Basics on analytical mechanics and dynamics of structures

PEF 6000 - Special topics on dynamics of structures

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1. Objectives and references
2. Fundamental concepts on analytical mechanics
3. Brief summary on linear dynamics of structures
4. Analysis in the state-space

- To introduce to basic aspects related to analytical mechanics and dynamics of structures;
- The topics addressed in the class are discussed in greater depth in other graduate courses (PEF5916 and PME5010);
- Examples of references:

1. Clough, R.W. & Penzien, J., 1975. *Dynamics of Structures*. McGraw Hill.
2. Lanczos, C., 1986. *The variational principles of mechanics*. Dover publications.
3. Mazzilli, C.E.N., André, J.C., Bucalem, M.L. & Cifú, S., 2016. *Lições em mecânica das estruturas: Dinâmica*. Edgard Blucher.
4. Meirovitch, L., 2003. *Methods of Analytical Dynamics*. Dover Publications.
5. Pesce, C.P., 1999. *Dinâmica dos corpos rígidos*.

- A complete set of lectures on Analytical Mechanics can be found here.

1. Objectives and references
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- Mechanical system: System of interacting particles;
- Configuration space: Let N being the number of independent particles (free of constraints) of a certain mechanical system. We can define a space (configuration space) characterized by the coordinates x_i, y_i and z_i $(i = 1, 2 \ldots, N)$ of each particle. The temporal evolution of the mechanical system is a curve in the configuration space;
- Constraint equations: Commonly, we have constraint equations that link the motion of some particles. We define c as the number of constraint equations;
- Focus is placed on holonomic constraints (constraint equations depend on the generalized coordinates and not on the generalized velocities). $f_k = f_c(x_i, y_i, z_i, t)$ $(k = 1, \ldots, c; i = 1, \ldots, N) \rightarrow$ Holonomic and rheonomic constraint. $f_k = f_c(x_i, y_i, z_i)$ $(k = 1, \ldots, c; i = 1, \ldots, N) \rightarrow$ Holonomic and scleronomic constraint and focus of the class..
- Number of degrees of freedom (dof): Number of generalized coordinates necessary for the complete description of the mechanical system. $n = 3N - c$;

- Generalized coordinate q_i: Set of independent variables that defines the configuration of a mechanical system. The temporal derivative of the generalized coordinates corresponds to the generalized velocities. The generalized coordinates must define, in a biunivic way, the motion in the physical coordinates. The choice of the generalized coordinates is not unique in general.;
- Virtual displacement: Arbitrary change in the position of the particles that satisfies the constraints of the problem. The virtual displacement does not consider the flux of time. For a system with n dofs described by $q_1, q_2 \dots, q_n$, if the position of a certain particle is $\boldsymbol{r_i} = \boldsymbol{r_i}(q_1, q_2, \dots, q_n, t)$, the associated virtual displacement is

$$\boldsymbol{\delta r_i} = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \delta q_j \tag{1}$$

where δq_j is the variation of the generalized coordinate q_j.

- Consider a system with N point masses $m_i, i = 1, 2, \ldots, N$ defined by the corresponding position vectors $\boldsymbol{r_i}$.
- Second Newton's law, assuming that m_i is independent of time:

$$\frac{d}{dt}(m_i \dot{\boldsymbol{r}}_i) = \boldsymbol{F_i} \rightarrow m_i \ddot{\boldsymbol{r}}_i = \boldsymbol{F_i} = \boldsymbol{F_i^a} + \boldsymbol{F_i^{ic}} + \boldsymbol{F_i^{nc}} \tag{2}$$

  $\boldsymbol{F_i^a}$ is the applied force, $\boldsymbol{F_i^{ic}}$ and $\boldsymbol{F_i^{nc}}$ are the forces associated with ideal and non-ideal constraints.
- Restricted d'Alembert's principle:

$$-m_i \ddot{\boldsymbol{r}}_i + \boldsymbol{F_i^a} + \boldsymbol{F_i^{ic}} + \boldsymbol{F_i^{nc}} = 0 \tag{3}$$

