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PROBLEMS

10.1 Pennes[1] obtained experimental data on temperature distribution in the forearm of several subjects. The average center and skin surface temperatures were found to be $36.1\,^\circ\mathrm{C}$ and $33.6\,^\circ\mathrm{C}$, respectively. Use the Pennes model to predict these two temperatures based on the following data:

c_b = specific heat of blood = $3.8\ \mathrm{J/g}-^\circ\mathrm{C}$

h = heat transfer coefficient = $4.18\ \mathrm{W/m^2}-^\circ\mathrm{C}$

k_b = thermal conductivity of blood = $0.5\ \mathrm{W/m}-^\circ\mathrm{C}$

k = thermal conductivity of muscle = $0.5\ \mathrm{W/m}-^\circ\mathrm{C}$

q'''_m = metabolic heat production = $0.000418\ \mathrm{W/cm^3}$

R = average forearm radius = 4 cm

$T_{a0} = 36.3\,^\circ\mathrm{C}$

$T_\infty = 26.6\,^\circ\mathrm{C}$

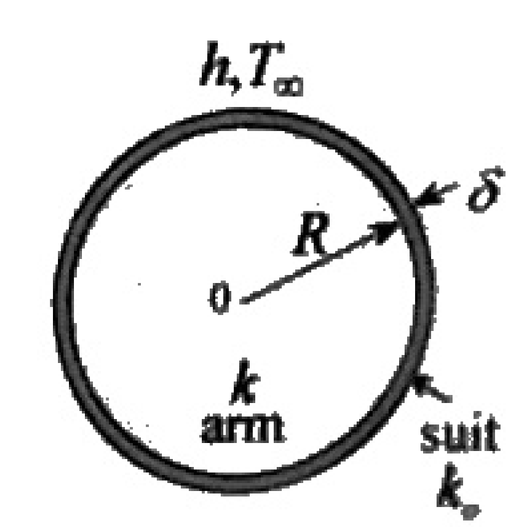
$\dot{w}_b$ = volumetric blood perfusion rate per unit tissue volume $= 0.0003\ \mathrm{cm^3/s/cm^3}$

ρ_b = blood density = $1050\ \mathrm{kg/m^3}$

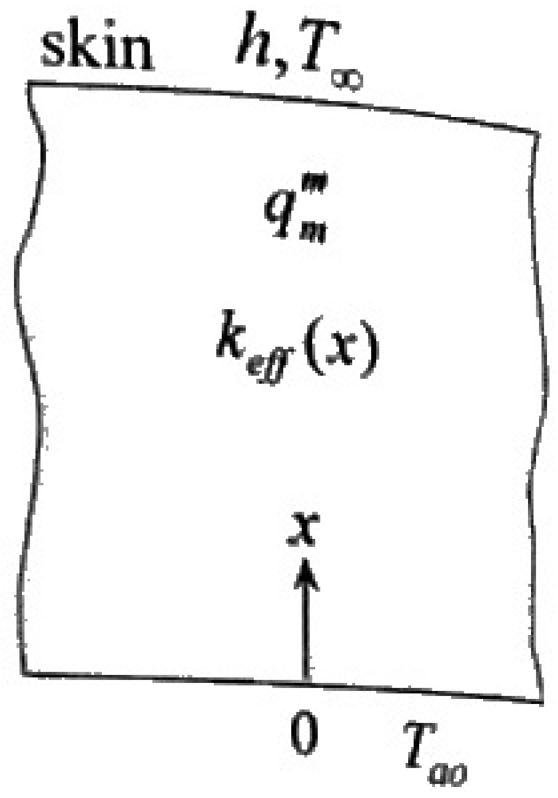
10.2 Blood perfusion rate plays an important role in regulating body temperature and skin heat flux. Use Pennes's data on the forearm of Problem 10.1 to construct a plot of skin surface temperature and heat flux for blood perfusion rates ranging from $\dot{w}_b = 0$ to $w_b = 0.0006\ (\mathrm{cm^3/s})/\mathrm{cm^3}$.

10.3 Certain clinical procedures involve cooling of human legs prior to surgery. Cooling is accomplished by maintaining surface temperature below body temperature. Model the leg as a cylinder of radius R with volumetric blood perfusion rate per unit tissue volume $\dot{w}_b$ and metabolic heat production rate q'''_m. Assume uniform skin surface temperature T_s. Use the Pennes bioheat equation to determine the steady state one-dimensional temperature distribution in the leg.

