# Equações de Diferenças

quinta-feira, 17 de setembro de 2020 12:44

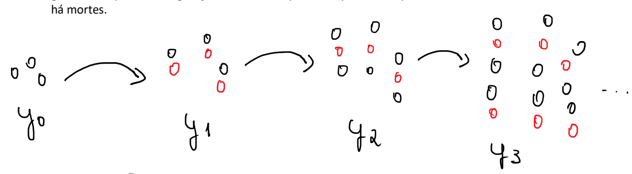
$$y_k = -a_1 y_{k-1} - a_2 y_{k-2} - \dots - a_n y_{k-n} + b_0 u_k + b_1 u_{k-1} + \dots + a_m u_{k-m}$$

Ex) $y_k = y_{k-1} + y_{k-2}$

$k=2$ (início) supondo $y_1 = y_0 = 1$

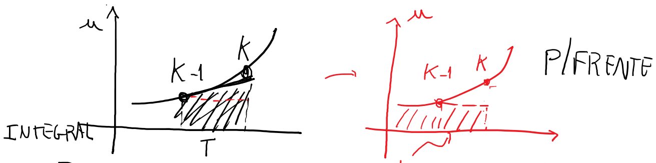
$y_2 = 2$
$y_3 = 3$
$y_4 = 5$
$\vdots$ SÉRIE FIBONACCI

EX2) Modelo de crescimento populacional
Suponha que u[k] representa um par de coelhos, e que os coelhos sejam sempre gerados em pares. Uma geração nova só se reproduz depois de um período. Não há mortes.




$$y[k] = \underbrace{y[k-1]}_{\text{GERAÇÃO ANTERIOR}} + \underbrace{y[k-2]}_{\text{RECEM NASCIDOS DOS COELHOS MADUROS}}$$

Ex) INTEGRAÇÃO TRAPEZOIDAL




$$y_k = y_{k-1} + \frac{T}{2}\cdot(u_k + u_{k-1})$$

$$y_k = y_{k-1} + T \cdot u_{k-1}$$

FUNÇÃO DE TRANSFERÊNCIA

$$Y(z) = \sum_{k=0}^{\infty} y_k \cdot z^{-k} \quad \text{(TRANSF. Z DO SINAL } y\text{)}$$

EX TRAPEZOIDAL)

$$\sum_0^{\infty} y_k z^{-k} = \sum_0^{\infty} y_{k-1} z^{-k} + \frac{T}{2}\left(\sum_0^{\infty} u_k z^{-k} + \sum_0^{\infty} u_{k-1} z^{-k}\right)$$

$\Downarrow$ $Y(z)$

$\Downarrow$ $k-1 = j$

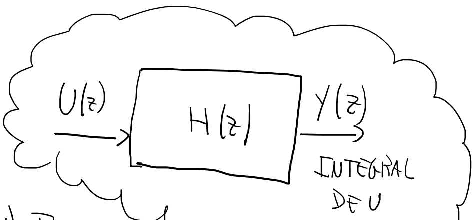
$$\sum_{j=-1}^{\infty} y_j \cdot z^{-j} \cdot z^{-1}$$

$$\left(\sum_0^{\infty} y_j z^{-j}\right) \cdot z^{-1}$$

$$Y(z) \cdot z^{-1}$$

$\Downarrow$ $U(z)$

$\Downarrow$ $U(z) \cdot z^{-1}$




$$Y(z)\cdot(1 - z^{-1}) = U(z)\cdot(1 + z^{-1})\frac{T}{2}$$

SAÍDA. ENT.

$$\boxed{\frac{Y(z)}{U(z)} = \frac{T}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right) = H(z)}$$

$$\text{SAÍDA} \to \boxed{\frac{Y(z)}{U(z)} = \frac{T}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right) = H(z)} \quad \text{ENT.} \to U(z)$$

GERAL

$$y_k = -a_1 y_{k-1} - a_2 y_{k-2} - \dots - a_m y_{k-m} + b_0 u_k + b_1 u_{k-1} + \dots + a_m u_{k-m}$$

$$Y(z) = \left[-a_1 z^{-1} - \dots - a_m z^{-m}\right] Y(z) + \left[b_0 + \dots + b_m z^{-m}\right] . U(z)$$

$$\boxed{\frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_m z^{-m}}}$$

SIGNIFICADO FÍSICO

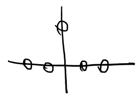
1) se $b_i = 0$ exceto $b_1 = 1$, $a_i = 0$ $\Rightarrow \frac{Y(z)}{U(z)} = z^{-1} \Rightarrow \boxed{y_k = u_{k-1}}$