- $(-m_i \ddot{\boldsymbol{r}}_i + \boldsymbol{F_i^a} + \boldsymbol{F_i^{ic}} + \boldsymbol{F_i^{nc}}).\delta \boldsymbol{r}_i = 0$
- As the virtual work of the forces associated with ideal constraints is null and defining the effective force $\boldsymbol{F_i^e} = \boldsymbol{F_i^a} + \boldsymbol{F_i^{nc}}$, we have $(-m_i \ddot{\boldsymbol{r}}_i + \boldsymbol{F_i^e}).\delta \boldsymbol{r}_i = 0$
- For the system with N particles

$$\sum_{i=1}^{N}(-m_i \ddot{\boldsymbol{r}}_i + \boldsymbol{F_i^e}).\delta \boldsymbol{r}_i = 0 \tag{4}$$

- We define the effective force as the sum of a term arisen from a potential function (conservative force $\boldsymbol{F_i^c}$) with a non-conservative one $\boldsymbol{F_i^{nc}}$;
- The virtual work of the conservative and non-conservative forces are $\boldsymbol{F_i^c}\boldsymbol{\delta r_i} = -\delta\mathcal{U}_i$ and $\boldsymbol{F_i^{nc}}\boldsymbol{\delta r_i} = \delta W^{nc}$, respectively;
- With the above definitions, Eq. 4 reads

$$\sum_{i=1}^{N}(-\delta\mathcal{U}_i + \delta W_i^{nc} - m_i\ddot{\boldsymbol{r}}_i.\boldsymbol{\delta r_i}) = 0 \rightarrow -\delta\mathcal{U} + \delta W^{nc} - \sum_{i=1}^{N} m_i\ddot{\boldsymbol{r}}_i.\boldsymbol{\delta r_i} = 0 \qquad (5)$$

- The variation of kinetic energy is

$$\delta\mathcal{T} = \sum_{i=1}^{N}\delta\left(\frac{1}{2}m_i\dot{\boldsymbol{r}}_i.\dot{\boldsymbol{r}}_i\right) = \sum_{i=1}^{N} m_i\dot{\boldsymbol{r}}_i.\boldsymbol{\delta}\dot{\boldsymbol{r}}_i \qquad (6)$$

- Mathematical identity:

$$\sum_{i=1}^{N} m_i\frac{d}{dt}(\dot{\boldsymbol{r}}_i.\boldsymbol{\delta r_i}) = \sum_{i=1}^{N}(m_i\ddot{\boldsymbol{r}}_i.\boldsymbol{\delta r_i} + m_i\dot{\boldsymbol{r}}_i.\delta\dot{\boldsymbol{r}}_i) \qquad (7)$$

- From Eq. 7, we have

$$-\sum_{i=1}^{N} m_i \ddot{\boldsymbol{r}}_{\boldsymbol{i}}.\boldsymbol{\delta r_i} = -\sum_{i=1}^{N} m_i \frac{d}{dt}(\dot{\boldsymbol{r}}_{\boldsymbol{i}}.\boldsymbol{\delta r_i}) + \delta\mathcal{T} \tag{8}$$

- Using Eq. 8 in Eq. 5

$$-\delta\mathcal{U} + \delta W^{nc} + \delta\mathcal{T} = \sum_{i=1}^{N} m_i \frac{d}{dt}(\dot{\boldsymbol{r}}_{\boldsymbol{i}}.\boldsymbol{\delta r_i}) \tag{9}$$

- Now, we integrate Eq. 9 from t_1 to t_2. In these instants $\boldsymbol{r_i}$ are known and, hence $\boldsymbol{\delta r_i}(t_1)$=$\boldsymbol{\delta r_i}(t_2) = 0$

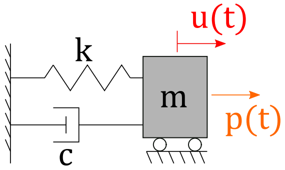
$$\int_{t_1}^{t_2} (\delta\mathcal{T} - \delta\mathcal{U} + \delta W^{nc})dt = \sum_{i=1}^{N} m_i [\boldsymbol{r_i}.\boldsymbol{\delta r_i}]_{t_1}^{t_2} = 0 \tag{10}$$

