5K_D = 5$$

from which we determine that

$$K_D = 10$$

Using a conventional design technique, the digital controller transfer function is determined as

$$G_D(w) = 10\left(\frac{1 + \dfrac{w}{1.994}}{1 + \dfrac{w}{12.5}}\right)$$

Therefore, the open-loop transfer function becomes

$$G_D(w)G(w) = 10\left(\frac{1 + \dfrac{w}{1.994}}{1 + \dfrac{w}{12.5}}\right)\frac{0.5(1 + 0.001666w)(1 - 0.05w)}{w(1 + 0.5016w)}$$

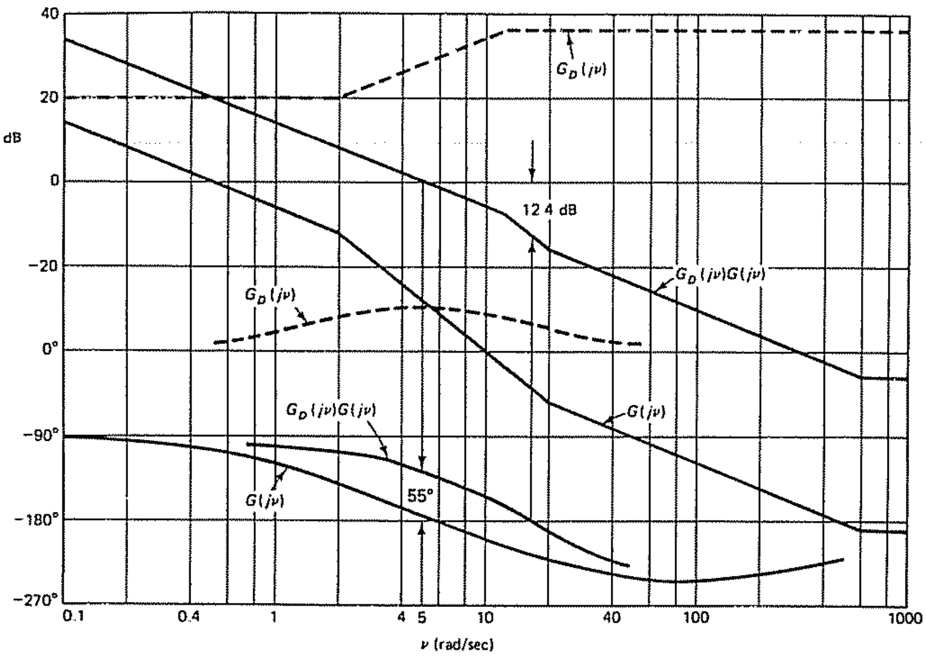


Figure 4–55 Bode diagram for the system considered in Problem A–4–11.

This open-loop transfer function gives the phase margin of approximately 55° and the gain margin of approximately 12.4 dB. The static velocity error constant K_v is 5 $\sec^{-1}$. Hence, all requirements are satisfied and the designed controller transfer function $G_D(w)$ is satisfactory.

Next, we transform $G_D(w)$ into $G_D(z)$. The following bilinear transformation should be used:

$$w = \frac{2}{T}\frac{z-1}{z+1} = \frac{2}{0.1}\frac{z-1}{z+1} = 20\left(\frac{z-1}{z+1}\right)$$

Then

$$G_D(z) = 10\left(\frac{1 + \dfrac{20}{1.994}\dfrac{z-1}{z+1}}{1 + \dfrac{20}{12.5}\dfrac{z-1}{z+1}}\right)$$

$$= 42.423\left(\frac{z - 0.8187}{z - 0.2308}\right) = 42.423\left(\frac{1 - 0.8187z^{-1}}{1 - 0.2308z^{-1}}\right)$$

The open-loop pulse transfer function now becomes

$$G_D(z)G(z) = \frac{0.1987(z + 0.9355)}{(z - 0.2308)(z - 1)}$$

Figure 4–56 shows the root-locus diagram for the system. Using the magnitude condition, we find that the closed-loop poles are located at $z = 0.516 \pm j0.388$. On

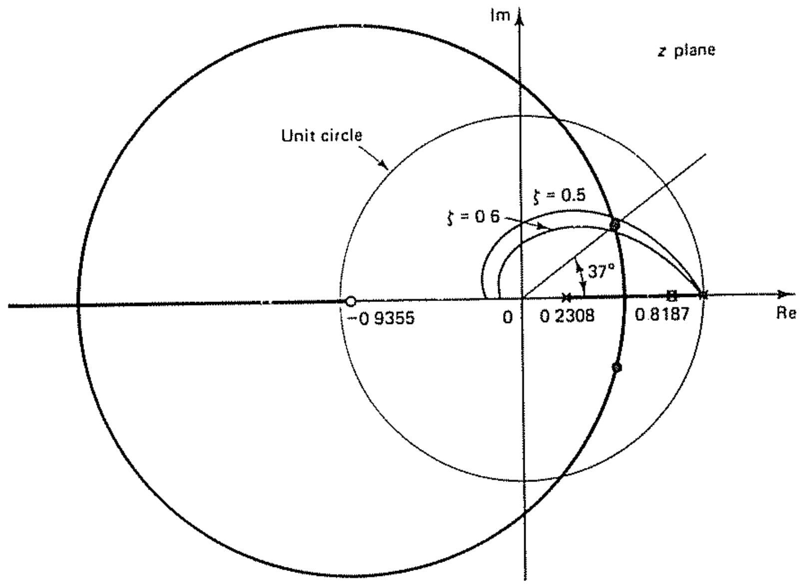


Figure 4–56 Root-locus diagram for the system designed in Problem A–4–11.

the root-locus diagram, constant ζ loci for $\zeta = 0.5$ and 0.6 are superimposed. From the diagram it can be seen that the damping ratio ζ of the closed-loop poles is approximately 0.55

The line connecting the closed-loop pole in the upper half of the z plane and the origin has an angle of 37°. Hence, the number of samples per cycle of damped sinusoidal oscillation is $360°/37° = 9.73$.

Problem A–4–12

Consider the digital control system shown in Figure 4–57, where the plant transfer function is $1/s^2$. Design a digital controller in the w plane such that the phase margin is 50° and the gain margin is at least 10 dB. The sampling period is 0.1 sec, or $T = 0.1$. After designing the controller, obtain the static velocity error constant K_v. Also, obtain the response of the designed system to a unit-step input.