delay de 1T

$$u \to \boxed{z^{-1}} \xrightarrow{y} u_{k-1}$$

2) IMPULSO

$$\delta_k = \begin{cases} 1 & \text{se } K = 0 \\ 0 & \text{se } K > 0 \end{cases}$$



$$Z[\delta_k] = 1$$

$$\underset{\text{IMPULSO}}{U(z)} \to \boxed{H(z)} \to Y(z)$$

$$U(z) = 1 \Rightarrow Y(z) = H(z)$$

$\Downarrow$

A FUNÇÃO DE TRANSF. $H(z)$ É A TRANSFORMADA Z DA RESPOSTA A UM IMPULSO UNITÁRIO

$$U(z) \to \boxed{H(z)} \to Y(z)$$

PLANTA + AD + DA

CONTROLE

⋮

ESTABILIDADE

ESTABILIDADE S.L. $\rightarrow h[K] \to 0$ p/ $K \to \infty$

EXPANSÃO FRAÇÕES PARCIAIS

$\boxed{\text{TERMOS}}$ - $\frac{1}{1-az^{-1}}$ ou $\frac{z^{-m}}{1-az^{-1}}$ $\infty$ POLOS $\boxed{z = a}$

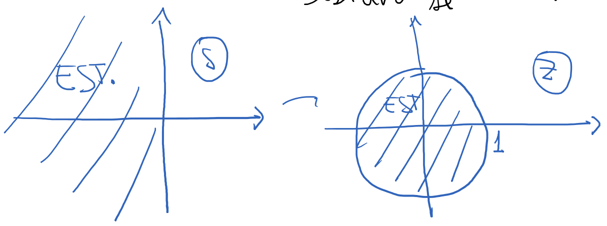
$$\frac{1}{1-az^{-1}} \to a^K \qquad \frac{z^{-m}}{1-az^{-1}} \to a^{K-m}$$

CONDIÇÃO ESTABILIDADE

estável se $|a| < 1$ ou $\boxed{|polo| < 1}$

EST ↑ (S) (Z)

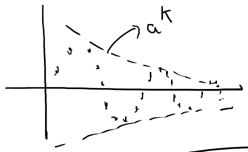
estável se $|a| < 1$ ou $\boxed{|polo| < 1}$




TERMO c/ POL. 2º GRAU)

$$\frac{a \,\text{sen}\,\omega T\, z^{-1}}{1 - 2a\cos\omega T\, z^{-1} + a^2 z^{-2}} \rightarrow \text{polos} = a(\cos\omega T \pm j\,\text{sen}\,\omega T)$$

$|\text{polos}| = a$ → estável $\boxed{|\text{polo}| < 1}$ GERAL




$$\downarrow$$

$$a^K \,\text{sen}\,\omega K T$$

estável se $|a| < 1$ → estável

Ex) $\boxed{y[K] = 0{,}5\, y[K-1] - 0{,}3\, y[K-2]}$

$y_0 = 1$
$y_1 = 1$

$y(2) = 0{,}2$
$y(3) = -0{,}2$
$y(4) = -0{,}16$
$y(5) = -0{,}02$
$\vdots$
$\rightarrow 0$

$$Y(z) = 0{,}5 z^{-1} Y(z) - 0{,}3 z^{-2} \cdot Y(z)$$

$$Y(z) \cdot \underbrace{(1 - 0{,}5z^{-1} + 0{,}3z^{-2})}_{\text{eq. CARACTERÍSTICA}} = \text{constante}$$

$$Y(z) = \frac{\text{constante}}{1 - 0{,}5z^{-1} + 0{,}3z^{-2}} \quad *$$

POLOS

$P = 0{,}25 \pm 0{,}48j \Rightarrow |P| = 0{,}58 \Rightarrow$ estável

$$y[K+2] = 0{,}5\, y[K+1] - 0{,}3\, y[K]$$

$$z^2 \cdot Y(z) - z^2 \cancel{y(0)}^{\,1} - z \cdot \cancel{y(1)}^{\,1} = 0{,}5\left(Y(z) \cdot z - z \cdot \cancel{y(0)}^{\,1}\right) - 0{,}3\,Y(z)$$

$$Y(z)\left[z^2 - 0{,}5z + 0{,}3\right] = z^2 - 0{,}5z$$

$$Y(z) = \frac{z^2 - 0{,}5z}{z^2 - 0{,}5z + 0{,}3} \quad *$$

| | |
|---|---|
| $x(t+T)$ or $x(k+1)$ | $zX(z) - zx(0)$ |
| $x(t+2T)$ | $z^2X(z) - z^2x(0) - zx(T)$ |
| $x(k+2)$ | $z^2X(z) - z^2x(0) - zx(1)$ |