10.4 A manufacturer of suits for divers is interested in evaluating the effect of thermal conductivity of suit material on skin temperature. Use Pennes model for the forearm to predict skin surface temperature of a diver wearing a tight suit of thickness δ and thermal conductivity k_o. The volumetric blood perfusion rate per unit tissue volume is $\dot{w}_b$ and metabolic heat production rate is q'''_m. The ambient temperature is T_∞ and the heat transfer coefficient is h. Neglect curvature of the suit layer.



10.5 Consider the single layer model of the peripheral tissue of Example 10.2. Tissue thickness is L and blood supply temperature to the deep layer at $x = 0$ is T_{a0}. The skin surface exchanges heat by convection. The ambient temperature is T_∞ and the heat transfer coefficient is h. Assume that the vascular geometry function $V(\xi)$ can be approximated by



$$V(\xi) = A + B\xi + C\xi^2,$$

where

$$A = 6.32\times10^{-5},\ B = -15.9\times10^{-5},\ \text{and}\ C = 10\times10^{-5}.$$

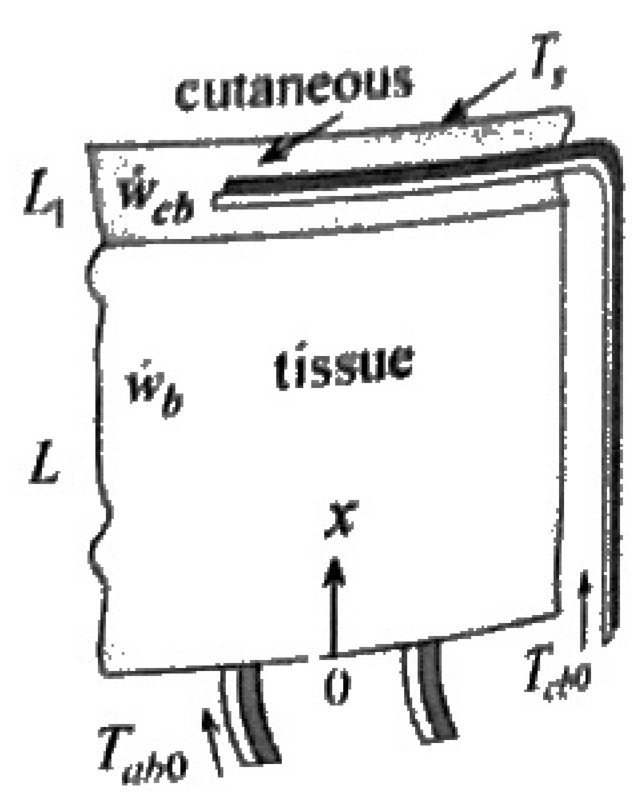
Use the Weinbaum-Jiji simplified bioheat equation to obtain a solution to the temperature distribution in the tissue. Express the result in non-dimensional form using the following dimensionless quantities:

$$\theta = \frac{T - T_\infty}{T_{a0} - T_\infty},\quad \xi = \frac{x}{L},\quad \gamma = \frac{q'''_m L^2}{k(T_{a0} - T_\infty)},\quad Bi = \frac{hL}{k},$$

$$Pe_0 = \frac{2\rho_b c_b a_0 u_0}{k_b}.$$

Construct a plot showing the effect of Biot number (surface convection) on tissue temperature distribution $\theta(\xi)$ for $Pe_0 = 180$, $\gamma = 0.6$ and $Bi = 0.1$ and 1.0.

10.6 In Example 10.3 skin surface is maintained at specified temperature. To examine the effect of surface convection on skin surface heat flux, repeat Example 10.3 assuming that the skin exchanges heat with the surroundings by convection. The ambient temperature is T_∞ and the heat transfer coefficient is h.