Solution We shall first obtain the z transform of the plant that is preceded by the zero-order hold:

$$G(z) = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s}\frac{1}{s^2}\right] = (1 - z^{-1})\mathcal{Z}\left[\frac{1}{s^3}\right]$$

$$= (1 - z^{-1})\frac{T^2(1 + z^{-1})z^{-1}}{2(1 - z^{-1})^3}$$

$$= \frac{0.005(1 + z^{-1})z^{-1}}{(1 - z^{-1})^2} = \frac{0.005(z + 1)}{(z - 1)^2}$$

Next, using the bilinear transformation given by

$$z = \frac{1 + (Tw/2)}{1 - (Tw/2)} = \frac{1 + 0.05w}{1 - 0.05w}$$

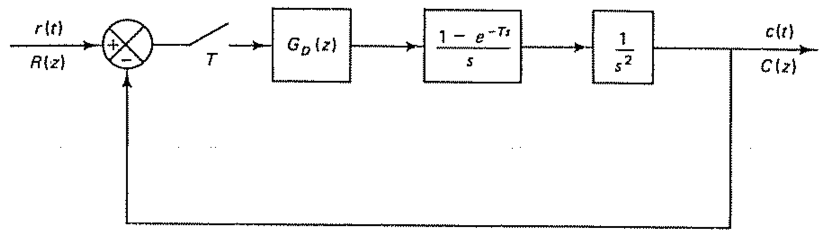


Figure 4–57 Digital control system.

we transform $G(z)$ into $G(w)$:

$$G(w) = \frac{0.005\left(\dfrac{1 + 0.005w}{1 - 0.05w} + 1\right)}{\left(\dfrac{1 + 0.05w}{1 - 0.05w} - 1\right)^2} = \frac{1 - 0.05w}{w^2}$$

Thus,

$$G(j\nu) = \frac{1 - 0.05j\nu}{(j\nu)^2}$$

Figure 4–58 shows the Bode diagram of $G(j\nu)$ thus obtained. Notice that the phase margin is $-2°$. It is necessary to add a lead network to give the required phase margin

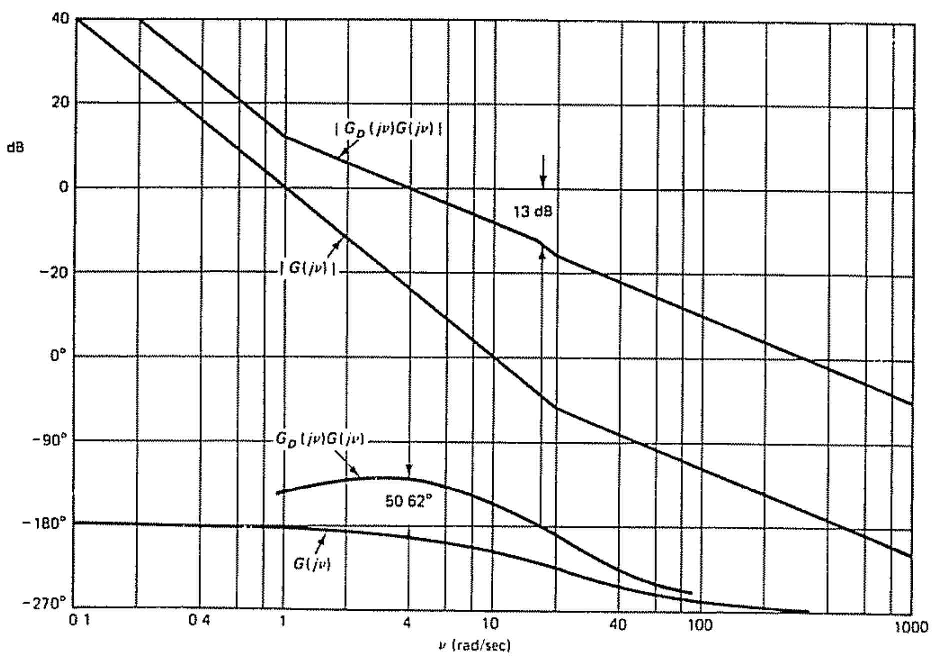


Figure 4–58 Bode diagram for the system considered in Problem A-4-12.

and gain margin. By applying a conventional design technique, it can be seen that the following lead network will satisfy the requirements:

$$G_D(w) = 4\frac{1 + w}{1 + (w/16)} = 64\left(\frac{w + 1}{w + 16}\right)$$

The addition of this lead network modifies the Bode diagram. The gain crossover frequency is shifted to $\nu = 4$. Note that the maximum phase lead ϕ_m that this lead network can produce is 61.93°, since

$$\phi_m = \sin^{-1}\frac{1 - \frac{1}{16}}{1 + \frac{1}{16}} = \sin^{-1} 0.8824 = 61.93^\circ$$

At the gain crossover frequency $\nu = 4$, the phase angle of $G_D(j\nu)G(j\nu)$ becomes $-191.31^\circ + 61.93^\circ = -129.38^\circ$. Thus, the phase margin is 50.62°. The gain margin is found to be approximately 13 dB. Hence, the given design specifications are satisfied.

We now transform the controller transfer function $G_D(w)$ into $G_D(z)$. By using the bilinear transformation

$$w = \frac{2}{T}\frac{z - 1}{z + 1} = \frac{2}{0.1}\frac{z - 1}{z + 1} = 20\left(\frac{z - 1}{z + 1}\right)$$

we obtain

$$G_D(z) = 64\frac{20\left(\frac{z - 1}{z + 1}\right) + 1}{20\left(\frac{z - 1}{z + 1}\right) + 16} = 37.333\left(\frac{z - 0.9048}{z - 0.1111}\right)$$

Hence, the open-loop pulse transfer function becomes

$$G_D(z)G(z) = 37.333\left(\frac{z - 0.9048}{z - 0.1111}\right)\frac{0.005(z + 1)}{(z - 1)^2}$$
$$= \frac{0.1867(1 - 0.9048z^{-1})(1 + z^{-1})z^{-1}}{(1 - 0.1111z^{-1})(1 - z^{-1})^2}$$

The static velocity error constant K_v is obtained as follows:

$$K_v = \lim_{z \to 1}\left[\frac{1 - z^{-1}}{T} G_D(z)G(z)\right]$$
$$= \lim_{z \to 1}\left[\frac{1 - z^{-1}}{0.1}\,\frac{0.1867(1 - 0.9048z^{-1})(1 + z^{-1})z^{-1}}{(1 - 0.1111z^{-1})(1 - z^{-1})^2}\right] = \infty$$

Thus, the static velocity error constant K_v is infinity. There is no steady-state error in the ramp response.

The closed-loop pulse transfer function of the system is

$$\frac{C(z)}{R(z)} = \frac{0.1867z^{-1} + 0.0178z^{-2} - 0.1689z^{-3}}{1 - 1.9244z^{-1} + 1.2400z^{-2} - 0.2800z^{-3}}$$

Figure 4–59 shows the unit-step response. Notice that the zero of the digital controller at $z = 0.9048$ is close to the double pole at $z = 1$. A pole–zero pair near point $z = 1$ creates a long tail with small amplitude in the response.