# Estabilidade (cont)

quinta-feira, 17 de setembro de 2020 12:55

## CRITÉRIO DE JURY

If the characteristic polynomial of the system is given by

$$f(z) = a_n + a_{n-1}z^1 + a_{n-2}z^2 + \cdots + a_1 z^{n-1} + a_0 z^n$$

then the table is constructed as follows:[1]

| row | $z^n$ | $z^{n-1}$ | $z^{n-2}$ | $z^{\cdots}$ | $z^1$ | $z^0$ |
|---|---|---|---|---|---|---|
| 1 | $a_0$ | $a_1$ | $a_2$ | ... | $a_{n-1}$ | $a_n$ |
| 2 | $a_n$ | $a_{n-1}$ | $a_{n-2}$ | ... | $a_1$ | $a_0$ |
| 3 | $b_0$ | $b_1$ | ... | $b_{n-2}$ | $b_{n-1}$ | |
| 4 | $b_{n-1}$ | $b_{n-2}$ | ... | $b_1$ | $b_0$ | |
| 5 | $c_0$ | $c_1$ | ... | $c_{n-2}$ | | |
| 6 | $c_{n-2}$ | $c_{n-3}$ | ... | $c_0$ | | |
| ... | ... | ... | ... | ... | ... | ... |
| 2n-5 | $p_0$ | $p_1$ | $p_2$ | p3 | | |
| 2n-4 | $p_3$ | $p_2$ | $p_1$ | p0 | | |
| 2n-3 | $q_2$ | $q_1$ | $q_0$ | | | |

That is, the first row is constructed of the polynomial coefficients in order, and the second row is the first row in reverse order and conjugated.

The third row of the table is calculated by subtracting $\frac{a_n}{a_0}$ times the second row from the first row, and the fourth row is the third row with the first n elements reversed (as the final element is zero).

$$\begin{array}{cccc}
a_0 \quad a_1 & & \cdots\ a_{n-1} & a_n \\
a_n \quad a_{n-1} & & \cdots\ a_1 & a_0 \\
\left(a_0 - a_n\frac{a_n}{a_0}\right) \left(a_1 - a_{n-1}\frac{a_n}{a_0}\right) & & \cdots \left(a_{n-1} - a_1\frac{a_n}{a_0}\right) & 0 \\
\left(a_{n-1} - a_1\frac{a_n}{a_0}\right) \cdots & \left(a_1 - a_{n-1}\frac{a_n}{a_0}\right) & \left(a_0 - a_n\frac{a_n}{a_0}\right) & 0
\end{array}$$

The expansion of the table is continued in this manner until a row containing only one non zero element is reached.

estável se $a_0, b_0, c_0, \ldots > 0$

$|polos| < 1$

$$a(z) = z^3 - 2{,}1z^2 + 1{,}6z - 0{,}4$$

| | | | |
|---|---|---|---|
| 1 | -2,1 | 1,6 | -0,4 |
| -0,4 | 1,6 | -2,1 | 1 |
| 0,84 | -1,46 | 0,76 | |
| 0,76 | -1,46 | 0,84 | |
| 0,15 | -0,14 | | |
| -0,14 | 0,15 | | |
| 0,025 | | | |

$$b_0 = \frac{a_0^2 - a_n^2}{a_0}$$

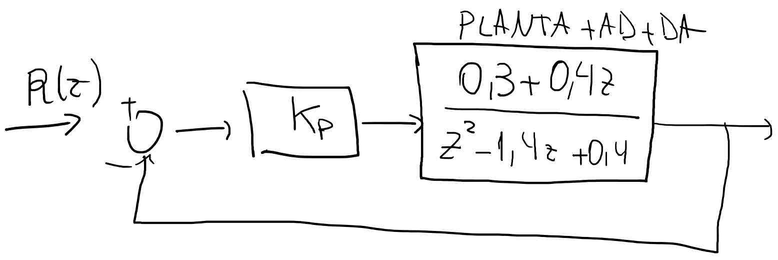
$$C_K = \frac{b_K b_0 - b_{m-1} b_{m-1-K}}{b_0}$$