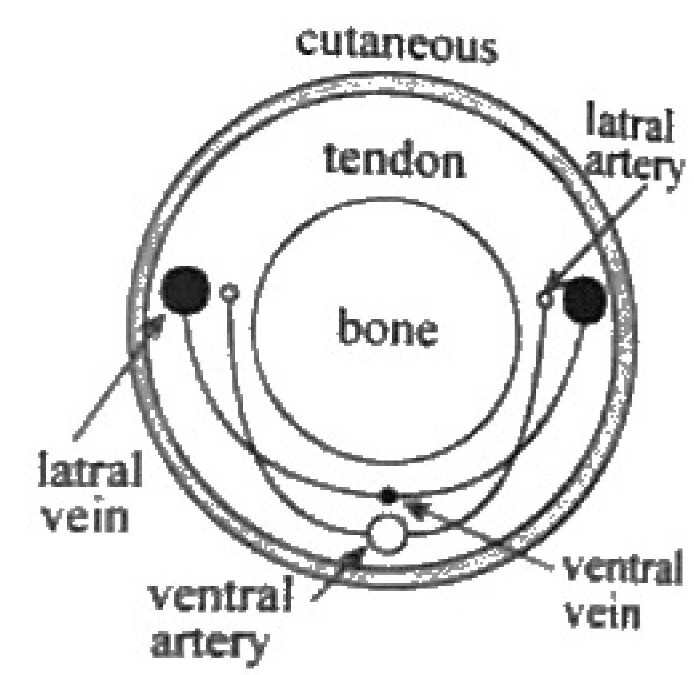
10.7 The vascular geometry of the peripheral tissue of Example 10.2 is approximated by a polynomial function. To evaluate the sensitivity of temperature distribution to the assumed vascular geometry function, consider a linear representation of the form

$$V(\xi) = A + B\xi,$$

where $A = 6.32\times10^{-5}$ and $B = -6.32\times10^{-5}$. Determine k_{eff}/k and $\theta(\xi)$ for $\gamma = 0.6$ and $Pe_0 = 180$. Compare your result with Example 10.2.

10.8 A digit consists mostly of bone surrounded by a thin cutaneous layer. A simplified model for analyzing the temperature distribution and heat transfer in digits is a cylindrical bone covered by a uniform cutaneous layer. Neglecting axial and angular variations, the problem reduces to one-dimensional temperature distribution. Consider the case of a digit with negligible metabolic heat production. The skin surface exchanges heat with the ambient by convection. The heat transfer coefficient is h and the ambient temperature is T_∞. Using the Pennes equation determine the steady state temperature distribution and heat transfer rate. Note that in the absence of metabolic heat production the bone in this model is at uniform temperature.

10.9 Fin approximation can be applied in modeling organs such as the elephant ear, rat tail, chicken legs, duck beak and human digits. Temperature distribution in these organs is three-dimensional. However, the problem can be significantly simplified using fin approximation. As an example, consider the rat tail. Anatomical studies have shown that it consists of three layers: bone, tendon and cutaneous layer. There are three major axial artery-vein pairs: one ventral and two lateral. These pairs are located in the tendon near the cutaneous layer as shown. The ventral vein is small compared to the lateral veins, and the lateral arteries are small compared to the ventral artery. Blood perfusion from the arteries to the veins takes place mostly in the cutaneous layer through a network of small vessels. Assume that blood is supplied to the cutaneous layer at uniform temperature T_{a0} all along the tail. Blood equilibrates at the local cutaneous temperature T before returning to the veins. Assume further that (1) cutaneous layer, tendon and bone have the same conductivity, (2) negligible angular variation, (3) uniform blood perfusion along the tail, (4) negligible metabolic heat, (5) steady state, (6) uniform outer radius and (7) negligible temperature variation in the radial direction (fin approximation is valid, $Bi << 1$). Surface heat exchange is by convection. The heat transfer coefficient is h and the ambient temperature is T_∞. Using Pennes model for the cutaneous layer, show that the heat equation for the rat tail is given by

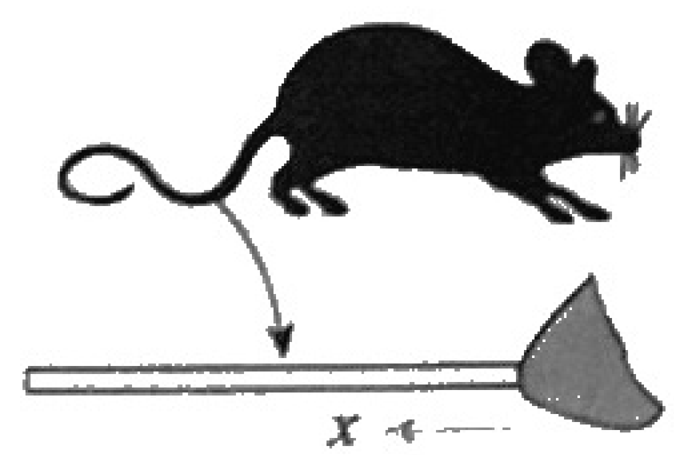


$$\frac{d^2\theta}{d\xi^2} - (m+\beta)\theta + m = 0,$$

where

$$\theta = \frac{T - T_{a0}}{T_\infty - T_{a0}}, \quad \xi = \frac{x}{L},$$

$$m = \frac{2hL^2}{kR}, \quad \beta = \frac{\dot{w}_b \rho_b c_b L^2}{k}.$$



Here L is tail length, R tail radius and x is axial distance along the tail.