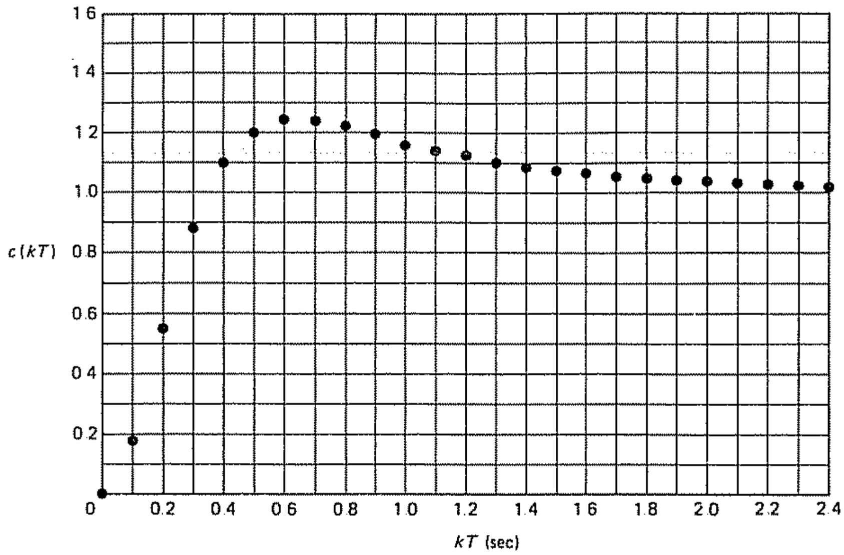


Figure 4-59 Plot of $c(kT)$ versus kT for the system designed in Problem A-4-12.

Problem A-4-13

Consider the digital control system shown in Figure 4-60. The plant transfer function involves a transportation lag e^{-5s}. The delay time is 5 sec, or $L = 5$. The desired output $c(t)$ in response to a unit-step input is as shown in Figure 4-61(a). The curve rises from zero to the final value in 10 sec (measured from $t = 5$ to $t = 15$) and there is neither overshoot nor steady-state error. The settling time is 15 sec (measured from $t = 0$ to $t = 15$). It is required that there be no intersampling ripples in the output after the settling time is reached. Design a digital controller $G_D(z)$.

Solution Let us choose the sampling period to be 5 sec, or $T = 5$ sec. (We may, of course, choose the sampling period to be 2.5 sec, 1 sec, or another value. In this example, however, to simplify our presentation, we set the sampling period at 5 sec.)

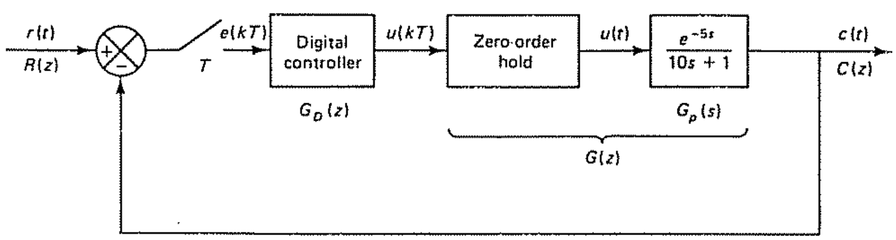


Figure 4-60 Digital control system

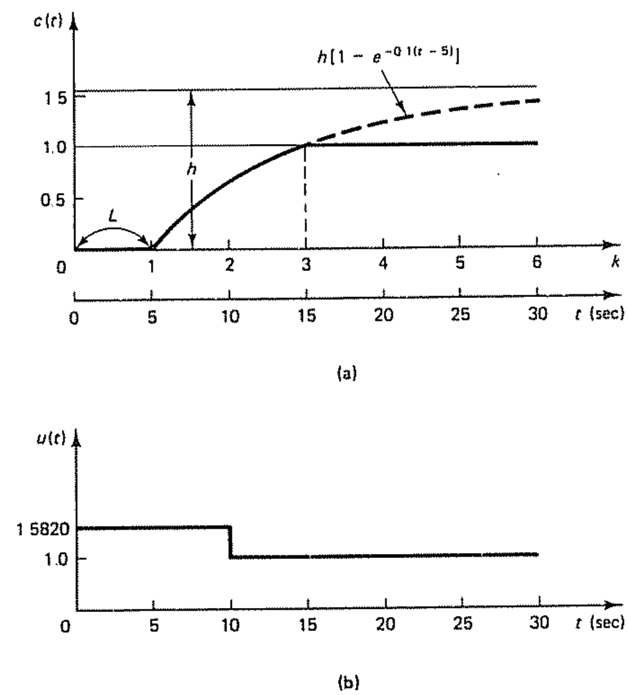


Figure 4–61 (a) Desired output $c(t)$ in response to a unit-step input; (b) plot of $u(t)$ versus t.

The z transform of the plant that is preceded by the zero-order hold is

$$G(z) = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s}\frac{e^{-5s}}{10s + 1}\right]$$

$$= (1 - z^{-1})z^{-1}\mathcal{Z}\left[\frac{1}{s(10s + 1)}\right]$$

$$= \frac{0.3935z^{-2}}{1 - 0.6065z^{-1}}$$

Notice that there is no unstable or critically stable pole involved in $G(z)$. Therefore, there is no stability problem involved in this case.

Let us define the closed-loop pulse transfer function as $F(z)$:

$$\frac{C(z)}{R(z)} = \frac{G_D(z)G(z)}{1 + G_D(z)G(z)} = F(z) \tag{4–80}$$

In the present case the output $c(t)$ in the unit-step response is specified as shown in Figure 4–61(a). Since $h[1 - e^{-0.1(15-5)}] = h(1 - e^{-1}) = 1$, we have $h = 1.5820$. from the deadbeat response curve shown in Figure 4–61(a), we obtain

$$c(0) = 0$$

$$c(1) = 0$$

$$c(2) = h(1 - e^{-0.5}) = 1.5820 \times 0.3935 = 0.6225$$
$$c(k) = 1, \qquad k = 3, 4, 5, \ldots$$

from which we get

$$\begin{aligned} C(z) &= 0.6225z^{-2} + z^{-3} + z^{-4} + z^{-5} + \cdots \\ &= 0.6225z^{-2} + z^{-3}\frac{1}{1 - z^{-1}} \\ &= \frac{0.6225z^{-2} + 0.3775z^{-3}}{1 - z^{-1}} \end{aligned}$$