- Contrary to Newtonian mechanics, the use of analytical mechanics allows obtaining the equations of motion from the scalar quantities kinetic energy $\mathcal{T}$, potential energy $\mathcal{U}$ and virtual work of the non-conservative forces δW^{nc};
- As shown (and discussed in greater depth in other graduate courses), the equations of motion are obtained by imposing

$$\int_{t_1}^{t_2} (\delta\mathcal{T} - \delta\mathcal{U} + \delta W^{nc})dt = 0 \qquad (11)$$

  where $\delta\mathcal{T}$ and $\delta\mathcal{U}$ are the first variations of kinetic and potential energies, respectively.
- The focus herein is not the derivation of the extended Hamilton's principle. The objective of this course is on the practical use of this principle for obtaining equations of motion of mechanical systems.

We will find the equation of motion for the problem below sketched.



- Kinetic energy and its variation: $\mathcal{T} = \frac{1}{2}m\dot{u}^2 \rightarrow \delta\mathcal{T} = m\dot{u}\delta\dot{u}$;
- Potential energy and its variation: $\mathcal{U} = \frac{1}{2}ku^2 \rightarrow \delta\mathcal{U} = ku\delta u$;
- Virtual work of the non-conservative forces: $\delta W^{nc} = (-c\dot{u} + p(t))\delta u$

- Extended Hamilton's principle:

$$\int_{t_1}^{t_2} (\delta\mathcal{T} - \delta\mathcal{U} + \delta W^{nc})dt = 0 \leftrightarrow$$
$$\leftrightarrow \int_{t_1}^{t_2} [m\dot{u}\delta\dot{u} - (ku + c\dot{u} - p(t))\delta u]dt = 0 \quad (12)$$

- Integrating Eq. 12 by parts and recalling that $\delta u(t_1) = \delta u(t_2) = 0$, we obtain:

$$\int_{t_1}^{t_2} (-m\ddot{u} - c\dot{u} - ku + p(t))\delta u dt + \underbrace{+ [m\dot{u}\delta u]_{t_1}^{t_2}}_{0} = 0 \quad (13)$$

- Since δu is arbitrary, Eq. 13 holds if $m\ddot{u} + c\dot{u} + ku = p(t)$.

- The kinetic energy $\mathcal{T}$ is function of the generalized coordinates q_i and the corresponding generalized velocities $\dot{q}_i$. Mathematically, $\mathcal{T}$ and $\delta\mathcal{T}$ read

$$\mathcal{T} = \mathcal{T}(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2 \ldots, \dot{q}_n) \rightarrow \delta\mathcal{T} = \sum_{i=1}^{n} \left(\frac{\partial \mathcal{T}}{\partial q_i}\delta q_i + \frac{\partial \mathcal{T}}{\partial \dot{q}_i}\delta \dot{q}_i \right) \tag{14}$$

- On the other hand, the potential energy is function of the generalized coordinates.

$$\mathcal{U} = \mathcal{U}(q_1, q_2, \ldots, q_n) \rightarrow \delta\mathcal{U} = \sum_{i=1}^{n} \frac{\partial \mathcal{U}}{\partial q_i}\delta q_i \tag{15}$$

- Consider that the mechanical system is loaded by N_f non-conservative forces $\boldsymbol{F_j}, j = 1, 2, \ldots, N_f$. The virtual work of the non-conservative forces is:

$$\delta W^{nc} = \sum_{j=1}^{N_f} \boldsymbol{F_j}.\boldsymbol{\delta r_j} = \sum_{j=1}^{N_f} \boldsymbol{F_j}.\sum_{i=1}^{n} \frac{\partial \boldsymbol{r_j}}{\partial q_i}\delta q_i = \sum_{i=1}^{n} \underbrace{\left(\sum_{j=1}^{N_f} \boldsymbol{F_j}.\frac{\partial \boldsymbol{r_j}}{\partial q_i} \right)}_{Q_i} \delta q_i \tag{16}$$

- Equations 14-16 are substituted into the extended Hamilton's principle. Using integration by parts and recalling that $\delta q_i(t_1) = \delta q_i(t_2) = 0, i = 1, 2, \ldots, n$, we have:

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left[-\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_i}\right) + \frac{\partial \mathcal{T}}{\partial q_i} - \frac{\partial \mathcal{U}}{\partial q_i} + Q_i \right] \delta q_i = 0 \tag{17}$$

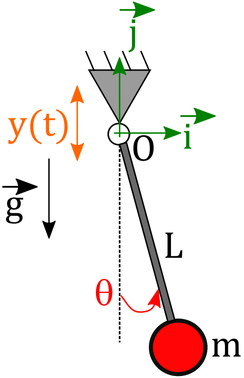
- Equation 17 holds if:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{T}}{\partial q_i} + \frac{\partial \mathcal{U}}{\partial q_i} = Q_i, i = 1, 2, \ldots, n \tag{18}$$

- Equation 18 is known as Euler-Lagrange's equation. If we define the Lagrangian as $\mathcal{L} = \mathcal{T} - \mathcal{U}$, Eq. 18 is rewritten as:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i, i = 1, 2, \ldots, n \tag{19}$$

Obtain the equation of motion for the pendulum under support excitation. The rigid and massless arm has length L. Vertical displacement $y(t)$ is applied to the support



- Velocity of the mass:
  $\boldsymbol{v_m} = \boldsymbol{v_O} + \boldsymbol{\omega} \times (\boldsymbol{m} - \boldsymbol{O}) =$
  $\dot{y}\boldsymbol{j} + \dot{\theta}\boldsymbol{k} \times (L\sin\theta\boldsymbol{i} - L\cos\theta\boldsymbol{j}) =$
  $(\dot{\theta}L\cos\theta)\boldsymbol{i} + (\dot{y} + \dot{\theta}L\sin\theta)\boldsymbol{j}$;

- The origin of the fixed referential coincides with the hinge point O when $y(t) = 0$;
- Kinetic energy: $\mathcal{T} = \frac{1}{2}m\boldsymbol{v_m}.\boldsymbol{v_m} =$ $\frac{1}{2}m(\dot{y}^2 + 2\dot{y}\dot{\theta}L\sin\theta + (\dot{\theta}L)^2)$;
- Potential energy and its variation: $\mathcal{U} = mg(y - L\cos\theta)$;
- Virtual work of the non-conservative forces: $\delta W^{nc} = 0 \rightarrow Q_\theta = 0$
- Using Euler-Lagrange's equation: $mL^2\ddot{\theta} + m(g + \ddot{y})L\sin\theta = 0$

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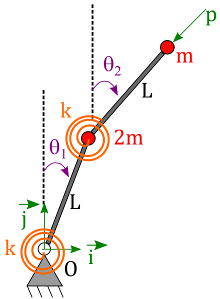
Classical problem, recently readdressed in d'Annibale & Ferretti (2020) and in Franzini & Mazzilli (2021). The massless bars have length L and are connected by means of torsional springs of stiffness k. A follower force p is applied to the tip.



- Position vectors of the masses: $\boldsymbol{r}_1 = L(\sin\theta_1 \boldsymbol{i} + \cos\theta_1 \boldsymbol{j})$ and $\boldsymbol{r}_2 = \boldsymbol{r}_1 + L(\sin\theta_2 \boldsymbol{i} + \cos\theta_2 \boldsymbol{j})$;
- $\boldsymbol{\delta r}_1 = L(\cos\theta_1 \boldsymbol{i} - \sin\theta_1 \boldsymbol{j})\delta\theta_1$;
- $\boldsymbol{\delta r}_2 = \boldsymbol{\delta r}_1 + L(\cos\theta_2 \boldsymbol{i} - \sin\theta_2 \boldsymbol{j})\delta\theta_2 =$;
- Kinetic energy: $\mathcal{T} = \frac{1}{2}(2m)\dot{\boldsymbol{r}}_1 . \dot{\boldsymbol{r}}_1 + \frac{1}{2}m\dot{\boldsymbol{r}}_2 . \dot{\boldsymbol{r}}_2$
- Potential energy: $\mathcal{U} = \frac{1}{2}k\theta_1^2 + \frac{1}{2}k(\theta_2 - \theta_1)^2$;