10.10 Consider the rat tail model described in Problem 10.9. Assume that the base of the tail is at the artery supply temperature T_{a0} and that the tip is insulated. Show that the axial temperature distribution and total heat transfer rate from the tail are given by

$$\theta(\xi) = \frac{m}{\beta+m}\left[(\tanh\sqrt{\beta+m})\sinh\sqrt{\beta+m}\,\xi - \cosh\sqrt{\beta+m}\,\xi + 1\right],$$

and

$$q = 2\pi hRL(T_{a0} - T_\infty)\left[1 - \frac{m}{\beta+m} + \frac{m\tanh\sqrt{\beta+m}}{(\beta+m)\sqrt{\beta+m}}\right],$$

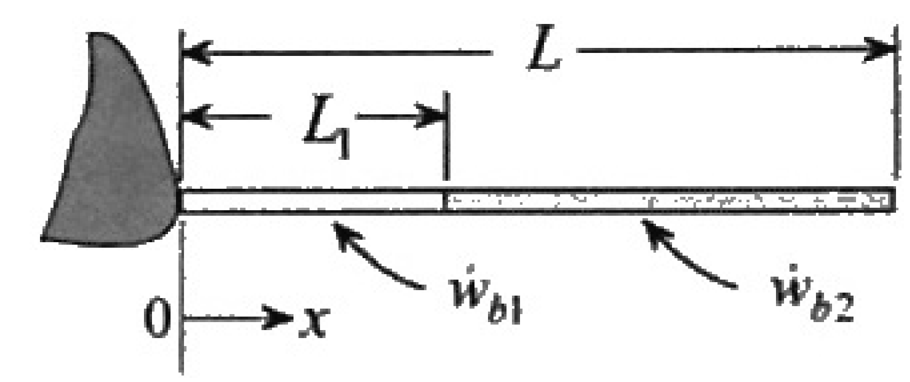
Construct a plot of the axial temperature distribution and calculate the heat transfer rate for the following data:

$$c_b = 3.8\frac{\text{J}}{\text{g-}^\circ\text{C}}, \quad h = 15.9\frac{\text{W - m}^2}{^\circ\text{C}}, \quad k = 0.5\frac{\text{W - m}}{^\circ\text{C}},$$

$$L = 22.5\,\text{cm}, \quad R = 0.365\,\text{cm}, \quad T_{a0} = 36\ ^\circ\text{C}, \; T_\infty = 22.5\ ^\circ\text{C},$$

$$\dot{w}_b = 0.01947\frac{\text{cm}^3/\text{s}}{\text{cm}^3}, \quad \rho = 1.05\frac{\text{g}}{\text{cm}^3}.$$

10.11 Studies have shown that blood perfusion along the rat tail is non-uniform. This case can be analyzed by dividing the tail into sections and assigning different blood perfusion rate to each section. Consider the rat tail described in Problem 10.9. Model the tail as having two sections. The first section extends a length L_1 from the base and the second has a length of $(L - L_1)$. Blood perfusion rate in the first section is $\dot{w}_{b1}$ and in the second section $\dot{w}_{b2}$. Determine the axial temperature distribution in the tail in terms of the following dimensionless quantities



$$\theta = \frac{T - T_{a0}}{T_\infty - T_{a0}}, \quad \xi = \frac{x}{L}, \; \xi_1 = \frac{L_1}{L}, \quad m = \frac{2hL^2}{kR},$$

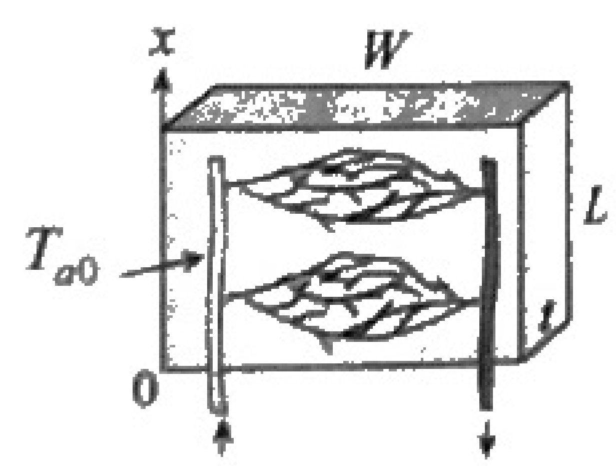
$$\beta_1 = \frac{\dot{w}_{b1}\rho_b c_b L^2}{k}, \quad \beta_2 = \frac{\dot{w}_{b2}\rho_b c_b L^2}{k}.$$