Noting that

$$C(z) = F(z)R(z) = F(z)\frac{1}{1 - z^{-1}} = \frac{0.6225z^{-2} + 0.3775z^{-3}}{1 - z^{-1}}$$

we obtain

$$F(z) = 0.6225z^{-2} + 0.3775z^{-3} = 0.6225(1 + 0.6065z^{-1})z^{-2}$$

Once $F(z)$ is determined, the pulse transfer function of the digital controller can be obtained from Equation (4–80):

$$G_D(z) = \frac{F(z)}{G(z)[1 - F(z)]}$$

Notice that from Equation (4–48) we have

$$1 - F(z) = (1 - z^{-1})N(z)$$

or

$$1 - 0.6225z^{-2} - 0.3775z^{-3} = (1 - z^{-1})N(z)$$

By dividing $(1 - 0.6225z^{-2} - 0.3775z^{-3})$ by $(1 - z^{-1})$, $N(z)$ can be determined as follows:

$$N(z) = 1 + z^{-1} + 0.3775z^{-2}$$

Consequently,

$$1 - F(z) = (1 - z^{-1})(1 + z^{-1} + 0.3775z^{-2})$$

and

$$\begin{aligned} G_D(z) &= \frac{0.6225(1 + 0.6065z^{-1})z^{-2}}{\dfrac{0.3935z^{-2}}{1 - 0.6065z^{-1}}(1 - z^{-1})(1 + z^{-1} + 0.3775z^{-2})} \\ &= \frac{1.5820(1 - 0.3678z^{-2})}{(1 - z^{-1})(1 + z^{-1} + 0.3775z^{-2})} \end{aligned}$$

This last equation gives the pulse transfer function of the digital controller. Since $c(t)$ must be unity at steady state, $u(t)$, a continuous-time signal, must be constant after the steady state is reached.

Let us determine $U(z)$:

$$\begin{aligned} U(z) &= \frac{C(z)}{G(z)} = \frac{0.6225z^{-2} + 0.3775z^{-3}}{(1 - z^{-1})\dfrac{0.3935z^{-2}}{1 - 0.6065z^{-1}}} = 1.5820\left(\frac{1 - 0.3678z^{-2}}{1 - z^{-1}}\right) \\ &= 1.5820 + 1.5820z^{-1} + z^{-2} + z^{-3} + z^{-4} + \cdots \end{aligned}$$

Taking the inverse z transform of $U(z)$, we find that $u(k)$ is constant for $k \geq 2$. Thus, there are no intersampling ripples in the output after the settling time is reached. The signal $u(t)$ versus t is plotted in Figure 4–61(b).

Problem A–4–14

Consider the digital control system shown in Figure 4–62. Design a digital controller $G_D(z)$ such that the closed-loop system will exhibit the minimum settling time with zero steady-state error in a unit-ramp response. The system should not exhibit intersampling ripples at steady state. The sampling period T is assumed to be 1 sec. After the controller is designed, investigate the response of the system to a Kronecker delta input and a unit-step input.

Solution The first step in the design is to determine the z transform of the plant that is preceded by the zero-order hold:

$$G(z) = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s}\frac{1}{s^2}\right] = (1 - z^{-1})\mathcal{Z}\left[\frac{1}{s^3}\right]$$
$$= \frac{(1 + z^{-1})z^{-1}}{2(1 - z^{-1})^2}$$

Now define the closed-loop pulse transfer function as $F(z)$:

$$\frac{C(z)}{R(z)} = \frac{G_D(z)G(z)}{1 + G_D(z)G(z)} = F(z)$$

Notice that if $G(z)$ is expanded into a series in z^{-1} then the first term will be $0.5z^{-1}$. Hence, $F(z)$ must begin with a term in z^{-1}:

$$F(z) = a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}$$

where $N \geq n$ and n is the order of the system. Since the system here is of the second order, $n = 2$.

Since the input is a unit ramp, from Equation (4–48) we require that

$$1 - F(z) = (1 - z^{-1})^2 N(z) \tag{4–81}$$

Notice that $G(z)$ has a critically stable double pole at $z = 1$. Therefore, from the stability requirement, $1 - F(z)$ must have a double zero at $z = 1$. However, the function $1 - F(z)$ already involves a term $(1 - z^{-1})^2$, and therefore it satisfies the stability requirement.

Since the system should not exhibit intersampling ripples at steady state, we require $U(z)$ to be of the following type of series in z^{-1}:

$$U(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_{N-1} z^{-N+1} + b(z^{-N} + z^{-N-1} + z^{-N-2} + \cdots)$$

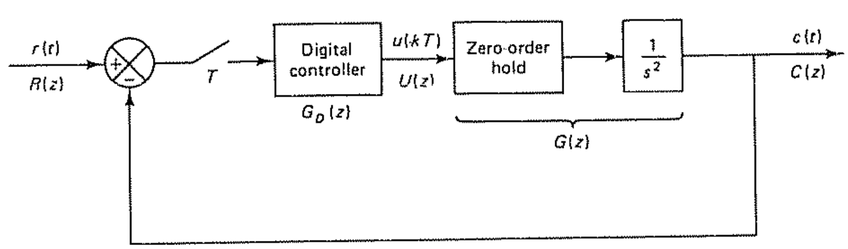


Figure 4–62 Digital control system

Because the plant transfer function $G_p(s)$ involves a double integrator, b must be zero. (Otherwise, the output increases parabolically, instead of linearly.) Consequently, we have

$$U(z) = b_0 + b_1 z^{-1} + \cdots + b_{N-1} z^{-N+1}$$

From Figure 4–62 $U(z)$ can be given by

$$U(z) = \frac{C(z)}{G(z)} = \frac{C(z)}{R(z)}\frac{R(z)}{G(z)} = F(z)\frac{R(z)}{G(z)}$$
$$= F(z)\frac{z^{-1}}{(1 - z^{-1})^2}\frac{2(1 - z^{-1})^2}{(1 + z^{-1})z^{-1}}$$
$$= F(z)\frac{2}{1 + z^{-1}}$$

For $U(z)$ to be a series in z^{-1} with a finite number of terms, $F(z)$ must be divisible by $1 + z^{-1}$:

$$F(z) = (1 + z^{-1})F_1(z) \tag{4–82}$$

Then $U(z)$ can be written as follows:

$$U(z) = 2F_1(z) \tag{4–83}$$

where $F_1(z)$ is a polynomial in z^{-1} with a finite number of terms.

By comparing Equations (4–81) and (4–82) and by making a simple analysis, we see that $F(z)$ must involve a term with at least z^{-3}. Hence, we assume

$$F(z) = a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}$$

This assumed form of $F(z)$ involves the minimum number of terms; the transient response will settle in three sampling periods.