- Virtual work of the non-conservative force:

$$
\begin{aligned}
&\delta W^{nc} = \boldsymbol{p}.\boldsymbol{\delta r}_2 = \\
&= -p(\sin\theta_2 \boldsymbol{i} + \cos\theta_2 \boldsymbol{j}).L((\cos\theta_1\delta\theta_1 + \cos\theta_2\delta\theta_2)\boldsymbol{i} - (\sin\theta_1\delta\theta_1 + \sin\theta_2\delta\theta_2)\boldsymbol{j}) = \\
&= -pL\sin(\theta_2 - \theta_1)\delta\theta_1 \rightarrow Q_{\theta_1} = -pL\sin(\theta_2 - \theta_1),\, Q_{\theta_2} = 0 \qquad (20)
\end{aligned}
$$

- After the derivatives and the application of Euler-Lagrange equation:

$$
\begin{aligned}
&3mL^2\ddot{\theta}_1 + mL^2\cos(\theta_2 - \theta_1)\ddot{\theta}_2 + mL^2\sin(\theta_1 - \theta_2)\dot{\theta}_2^2 + 2k\theta_1 - k\theta_2 = \\
&= pL\sin(\theta_1 - \theta_2) \qquad (21)
\end{aligned}
$$

$$
mL^2\cos(\theta_1 - \theta_2)\ddot{\theta}_1 + mL^2\ddot{\theta}_2 + k\theta_2 - k\theta_1 - mL^2\sin(\theta_1 - \theta_2)\dot{\theta}_1^2 = 0 \qquad (22)
$$

Now, we linearize the equations of motion around a certain position $(\theta_1^0; \theta_2^0)$. We consider $\theta_1(t) = \theta_1^0 + q_1(t)$ and $\theta_2(t) = \theta_2^0 + q_2(t)$, $q_1(t)$ and $q_2(t)$ disturbances superimposed to the point around which the mathematical model is linearized.

- $\dot{\theta}_1 = \dot{q}_1$, $\dot{\theta}_2 = \dot{q}_2$, $\ddot{\theta}_1 = \ddot{q}_1$ and $\ddot{\theta}_2 = \ddot{q}_2$;
- Using Taylor series and keeping only the first two terms: $\sin(\theta_1 - \theta_2) = \sin(\theta_1^0 - \theta_2^0) + \cos(\theta_1^0 - \theta_2^0)q_1 - \cos(\theta_1^0 - \theta_2^0)q_2$ and $\cos(\theta_1 - \theta_2) = \cos(\theta_1^0 - \theta_2^0) - \sin(\theta_1^0 - \theta_2^0)q_1 + \sin(\theta_1^0 - \theta_2^0)q_2$;
- Here, we linearize around the trivial condition $(\theta_1^0; \theta_2^0) = (0; 0)$, we have $\sin(\theta_1 - \theta_2) = q_1 - q_2$ and $\cos(\theta_1 - \theta_2) = 1$.
- Using these quantities, the linearized mathematical model is:

$$3mL^2\ddot{q}_1 + mL^2\ddot{q}_2 + (2k - pL)q_1 - (k - pL)q_2 = 0 \tag{23}$$

$$mL^2\ddot{q}_1 + mL^2\ddot{q}_2 - kq_1 + kq_2 = 0 \tag{24}$$

- In this problem, the linearized stiffness matrix is no longer symmetric.

- As mentioned, the equation of motion of a 1-dof linear oscillator can be obtained by using either the second Newton's law or concepts of analytical mechanics (extended Hamilton principle, Euler-Lagrange's equation);
- Consider the 1-dof system of mass m, spring stiffness k and damping constant c. The system is forced by an external load $p(t)$. The displacement (generalized coordinate) is $u = u(t)$. The equation of motion is given by:

$$m\ddot{u} + c\dot{u} + ku = p(t) \tag{25}$$

- Equation 25 is a second-order, linear, ordinary differential equation. The solution of Eq. 25 is the sum of the homogeneous solution u_h with the particular solution u_p. Mathematically, $u = u_h + u_p$. The initial conditions $u(0) = u_0$ and $\dot{u}(0) = \dot{u}_0$ must be imposed to the complete solution;
- Definitions $\omega = \sqrt{\frac{k}{m}}$ (undamped natural frequency), $\zeta = \frac{c}{2m\omega} = \frac{c}{2\sqrt{km}}$ (damping ratio) and $\omega_d = \omega\sqrt{1-\zeta^2}$ (damped natural frequency).