10.12 The cutaneous layer of a peripheral tissue is supplied by blood at temperature T_{c0} and a *total* flow rate $\dot{W}_{cb}$. The tissue is supplied by blood at temperature T_{a0} and *total* flow rate $\dot{W}_b$. Tissue thickness is L and cutaneous layer thickness is L_1. One mechanism for regulating surface heat loss is by controlling blood flow through the cutaneous layer. Use the Weinbaum-Jiji simplified bioheat equation to examine the effect of blood flow ratio, $R = \dfrac{\dot{W}_{cb}}{W_b} = \dfrac{L_1 \dot{w}_{cb}}{L\dot{w}_b}$, on surface heat flux. The skin is maintained at uniform temperature T_s. Assume a linear tissue vascular geometry function of the form

$$V(\xi) = A + B\xi .$$

10.13 Studies have concluded that the plates on the back of the dinosaur Stegosaurus served a thermoregulatory function as heat dissipating fins [19]. There are indications that the network of channels within the plates may be blood vessels. Model the plate as a rectangular fin of width W, length L and thickness t. Use the Pennes model to formulate the heat equation for this blood

perfused plate. The plate exchanges heat with the ambient air by convection. The heat transfer coefficient is h and the ambient temperature is T_∞. Assume that blood reaches each part of the plate at temperature T_{a0} and that it



equilibrates at the local temperature T. Assume further that (1) blood perfusion is uniform, (2) negligible metabolic heat production, (3) negligible temperature variation along plate thickness t (fin approximation is valid, $Bi << 1$), (4) steady state and (5) constant properties. Show that the heat equation for this model is given by

$$\frac{d^2\theta}{d\xi^2} - (m+\beta)\theta + m = 0,$$

where

$$\theta = \frac{T - T_{a0}}{T_\infty - T_{a0}}, \quad \xi = \frac{x}{L}, \quad m = \frac{2h(W+t)L^2}{kWt}, \quad \beta = \frac{\dot{w}_b \rho_b c_b L^2}{k}.$$

10.14 Modeling the plates on the back of the dinosaur Stegosaurus as rectangular fins, use the bioheat fin equation formulated in Problem 10.13 to show that the enhancement in heat transfer, η, due to blood flow is given by

$$\eta = \frac{q}{q_o} = \frac{\sqrt{m}}{\beta + m}\,\frac{\beta + \dfrac{m}{\sqrt{\beta+m}}\tanh\sqrt{\beta+m}}{\tanh\sqrt{m}},$$

where

$$m = \frac{2h(W+t)L^2}{kWt}, \quad \beta = \frac{\dot{w}_b \rho_b c_b L^2}{k}.$$

q = heat transfer rate from plate with blood perfusion
q_o = heat transfer rate from plate with no blood perfusion

Compute the enhancement η and total heat loss from 10 plates for the following data:

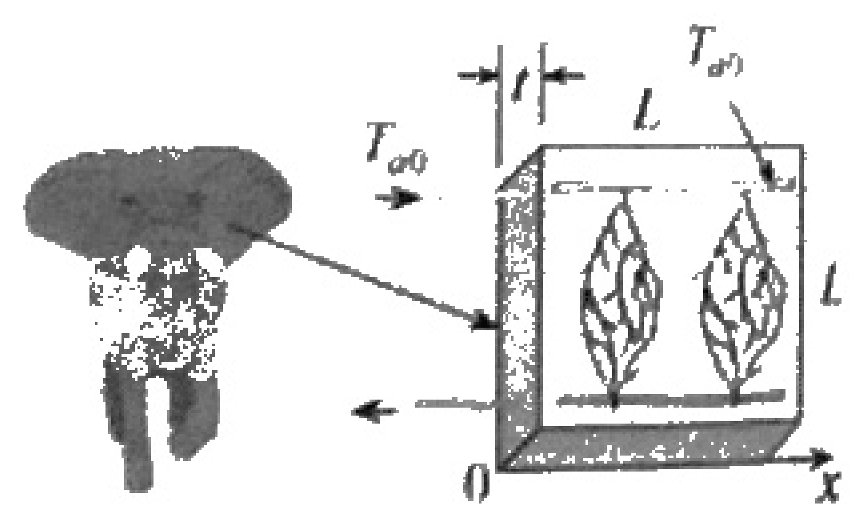
c_b = specific heat of blood = 3800 J/kg–°C
h = heat transfer coefficient = 14.9 W/m^2–°C
k = thermal conductivity = 0.6 W/m–°C
L = plate length = 0.45 m
t = plate thickness = 0.2 m
T_{a0} = blood supply temperature = 37 °C
T_∞ = ambient temperature = 27 °C
$\dot{w}_b$ = blood perfusion rate per unit tissue volume = 0.00045 (cm^3/s)/cm^3
W = plate width = 0.7 m
ρ_b = blood density = 1050 kg/m^3