We shall now determine constants a_1, a_2, and a_3. From Equation (4–81), we have

$$1 - a_1 z^{-1} - a_2 z^{-2} - a_3 z^{-3} = (1 - z^{-1})^2 N(z)$$

If we divide the left-hand side of this last equation by $(1 - z^{-1})^2$, the quotient is $1 + (2 - a_1)z^{-1}$. The remainder is $[2(2 - a_1) - (1 + a_2)]z^{-2} - [(2 - a_1) + a_3]z^{-3}$. Hence, $N(z)$ is determined as

$$N(z) = 1 + (2 - a_1)z^{-1}$$

and the remainder is set equal to zero:

$$[2(2 - a_1) - (1 + a_2)]z^{-2} - (2 - a_1 + a_3)z^{-3} = 0$$

To satisfy this last equation, we require that

$$2(2 - a_1) - (1 + a_2) = 0 \tag{4–84}$$

$$2 - a_1 + a_3 = 0 \tag{4–85}$$

From Equation (4–82), we have

$$a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} = (1 + z^{-1})F_1(z)$$

If we divide the left-hand side of this last equation by $1 + z^{-1}$, the quotient is $a_1 z^{-1} + (a_2 - a_1)z^{-2}$ The remainder is $(a_1 - a_2 + a_3)z^{-3}$. Hence,

$$F_1(z) = a_1 z^{-1} + (a_2 - a_1)z^{-2}$$

and the remainder is set equal to zero:

$$a_1 - a_2 + a_3 = 0 \tag{4-86}$$

By solving Equations (4–84), (4–85), and (4–86) for a_1, a_2, and a_3, we obtain

$$a_1 = 1.25, \qquad a_2 = 0.5, \qquad a_3 = -0.75$$

Hence,

$$N(z) = 1 + 0.75z^{-1}$$

and

$$F_1(z) = 1.25z^{-1} - 0.75z^{-2} = 1.25z^{-1}(1 - 0.6z^{-1})$$

and $F(z)$ is determined as follows:

$$\begin{aligned} F(z) &= 1.25z^{-1} + 0.5z^{-2} - 0.75z^{-3} \\ &= 1.25z^{-1}(1 + z^{-1})(1 - 0.6z^{-1}) \end{aligned}$$

The digital controller $G_D(z)$ is then determined from Equation (4–50):

$$\begin{aligned} G_D(z) &= \frac{F(z)}{G(z)(1 - z^{-1})^2 N(z)} \\ &= \frac{1.25z^{-1}(1 + z^{-1})(1 - 0.6z^{-1})}{\dfrac{(1 + z^{-1})z^{-1}}{2(1 - z^{-1})^2}(1 - z^{-1})^2(1 + 0.75z^{-1})} \\ &= \frac{2.5(1 - 0.6z^{-1})}{1 + 0.75z^{-1}} \end{aligned}$$

With the digital controller thus designed, the system output in response to a unit-ramp input is obtained as follows:

$$\begin{aligned} C(z) &= F(z)R(z) \\ &= (1.25z^{-1} + 0.5z^{-2} - 0.75z^{-3})\frac{z^{-1}}{(1 - z^{-1})^2} \\ &= 1.25z^{-2} + 3z^{-3} + 4z^{-4} + 5z^{-5} + \cdots \end{aligned}$$

Hence,

$$\begin{aligned} c(0) &= 0 \\ c(1) &= 0 \\ c(2) &= 1.25 \\ c(k) &= k, \qquad k = 3, 4, 5, \ldots \end{aligned}$$

Notice that from Equation (4–83) we have

$$\begin{aligned} U(z) &= 2F_1(z) \\ &= 2(1.25z^{-1})(1 - 0.6z^{-1}) \\ &= 2.5z^{-1} - 1.5z^{-2} \end{aligned}$$

Thus, the control signal $u(k)$ becomes zero for $k \geq 3$. Consequently, there are no intersampling ripples in the response at steady state. Figure 4–63 shows plots of $c(k)$ versus k, $u(k)$ versus k, and $u(t)$ versus t in the unit-ramp response.

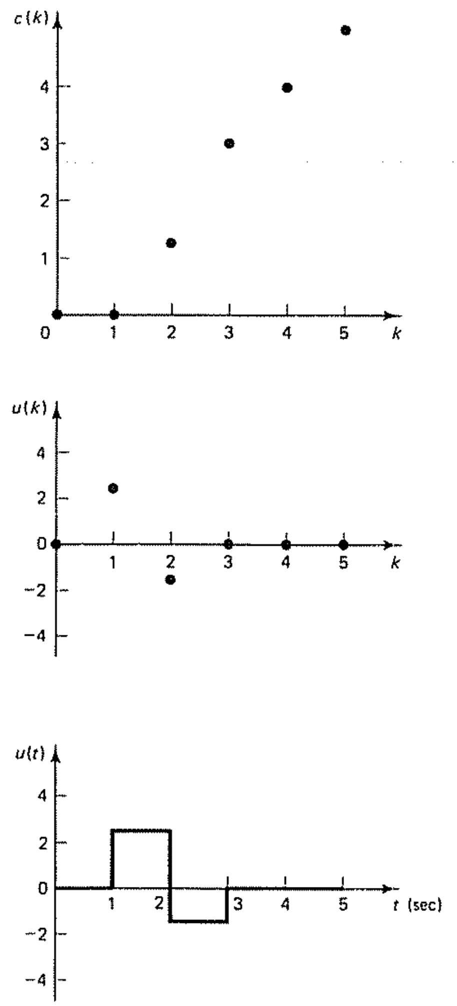


Figure 4–63 Plots of $c(k)$ versus k, $u(k)$ versus k, and $u(t)$ versus t in the unit ramp response of the system designed in Problem A–4–14.

Next, let us investigate the response of this system to a Kronecker delta input and a unit-step input. For a Kronecker delta input

$$C(z) = F(z)R(z) = F(z) = 1.25z^{-1} + 0.5z^{-2} - 0.75z^{-3}$$

Notice that $U(z)$ in this case becomes as follows:

$$U(z) = F(z)\frac{R(z)}{G(z)} = \frac{1.25z^{-1}(1 + z^{-1})(1 - 0.6z^{-1})}{(1 + z^{-1})z^{-1}/[2(1 - z^{-1})^2]}$$

$$= 2.5(1 - 0.6z^{-1})(1 - z^{-1})^2$$

$$= 2.5 - 6.5z^{-1} + 5.5z^{-2} - 1.5z^{-3}$$

The control signal $u(k)$ becomes zero for $k \geq 4$. Hence, there are no intersampling ripples after $t \geq 4T = 4$

For the unit-step input,

$$C(z) = F(z)R(z) = (1.25z^{-1} + 0.5z^{-2} - 0.75z^{-3})\frac{1}{1 - z^{-1}}$$
$$= 1.25z^{-1} + 1.75z^{-2} + z^{-3} + z^{-4} + z^{-5} + \cdots$$