- In free vibrations, the external oscillation is null $\rightarrow p(t) = 0$. Oscillations occur due to non-trivial initial conditions;
- Three cases appear, depending on the value of ζ. $\zeta < 1$ (sub-critical case) is the focus herein, since it is more common on engineering problems (civil, mechanical and naval engineering);
- In the sub-critical case: $u(t) = \rho e^{-\zeta\omega t}\cos(\omega_d t - \theta)$, ρ and θ depending on the initial conditions. Oscillations amplitudes exponentially decay with time.
- A system in free vibrations oscillates with its damped natural frequency ω_d!!!!;
- If $\zeta << 1 \rightarrow \omega_d \approx \omega$. For example, $\zeta = 0.10 \rightarrow \omega_d = 0.995\omega$.

- In this case $p(t) = p_0 \cos(\bar{\omega} t) = Re\{p_0 e^{i\bar{\omega} t}\}$;
- In the damped case, the homogeneous solution $u_h \to 0$ when $t \to \infty$ (steady-state response). In this case only the particular solution u_p appears;
- Finding u_p. We assume $u = u_p = Re\{Ue^{i\bar{\omega} t}\} \to \dot{u} = Re\{i\bar{\omega} Ue^{i\bar{\omega} t}\}$, $\ddot{u} = Re\{-\bar{\omega}^2 Ue^{i\bar{\omega} t}\}$. Now, we consider the complex quantities in the equations of motion, taking the real part at the end of the derivation:
- Using the above consideration

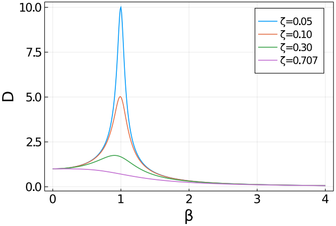
$$(-m\bar{\omega}^2 + i\bar{\omega} c + k)Ue^{i\bar{\omega} t} = p_0 e^{i\bar{\omega} t} \to U = \frac{p_0}{(-m\bar{\omega}^2 + i\bar{\omega} c + k)} = p_0 H(\bar{\omega}) \tag{26}$$

- $H(\bar{\omega})$ is the frequency response function. Notice that $H(\bar{\omega})$ and U are complex functions.

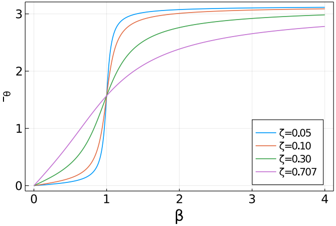
Following, we define $\beta = \frac{\bar{\omega}}{\omega}$. As a consequence, u reads:

$$u = \frac{p_0}{(-m\bar{\omega}^2 + i\bar{\omega}c + k)} e^{i\bar{\omega}t} = \frac{p_0}{k}\left(\frac{1}{(1-\beta^2) + i2\zeta\beta}\right) e^{i\bar{\omega}t} =$$
$$= \frac{p_0}{k}\left(\frac{1}{\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}}\right) e^{i(\bar{\omega}t - \bar{\theta})}; \tan\bar{\theta} = \frac{2\zeta\beta}{1-\beta^2} \tag{27}$$

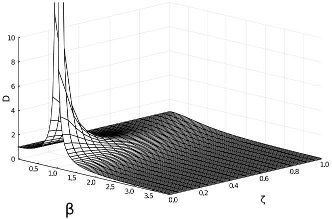
- The dynamic magnification factor D is defined as $D = \frac{1}{\sqrt{(1-\beta^2)^2+(2\zeta\beta)^2}}$. Notice that $\frac{p_0}{k} = u_s$ is response to a load equal to the amplitude of the varying load statically applied to the structure. $\bar{\theta}$ is the phase lag between excitation and displacement;
- As the external excitation is the real part of $p_0 e^{i\bar{\omega}t}$, we take the real part of Eq. 27, obtaining $u = \bar{\rho}\cos(\bar{\omega}t - \bar{\theta}) = u_s D\cos(\bar{\omega}t - \bar{\theta})$.
- If the excitation is poly-chromatic (i.e., composed of a number of harmonic functions), we can obtain the response for each individual component and superimpose the result. This is valid within the linear theory!.