10.15 Elephant ears serve as a thermoregulatory device by controlling blood supply rate to the ears and by flapping them. Increasing blood perfusion increases heat loss due to an increase in surface temperature. Flapping results in an increase in the heat transfer coefficient as air flow over the ears changes from natural to forced convection. Consider the following data on a 2000 kg elephant which generates 1640 W:

c_b = specific heat of blood = 3800 J/kg–°C

h_n = natural convection heat transfer coefficient = 2 W/m^2–°C
h_f = forced convection heat transfer coefficient (flapping) = 17.6 W/m^2–°C
k = thermal conductivity = 0.6 W/m–°C
L = equivalent length of square ear = 0.93 m
t = average ear thickness = 0.6 cm
T_{a0} = blood supply temperature = 36 °C
T_∞ = ambient temperature = 24 °C
$\dot{w}_b$ = blood perfusion rate per unit tissue volume = 0.0015 (cm^3/s)/cm^3
ρ_b = blood density = 1050 kg/m^3

Neglecting metabolic heat production in the ear, model the ear as a square fin using the bioheat equation formulated in Problem 10.13 to determine the total heat transfer rate from two sides of two ears with and without flapping. In addition compute the enhancement in heat transfer for the two cases. Define enhancement η as



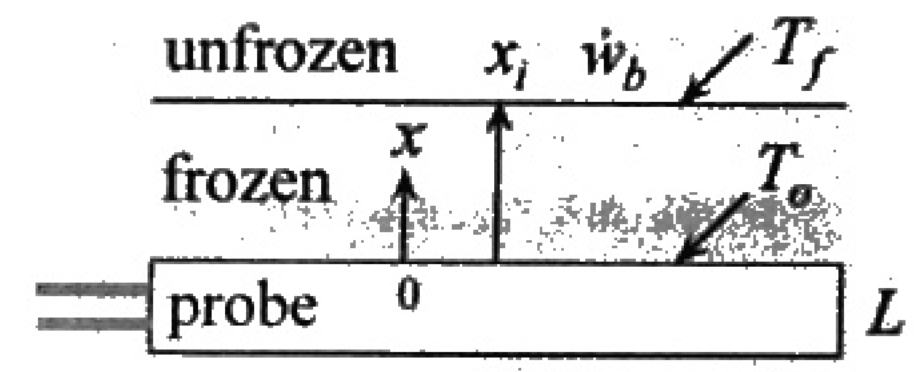
$$\eta = \frac{q}{q_o},$$

where

q = heat transfer rate from ear
q_o = heat transfer rate from ear with no blood perfusion and no flapping