The maximum overshoot is 75% in the unit-step response. Notice that

$$U(z) = F(z)\frac{R(z)}{G(z)} = \frac{1.25z^{-1}(1 + z^{-1})(1 - 0.6z^{-1})}{\frac{(1 + z^{-1})z^{-1}}{2(1 - z^{-1})^2}(1 - z^{-1})}$$
$$= 1.25(1 - 0.6z^{-1})(2)(1 - z^{-1})$$
$$= 2.5 - 4z^{-1} + 1.5z^{-2}$$

The control signal $u(k)$ becomes zero for $k \geq 3$. Consequently, there are no intersampling ripples after the settling time is reached. Figure 4–64(a) shows plots of $c(k)$ versus k, $u(k)$ versus k, and $u(t)$ versus t in the response to the Kronecker delta input. Figure 4–64(b) shows similar plots in the unit-step response. Notice that when the system is designed for the ramp input the response to a step input is no longer deadbeat.

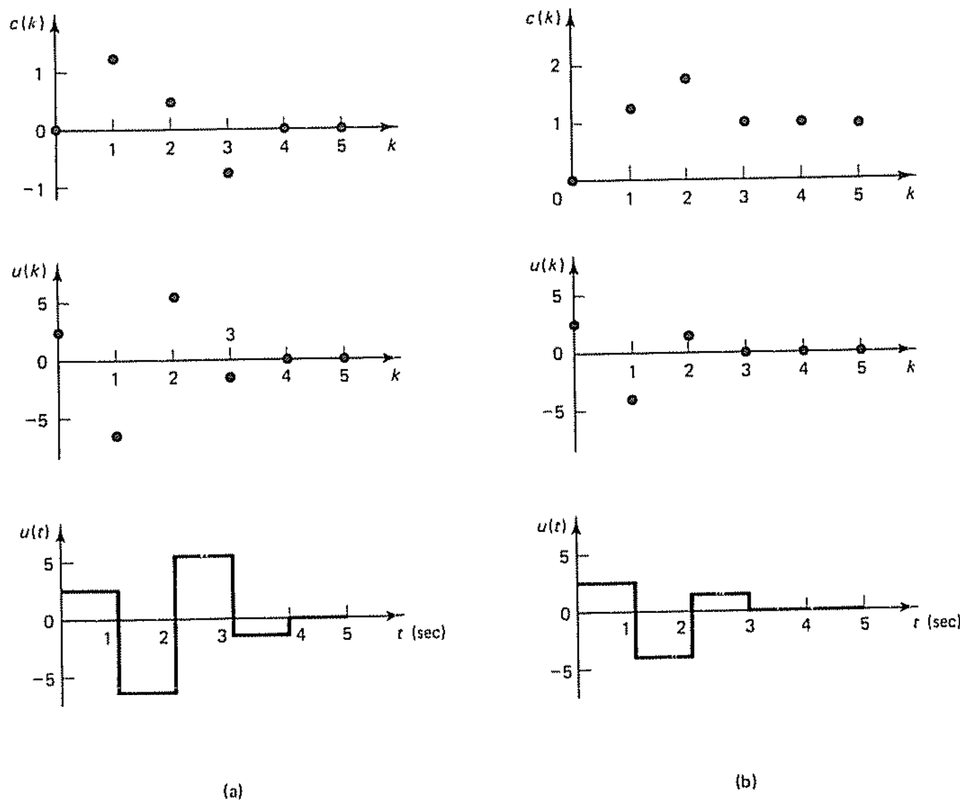


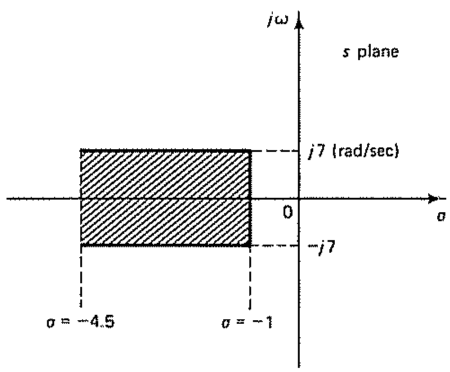
Figure 4–64 (a) Plots of $c(k)$ versus k, $u(k)$ versus k, and $u(t)$ versus t in the response to the Kronecker delta input of the system designed in Problem A–4–14; (b) plots of $c(k)$ versus k, $u(k)$ versus k, and $u(t)$ versus t in the unit-step response of the same system

PROBLEMS

Problem B–4–1

Consider the regions in the s plane shown in Figures 4–65(a) and (b). Draw the corresponding regions in the z plane. The sampling period T is assumed to be 0.3 sec. (The sampling frequency is $\omega_s = 2\pi/T = 2\pi/0.3 = 20.9$ rad/sec.)

(a)



(b)

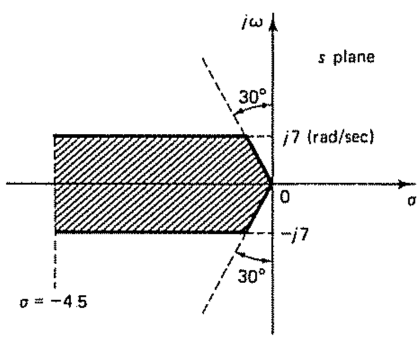


Figure 4–65 (a) Region in the s plane bounded by constant ω lines and constant σ lines; (b) region in the s plane bounded by constant ζ lines, constant ω lines, and a constant σ line.

Problem B–4–2

Consider the following characteristic equation:

$$z^3 + 2.1z^2 + 1.44z + 0.32 = 0$$

Determine whether or not any of the roots of the characteristic equation lie outside the unit circle centered at the origin of the z plane.

Problem B–4–3

Determine the stability of the following discrete-time system:

$$\frac{Y(z)}{X(z)} = \frac{z^{-3}}{1 + 0.5z^{-1} - 1.34z^{-2} + 0.24z^{-3}}$$

Problem B–4–4

Consider the discrete-time closed-loop control system shown in Figure 4–13. Determine the range of gain K for stability by use of the Jury stability criterion.

Problem B–4–5

Solve Problem B–4–4 by using the bilinear transformation coupled with the Routh stability criterion.

Problem B–4–6

Consider the system

$$\frac{Y(z)}{X(z)} = G(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

Suppose that the input sequence $\{x(k)\}$ is bounded; that is,

$$|x(k)| \le M_1 = \text{constant}, \qquad k = 0, 1, 2, \ldots$$

Show that, if all poles of $G(z)$ lie inside the unit circle in the z plane, then the output $y(k)$ is also bounded; that is,

$$|y(k)| \le M_2 = \text{constant}, \qquad k = 0, 1, 2, \ldots$$

Problem B–4–7

State the conditions for stability, instability, and critical stability in terms of the weighting sequence $g(kT)$ of a linear time-invariant discrete-time control system.