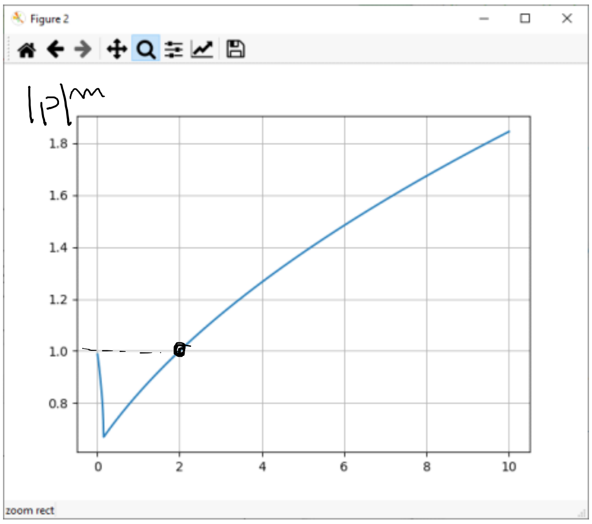
$$b_1 = \frac{a_0 a_1 - a_m a_{m-1}}{a_0}$$

$$b_K = \frac{a_0 a_K - a_m a_{m-K}}{a_0}$$

$a_0, b_0, c_0, d_0 > 0 \Rightarrow$ ESTÁVEL

Ex2)






$Kp < 2$

Solução de Eq diferenças

quinta-feira, 17 de setembro de 2020 14:21

$x(K+4) \Rightarrow z^4 X(z) - z^4 x(0) - z^3 x(1) - z^2 x(2) - z x(3)$

$x(K+3) \Rightarrow z^3 X(z) - z^3 x(0) - z^2 x(1) - z x(2)$

$x(K+2) \Rightarrow z^2 X(z) - z^2 x(0) - z x(1)$

$x(K+1) \Rightarrow z X(z) - z x(0)$

$x(K) \Rightarrow X(z)$

$x(K-1) \Rightarrow z^{-1} X(z)$

$x(K-n) \Rightarrow z^{-n} X(z)$

RESOLVER) $\begin{cases} x(K+2) + 3x(K+1) + 2x(K) = 0 \\ x(0) = 0 \\ x(1) = 1 \end{cases}$

$x(K+2) \rightarrow z^2 X(z) - z^2 . \cancel{x(0)}^{0} \; \underline{-z x(1)}$

$x(K+1) \rightarrow z X(z) - z . x(0)$

$z^2 X(z) + 3z X(z) + 2X(z) = z$

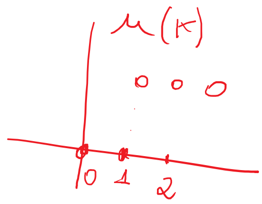
$X(z) = \dfrac{z}{z^2+3z+2} = \dfrac{z}{z+1} - \dfrac{z}{z+2}$

$X(z) = \dfrac{1}{1+z^{-1}} - \dfrac{1}{1+2z^{-1}} \Rightarrow \boxed{x(K) = (-1)^K - (-2)^K}$

$\downarrow \qquad\qquad \downarrow$

$(-1)^K \qquad (-2)^K$

2) $\begin{cases} x(K) - x(K-1) + 0{,}25\, x(K-2) = u(K) \\ \boxed{\begin{matrix} x(0) = 2 \\ x(1) = 3 \end{matrix}} \\ u(K) = 1 \text{ p/ } K \geq 1 \end{cases}$




$\begin{cases} u(n+2) = u'(n) \\ \text{c/ } u'(n) = 1 \text{ p/ } n \geq 0 \end{cases}$

Pois $u'(0) = u(2) = 1$

$K - 2 = n$

$x(n+2) - x(n+1) + 0{,}25\, x(n) = u(n+2) = u'(n) \qquad Z\{u'\} = \dfrac{z}{z-1}$

$z^2 X - z^2 x(0) - z x(1) - z . X - z . x(0) + 0{,}25 X = \dfrac{z}{z-1}$

$$z^2 X - z^2 x(0) - z\,x(1) - z \cdot X - z \cdot x(0) + 0{,}25 X = \frac{z}{z-1}$$

$$\boxed{X(z) = \frac{2z^3 - z^2}{(z-1)(z^2 - z + 0{,}25)}} \rightarrow \boxed{x(k) = 4\left(1 - 0{,}5^{k+1}\right)}$$

$k=0 \quad x(0) = 2$ ✓

$k=1 \quad x(0) = 3$ ✓

$k=2 \quad x(2) = 3{,}5$

$\vdots$