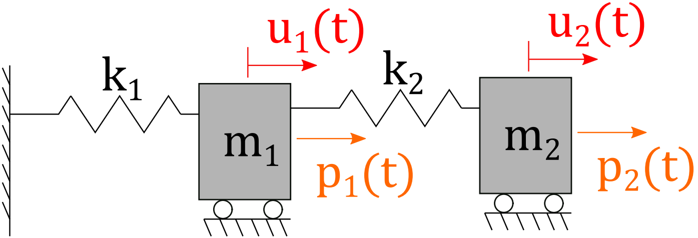
(a) $D(\beta)$.



(b) $\bar{\theta}(\beta)$.



(c) $D(\beta, \zeta)$.


k_1
k_2
$u_1(t)$
$u_2(t)$
m_1
m_2
$p_1(t)$
$p_2(t)$

In this case, the equations of motion are easily obtained by using Euler-Lagrange's equation.

- Kinetic energy:

$$\mathcal{T} = \frac{1}{2} m_1 \dot{u}_1^2 + \frac{1}{2} m_2 \dot{u}_2^2 \tag{28}$$

- Potential energy:

$$\mathcal{U} = \frac{1}{2} k_1 u_1^2 + \frac{1}{2} k_2 (u_2 - u_1)^2 \tag{29}$$

- Lagrangian $\mathcal{L} = \mathcal{T} - \mathcal{U}$;
- Virtual work of the non-conservative forces:

$$\delta W_{nc} = p_1 \delta u_1 + p_2 \delta u_2 = Q_1 \delta u_1 + Q_2 \delta u_2 \tag{30}$$

Euler-Lagrange's equation:

$$\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{u}_i}\right) - \frac{\partial \mathcal{L}}{\partial u_i} = Q_i, i = 1, 2 \tag{31}$$

$$m_1 \ddot{u}_1 + (k_1 + k_2)u_1 - k_2 u_2 = p_1(t) \tag{32}$$

$$m_2 \ddot{u}_2 - k_2 u_1 + k_2 u_2 = p_2(t) \tag{33}$$

We can define vectors $\boldsymbol{U} = \{u_1 \quad u_2\}^T$ and $\boldsymbol{P}(t) = \{p_1(t) \quad p_2(t)\}^T$. Using these definitions, Eqs. 32 and 33 can be written in the matrix form as

$$\boldsymbol{M}\ddot{\boldsymbol{U}} + \boldsymbol{K}\boldsymbol{U} = \boldsymbol{P}(t) \tag{34}$$

$\boldsymbol{M}$ and $\boldsymbol{K}$ being the mass matrix and the stiffness matrix, respectively. If linear damping is considered, the term $\boldsymbol{C}\dot{\boldsymbol{U}}$ is included on the left-hand side of Eq. 34.

- Any N-dof undamped linear oscillator is governed by the general Eq. 34. Modal analysis deals with the unforced response of this system ($\boldsymbol{P}(t) = 0$). In this scenario:

$$\ddot{\boldsymbol{U}} + \boldsymbol{A}\boldsymbol{U} = 0 \tag{35}$$

where $\boldsymbol{A} = \boldsymbol{M}^{-1}\boldsymbol{K}$.

- A general solution for Eq. 35 is $\boldsymbol{U} = \phi e^{i\omega t}$. Substituting this expression into Eq. 35, we have:

$$(\boldsymbol{A} - \omega^2 \boldsymbol{I})\phi e^{i\omega t} = 0 \tag{36}$$

- The existence of non-trivial solutions of Eq. 36 implies that $det(\boldsymbol{A} - \omega^2 \boldsymbol{I}) = 0$. Hence, the eigenvalues of A are the natural frequencies ω squared. The eigenvectors ϕ define the natural modes, associated with the "shape" of the response;
- The modal vectors ϕ play a key role in dynamics. It can be shown that they can be used for decoupling a system of N differential equations into N uncoupled oscillators.

