Lista 7: Transformada de Laplace (extraídos do livro de Boyce e DiPrima)

Problems

In each of Problems 1 through 3, sketch the graph of the given function. In each case determine whether f is continuous, piecewise continuous, or neither on the interval $0 \le t \le 3$.

1. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 2+t, & 1 < t \le 2 \\ 6-t, & 2 < t \le 3 \end{cases}$

2. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ (t-1)^{-1}, & 1 < t \le 2 \\ 1, & 2 < t \le 3 \end{cases}$

3. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 1, & 1 < t \le 2 \\ 3-t, & 2 < t \le 3 \end{cases}$

4. Find the Laplace transform of each of the following functions:

a. $f(t) = t$

b. $f(t) = t^2$

c. $f(t) = t^n$, where n is a positive integer

5. Find the Laplace transform of $f(t) = \cos(at)$, where a is a real constant.

Recall that

$$\cosh(bt) = \frac{1}{2}(e^{bt} + e^{-bt}) \text{ and } \sinh(bt) = \frac{1}{2}(e^{bt} - e^{-bt}).$$

In each of Problems 6 through 7, use the linearity of the Laplace transform to find the Laplace transform of the given function; a and b are real constants.

6. $f(t) = \cosh(bt)$

7. $f(t) = \sinh(bt)$

Recall that

$$\cos(bt) = \frac{1}{2}(e^{ibt} + e^{-ibt}) \text{ and } \sin(bt) = \frac{1}{2i}(e^{ibt} - e^{-ibt}).$$

In each of Problems 8 through 11, use the linearity of the Laplace transform to find the Laplace transform of the given function; a and b are real constants. Assume that the necessary elementary integration formulas extend to this case.

8. $f(t) = \sin(bt)$

9. $f(t) = \cos(bt)$

10. $f(t) = e^{at}\sin(bt)$

11. $f(t) = e^{at}\cos(bt)$

In each of Problems 12 through 15, use integration by parts to find the Laplace transform of the given function; n is a positive integer and a is a real constant.

12. $f(t) = te^{at}$

13. $f(t) = t\sin(at)$

14. $f(t) = t^n e^{at}$

15. $f(t) = t^2\sin(at)$

In each of Problems 16 through 18, find the Laplace transform of the given function.

16. $f(t) = \begin{cases} 1, & 0 \le t < \pi \\ 0, & \pi \le t < \infty \end{cases}$

17. $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & 1 \le t < \infty \end{cases}$

18. $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 2-t, & 1 \le t < 2 \\ 0, & 2 \le t < \infty \end{cases}$

In each of Problems 19 through 21, determine whether the given integral converges or diverges.

19. $\displaystyle\int_0^\infty (t^2+1)^{-1}dt$

20. $\displaystyle\int_0^\infty te^{-t}dt$

21. $\displaystyle\int_1^\infty t^{-2}e^t dt$

22. Suppose that f and f' are continuous for $t \ge 0$ and of exponential order as $t \to \infty$. Use integration by parts to show that if $F(s) = \mathcal{L}\{f(t)\}$, then $\lim_{s\to\infty} F(s) = 0$. The result is actually true under less restrictive conditions, such as those of Theorem 6.1.2.

23. The Gamma Function. The gamma function is denoted by $\Gamma(p)$ and is defined by the integral

$$\Gamma(p+1) = \int_0^\infty e^{-x}x^p dx. \tag{7}$$

The integral converges as $x \to \infty$ for all p. For $p < 0$ it is also improper at $x = 0$, because the integrand becomes unbounded as $x \to 0$. However, the integral can be shown to converge at $x = 0$ for $p > -1$.

a. Show that, for $p > 0$,

$$\Gamma(p+1) = p\Gamma(p).$$

b. Show that $\Gamma(1) = 1$.

c. If p is a positive integer n, show that

$$\Gamma(n+1) = n!.$$

Since $\Gamma(p)$ is also defined when p is not an integer, this function provides an extension of the factorial function to nonintegral values of the independent variable. Note that it is also consistent to define $0! = 1$.

d. Show that, for $p > 0$,

$$p(p+1)(p+2)\cdots(p+n-1) = \frac{\Gamma(p+n)}{\Gamma(p)}.$$

Thus $\Gamma(p)$ can be determined for all positive values of p if $\Gamma(p)$ is known in a single interval of unit length—say, $0 < p \le 1$. It is possible to show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Find $\Gamma\left(\frac{3}{2}\right)$ and $\Gamma\left(\frac{11}{2}\right)$.