Problem B–4–8

Consider the digital control system shown in Figure 4–66. Plot the root loci as the gain K is varied from 0 to ∞. Determine the critical value of gain K for stability. The sampling period is 0.1 sec, or $T = 0.1$. What value of gain K will yield a damping ratio ζ of the closed-loop poles equal to 0.5? With gain K set to yield $\zeta = 0.5$, determine the damped natural frequency ω_d and the number of samples per cycle of damped sinusoidal oscillation.

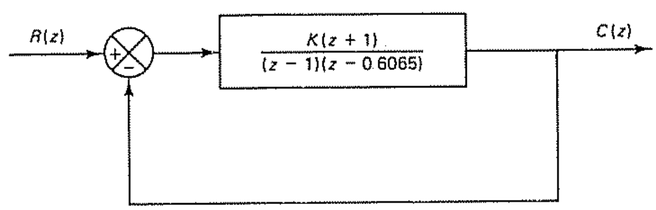


Figure 4–66 Digital control system for Problem B–4–8.

Problem B–4–9

Referring to the digital control system shown in Figure 4–67, design a digital controller $G_D(z)$ such that the damping ratio ζ of the dominant closed-loop poles is 0.5 and the number of samples per cycle of damped sinusoidal oscillation is 8. Assume that the sampling period is 0.1 sec, or $T = 0.1$. Determine the static velocity error constant. Also, determine the response of the designed system to a unit-step input.

Problem B–4–10

Consider the control system shown in Figure 4–68. Design a suitable digital controller that includes an integral control action. The design specifications are that the damping ratio ζ of the dominant closed-loop poles be 0.5 and that there be at least eight samples per cycle of damped sinusoidal oscillation. The sampling period is assumed to be 0.2

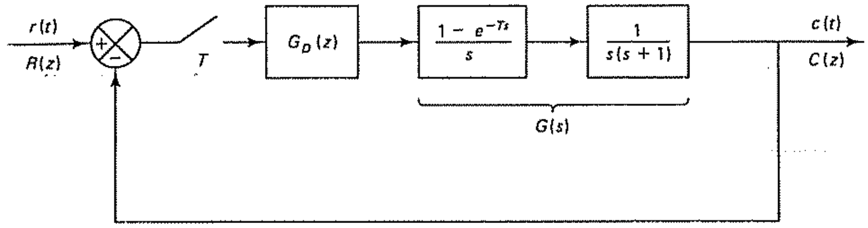


Figure 4–67 Digital control system for Problem B–4–9.

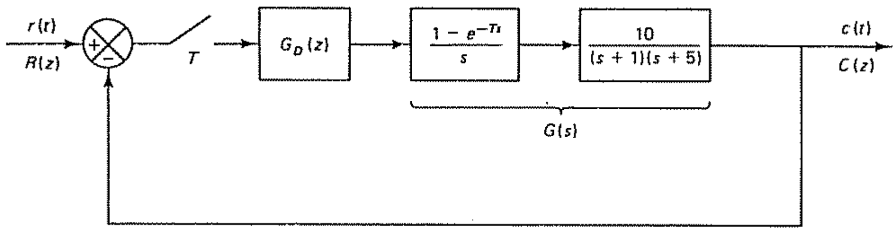


Figure 4–68 Digital control system for Problem B–4–10.

sec, or $T = 0.2$. After the digital controller is designed, determine the static velocity error constant K_v.

Problem B–4–11

Consider the digital control system shown in Figure 4–69, where the plant is of the first order and has a dead time of 5 sec. By choosing a reasonable sampling period T, design a digital PI controller such that the dominant closed-loop poles have a damping ratio ζ of 0.5 and the number of samples per cycle of damped sinusoidal oscillation is 10. After the controller is designed, determine the response of the system to a unit-step input.

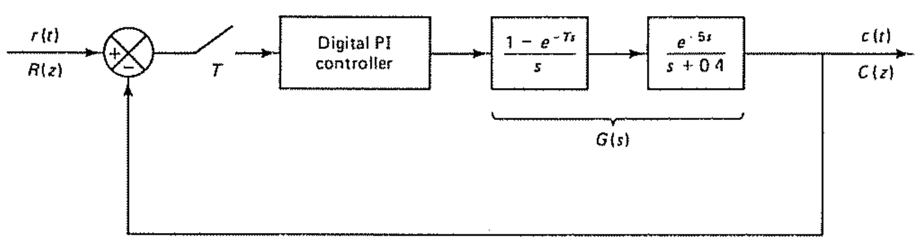


Figure 4–69 Digital control system for Problem B–4–11.

Problem B–4–12

Design a digital proportional-plus-derivative controller for the plant whose transfer function is $1/s^2$, as shown in Figure 4–70. It is desired that the damping ratio ζ of the dominant closed-loop poles be 0.5 and the undamped natural frequency be 4 rad/sec. The sampling period is 0.1 sec, or $T = 0.1$. After the controller is designed, determine the number of samples per cycle of damped sinusoidal oscillation.

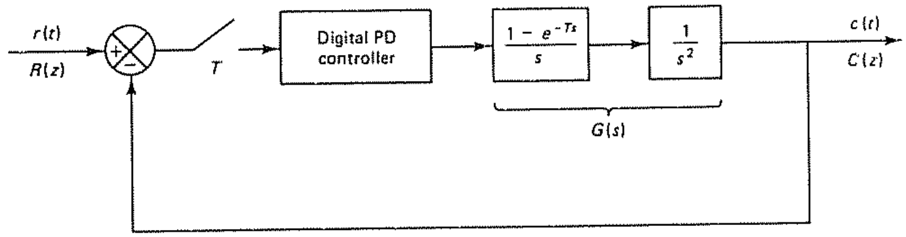


Figure 4–70 Digital control system for Problem B–4–12

Problem B–4–13

Referring to the system considered in Problem A–4–9, redesign the digital controller so that the static velocity error constant K_v is 12 sec^{-1}, without appreciably changing the locations of the dominant closed-loop poles in the z plane. The sampling period is assumed to be 0.2 sec, or $T = 0.2$. After the controller is redesigned, obtain the unit-step response and unit-ramp response of the digital control system.

Problem B–4–14

Consider the digital control system shown in Figure 4–71. Draw a Bode diagram in the w plane. Set the gain K so that the phase margin becomes equal to 50°. With the gain K so set, determine the gain margin and the static velocity error constant K_v. The sampling period is assumed to be 0.1 sec, or $T = 0.1$.