- The representation of the dynamics on the configuration space does not allow assessing the velocities involved in the dynamics;
- Hence, the representation in the state-space appears as an interesting approach;
- As a first example, consider the forced 1-dof linear oscillator $\ddot{u} + 2\zeta\omega\dot{u} + \omega^2 u = p(t)/m$. We define can define the state-variables as $x_1 = u$ and $x_2 = \dot{u}$.
- The original equation is rewritten as:

$$\dot{x}_1 = x_2 \tag{37}$$

$$\dot{x}_2 = -\omega^2 x_1 - 2\zeta\omega x_2 + p(t)/m \tag{38}$$

- Defining $\boldsymbol{x} = \{x_1 \quad x_2\}^T$, Eqs. 37 and 38 can be rewritten in matrix form as:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ p(t)/m \end{Bmatrix} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b} \tag{39}$$

- In a more general form $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\mu}, t)$, $\boldsymbol{\mu}$ being a vector with parameters of the mathematical model (mass, damping, stiffness...). A system with this form is said to be non-autonomous, since it explicitly depends on time. An autonomous system has no explicit dependence on time and, hence, is given by the general form $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\mu})$.

- The state-space representation transforms an ordinary differential equation of second order into a system of first-order differential equations;
- For linear undamped systems with N degrees of freedom, the general equation of motion is given by Eq. 34. If we define $\boldsymbol{x} = \{u_1 \quad u_2 \dots u_N \quad \dot{u}_1 \quad \dot{u}_2 \dots \dot{u}_N\}^T$, the corresponding first-order system of differential equations is:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0_{N\times N} & \boldsymbol{I}_{N\times N} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & 0_{N\times N} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2N} \end{Bmatrix} + \begin{Bmatrix} 0_{N\times 1} \\ \boldsymbol{P}(t) \end{Bmatrix} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b} \tag{40}$$

- Notice that the damping matrix $\boldsymbol{C}$ can be easily included in Eq. 40.
- Notice also that in non-linear systems, $\boldsymbol{M}$, $\boldsymbol{C}$ or $\boldsymbol{K}$ depend on the state-vector $\boldsymbol{x}$;
- For non-autonomous systems, the dimension of the state-space is $2N$. On the other hand, this dimension is $2N+1$ for non-autonomous systems;
- It is worth mentioning that we can transform a non-autonomous system into an autonomous one. For this, we can include the state-variable $x_{N+1} = t$ and its derivative as $\dot{x}_{N+1} = 1$.

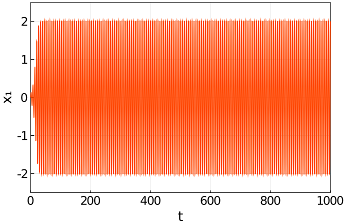
- Consider the van der Pol equation $\ddot{u} + \epsilon(u^2 - 1)\dot{u} + u = 0$
- Defining $x_1 = u$ and $x_2 = \dot{u}$, we have:

$$\dot{x}_1 = x_2 \tag{41}$$

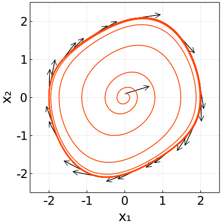
$$\dot{x}_2 = -x_1 - \epsilon(x_1^2 - 1)x_2 \tag{42}$$

- The solution of the van der Pol equation is a curve (orbit) in the phase-plane $(x_1(t); x_2(t))$. The tangent vector to this curve is $(\dot{x}_1; \dot{x}_2) = (x_2; -x_1 - \epsilon(x_1^2 - 1)x_2)$

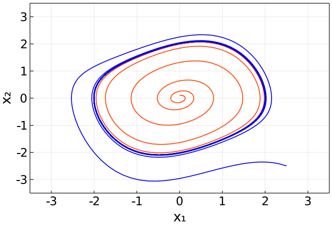
Example: van der Pol equation



(a) $u(t) = x_1(t)$.



(b) $x_2(t) \times x_1(t)$.



(c) $x_2(t) \times x_1(t)$.