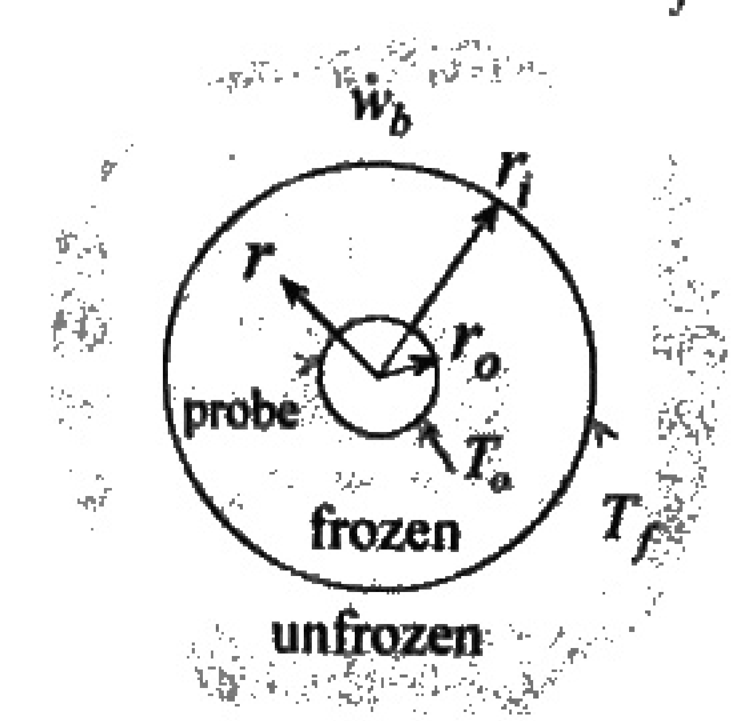
10.16 Cryosurgical probes are used in medical procedures to selectively freeze and destroy diseased tissue. The cryoprobe surface is maintained at a temperature below tissue freezing temperature causing a frozen front to form at the surface and propagate outward. Knowledge of the maximum frozen layer or lesion size is helpful to the surgeon in selecting the proper settings for the cryoprobe. Maximum lesion size corresponds to the steady state temperature

distribution. Consider a planar probe which is inserted in a large region of tissue. Probe thickness is L and its surface temperature is T_o. Using the s-vessel tissue cylinder model, determine the maximum lesion size for the following data:



c_b = specific heat of blood = 3800 J/kg–°C
L = probe thickness = 4 mm
k = thermal conductivity of unfrozen tissue = 0.6 W/m–°C
k_s = thermal conductivity of frozen tissue = 1.8 W/m–°C
q'''_m = metabolic heat production = 0.021 W/cm^3
T_{ab0} = artery blood temperature = 36.5 °C
T_f = tissue freezing temperature = 0 °C
T_o = probe surface temperature = –42 °C
$\dot{w}_b$ = blood perfusion rate per unit tissue volume = 0.0032 (cm^3/s)/cm^3
ρ_b = blood density = 1050 kg/m^3
ΔT^* = 0.75

10.17 A cylindrical cryosurgical probe consists of a tube whose surface temperature is maintained below tissue freezing temperature T_f. The frozen region or lesion around the cryoprobe reaches its maximum size at steady state. Predicting the maximum lesion size is important in avoiding damaging healthy tissue and in targeting diseased areas. Consider a cylindrical probe which is inserted in a large tissue region. Probe radius is r_o and its surface temperature is T_o. Metabolic heat production is q'''_m and blood perfusion rate is $\dot{w}_b$. Let T be the temperature distribution in the unfrozen tissue and T_s the



temperature distribution in the frozen tissue. Using the *s*-vessel tissue cylinder model, determine the steady state temperature distribution in the two regions and the maximum lesion size. Note that the conductivity of frozen tissue k_s is significantly different from the conductivity k of unfrozen tissue. Express the result in dimensionless form using the following dimensionless quantities:

$$\theta = \frac{T - T_{ab0}}{T_f - T_{ab0}}, \quad \theta_s = \frac{T_s - T_o}{T_f - T_o}, \quad \xi = \frac{r}{r_o}, \quad \xi_i = \frac{r_i}{r_o},$$

$$\beta = \frac{\dot{w}_b \rho_b c_b \Delta T^* r_o^2}{k}, \quad \gamma = \frac{q_m''' r_o^2}{k(T_{ab0} - T_f)}.$$

10.18 A brain surgical procedure requires the use of a spherical cryosurgical probe to create a frozen region (lesion) 6 mm in radius. A 3 mm radius spherical cryoprobe is selected for insertion into the diseased area. Using the Pennes model, determine the required probe surface temperature such that the maximum lesion size does not exceed 6 mm. Note that maximum size corresponds to the steady state temperature distribution in the frozen and unfrozen regions around the probe. The following data is given

c_b = specific heat of blood = 3800 J/kg−°C
k = thermal conductivity of unfrozen tissue = 0.6 W/m−°C
k_s = thermal conductivity of frozen tissue = 1.8 W/m−°C
q_m''' = metabolic heat production = 0.011 W/cm^3
T_{a0} = artery blood temperature = 36.5 °C
T_f = tissue freezing temperature = 0 °C
$\dot{w}_b$ = blood perfusion rate per unit tissue volume = 0.0083 (cm^3/s)/cm^3
ρ_b = blood density = 1050 kg/m^3