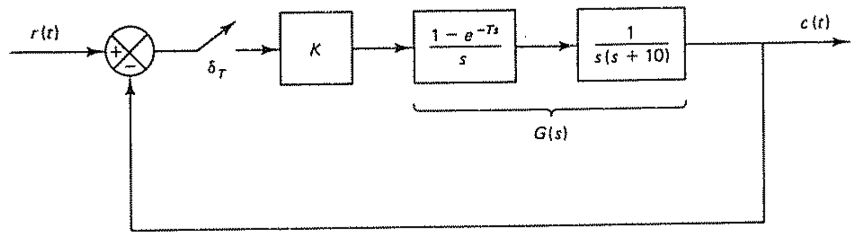


Figure 4–71 Digital control system for Problem B–4–14

Problem B–4–15

Using the Bode diagram approach in the w plane, design a digital controller for the system shown in Figure 4–72. The design specifications are that the phase margin be 50°, the gain margin be at least 10 dB, and the static velocity error constant K_v be 20 sec^{-1}. The sampling period is assumed to be 0.1 sec, or $T = 0.1$. After the controller is designed, calculate the number of samples per cycle of damped sinusoidal oscillation.

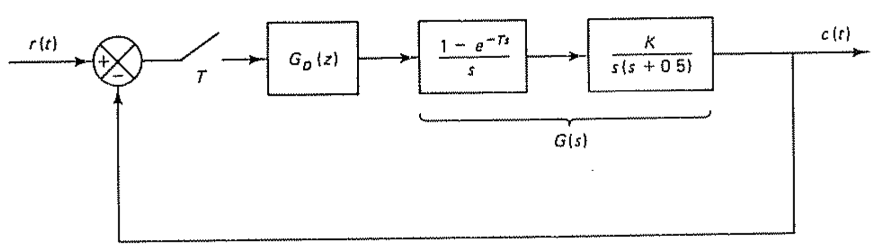


Figure 4–72 Digital control system for Problem B–4–15

Problem B–4–16

Consider the digital control system shown in Figure 4–73. Using the Bode diagram approach in the w plane, design a digital controller such that the phase margin is 60°, the gain margin is 12 dB or more, and the static velocity error constant is 5 sec^{-1}. The sampling period is assumed to be 0.1 sec, or $T = 0.1$.

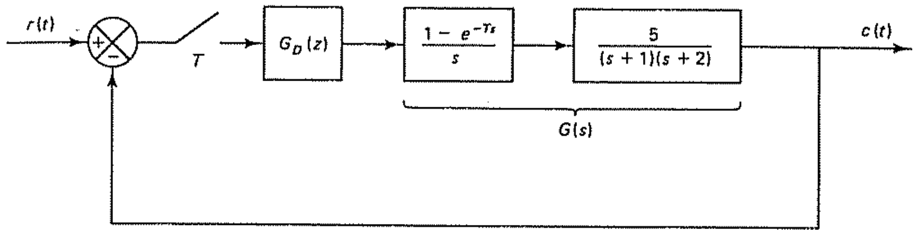


Figure 4–73 Digital control system for Problem B–4–16.

Problem B–4–17

Consider the system shown in Figure 4–74. Design a digital controller using a Bode diagram in the w plane such that the phase margin is 50° and the gain margin is at least 10 dB. It is desired that the static velocity error constant K_v be 10 sec^{-1}. The sampling period is specified as 0.1 sec, or $T = 0.1$. After the controller is designed, determine the number of samples per cycle of damped sinusoidal oscillation.

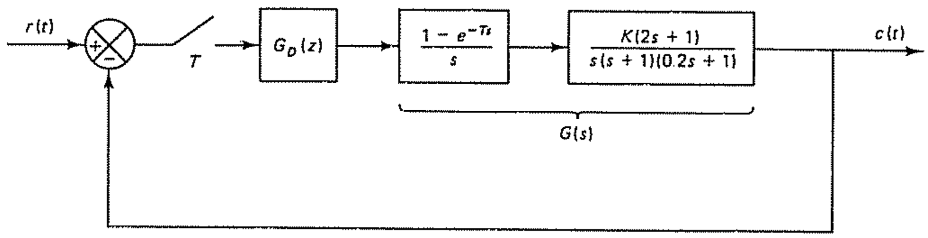


Figure 4–74 Digital control system for Problem B–4–17.

Problem B–4–18

Consider the digital control system shown in Figure 4–75. Design a digital controller $G_D(z)$ such that the system output will exhibit a deadbeat response to a unit step input (that is, the settling time will be the minimum possible and the steady-state error will be zero; also, the system output will not exhibit intersampling ripples after the settling time is reached). The sampling period T is assumed to be 1 sec, or $T = 1$.

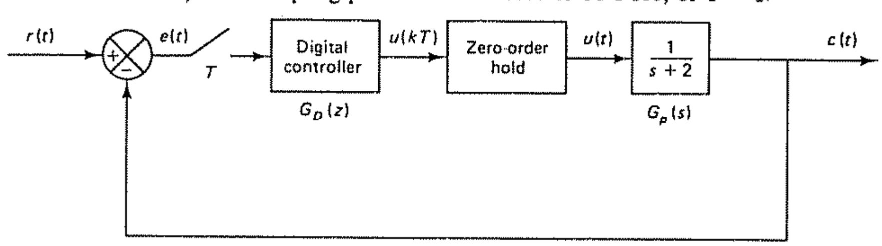


Figure 4–75 Digital control system for Problem B–4–18.

5

State-Space Analysis

5–1 INTRODUCTION

In Chapters 3 and 4 we were concerned with conventional methods for the analysis and design of control systems. Conventional methods such as the root-locus and frequency-response methods are useful for dealing with single-input–single-output systems. Conventional methods are conceptually simple and require only a reasonable number of computations, but they are applicable only to linear time-invariant systems having a single input and single output. They are based on the input–output relationship of the system, that is, the transfer function or the pulse transfer function. They do not apply to nonlinear systems except in simple cases. Also, the conventional methods do not apply to the design of optimal and adaptive control systems, which are mostly time varying and/or nonlinear.

A modern control system may have many inputs and many outputs, and these may be interrelated in a complicated manner. The state-space methods for the analysis and synthesis of control systems are best suited for dealing with multiple-input–multiple-output systems that are required to be optimal in some sense.

Concept of the State-Space Method. The state-space method is based on the description of system equations in terms of n first-order difference equations or differential equations, which may be combined into a first-order vector-matrix difference equation or differential equation. The use of the vector-matrix notation greatly simplifies the mathematical representation of the systems of equations.

System design by use of the state-space concept enables the engineer to design control systems with respect to given performance indexes. In addition, design in the state space can be carried out for a *class* of inputs, instead of a specific input