10.19 Analytical prediction of the growth of the frozen region around cryosurgical probes provides important guidelines for establishing probe application time. Consider the planar probe described in Problem 10.16. Use the *s*-vessel tissue cylinder model and assume a

quasi-steady approximation to show that the dimensionless interface location ξ_i is given by:

$$\tau = \frac{1}{\lambda^2}\left[\lambda\xi_i - \ln(1 + \lambda\xi_i)\right],$$

where

$$\tau = \frac{k_s(T_f - T_o)}{\rho L^2 \mathcal{L}} t, \quad \lambda = \sqrt{\beta}\, \frac{k(T_f - T_{ab0})}{k_s(T_f - T_o)}\left[1 + \frac{\gamma}{\beta}\right], \quad \xi_i = \frac{x_i}{L},$$

$$\beta = \frac{\dot{w}_b \rho_b c_b \Delta T^* L^2}{k}, \quad \gamma = \frac{q_m''' L_o^2}{k(T_{ab0} - T_f)}.$$

where $\mathcal{L} = 333{,}690$ J/kg is the latent heat of fusion and $\rho_s = 1040\,\text{kg/m}^3$ is the density of frozen tissue. How long should the probe of Problem 10.16 be applied so that the frozen layer is 3.5 mm thick?

10.20 Consider the cylindrical cryosurgical described in Problem 10.17. Using the *s*-vessel tissue cylinder model and assuming quasi-steady interface motion, determine lesion size as a function of time.

10.21 The spherical cryosurgical probe of Problem 10.18 is used to create a lesion corresponding to 95% of its maximum size. Using the Pennes model and assuming quasi-steady interface motion, determine the probe application time. Probe temperature is $T_o = -29.6$°C, latent heat of fusion is $\mathcal{L} = 333{,}690$ J/kg and the density of frozen tissue is $\rho_s = 1040\,\text{kg/m}^3$.

10.22 Prolonged exposure to cold environment of elephants can result in frost bite on their ears. Model the elephant ear as a sheet of total surface area (two sides) A and uniform thickness δ. Assume uniform blood perfusion $\dot{w}_b$ and uniform metabolic heat q_m'''. The ear loses heat by convection. The ambient temperature is T_∞ and the heat transfer coefficient is h. Using lumped capacity approximation and the Pennes model, show that the dimensionless transient heat equation is given by

$$\frac{d\theta}{d\tau} = (1+\gamma) - (1+\beta)\theta,$$

where

$$\theta = \frac{T - T_{a0}}{T_\infty - T_{a0}}, \quad \tau = \frac{2h}{\delta \rho c} t, \quad \beta = \frac{\dot{w}_b \rho_b c_b \delta}{2h}, \quad \gamma = \frac{q_m''' \delta}{2h(T_\infty - T_{a0})}.$$

Here ρ is tissue density and c tissue specific heat. The subscript b refers to blood. Determine the maximum time a zoo elephant can remain outdoors on a cold winter day without resulting in frost bite when the ambient temperature is lower than freezing temperature T_f. Assume that initially the ears are at uniform temperature T_i.

11

MICROSCALE CONDUCTION

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11.1 Introduction

Heat conduction at the microscale can be dramatically different than at the macroscale. The differences become clear by comparing the thermal conductivity of a "bulk" material (that is, a large sample that is by definition free of microscale effects) to the effective thermal conductivity of a microscopic sample of the same material. The values of thermal conductivity that are readily available in textbooks and standard reference books (so-called "handbook values") apply to bulk samples, but must be used with great caution for microscale samples. As one example, according to a standard reference [1], the thermal conductivity k of pure silicon at room temperature is 148 W/m-°C. This value is appropriate for silicon samples with characteristic lengths ranging from meters to millimeters to microns. However, a silicon nanowire of diameter 56 nm ($1 \text{ nm} = 10^{-9}$ m) has thermal conductivity of only 26 W/m-°C [2], a reduction by more than a factor of five compared to the bulk value. The reductions are even more dramatic at low temperature: at 20 K the values are 4940 W/m-°C for bulk Si [1], and 0.72 W/m-°C for the Si nanowire [2], a difference of more than a factor of 6000! By the end of this chapter, readers should understand the physical reasons for this tremendous reduction, and be able to evaluate it numerically with approximate calculations.

Although the great majority of microsystems follow this same pattern of having thermal conductivity less than their bulk counterparts, there are also certain materials that exhibit nanoscale thermal conductivity greater than