Lista de exercícios 3 – Equação de Euler (do livro de Boyce e Di Prima)

Problems

In each of Problems 1 through 8, determine the general solution of the given differential equation that is valid in any interval not including the singular point.

1. $x^2y'' + 4xy' + 2y = 0$

2. $(x+1)^2y'' + 3(x+1)y' + 0.75y = 0$

3. $x^2y'' - 3xy' + 4y = 0$

4. $x^2y'' - xy' + y = 0$

5. $x^2y'' + 6xy' - y = 0$

6. $2x^2y'' - 4xy' + 6y = 0$

7. $x^2y'' - 5xy' + 9y = 0$

8. $(x-2)^2y'' + 5(x-2)y' + 8y = 0$

In each of Problems 9 through 11, find the solution of the given initial-value problem. Plot the graph of the solution and describe how the solution behaves as $x \to 0$.

G **9.** $2x^2y'' + xy' - 3y = 0, \quad y(1) = 1, \quad y'(1) = 4$

G **10.** $4x^2y'' + 8xy' + 17y = 0, \quad y(1) = 2, \quad y'(1) = -3$

G **11.** $x^2y'' - 3xy' + 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 3$

In each of Problems 12 through 23, find all singular points of the given equation and determine whether each one is regular or irregular.

12. $xy'' + (1-x)y' + xy = 0$

13. $x^2(1-x)^2y'' + 2xy' + 4y = 0$

14. $x^2(1-x)y'' + (x-2)y' - 3xy = 0$

15. $x^2(1-x^2)y'' + \left(\frac{2}{x}\right)y' + 4y = 0$

16. $(1-x^2)^2y'' + x(1-x)y' + (1+x)y = 0$

17. $x^2y'' + xy' + (x^2 - \nu^2)y = 0 \quad$ (Bessel equation)

18. $(x+2)^2(x-1)y'' + 3(x-1)y' - 2(x+2)y = 0$

19. $x(3-x)y'' + (x+1)y' - 2y = 0$

20. $xy'' + e^xy' + (3\cos x)y = 0$

21. $y'' + (\ln|x|)y' + 3xy = 0$

22. $(\sin x)y'' + xy' + 4y = 0$

23. $(x \sin x)y'' + 3y' + xy = 0$

24. Find all values of α for which all solutions of $x^2y'' + \alpha xy' + \frac{5}{2}y = 0$ approach zero as $x \to 0$.

25. Find all values of β for which all solutions of $x^2y'' + \beta y = 0$ approach zero as $x \to 0$.

26. Find γ so that the solution of the initial-value problem $x^2y'' - 2y = 0$, $y(1) = 1$, $y'(1) = \gamma$ is bounded as $x \to 0$.

27. Consider the Euler equation $x^2y'' + \alpha xy' + \beta y = 0$. Find conditions on α and β so that:

- **a.** All solutions approach zero as $x \to 0$.
- **b.** All solutions are bounded as $x \to 0$.
- **c.** All solutions approach zero as $x \to \infty$.
- **d.** All solutions are bounded as $x \to \infty$.
- **e.** All solutions are bounded both as $x \to 0$ and as $x \to \infty$.

28. Using the method of reduction of order, show that if r_1 is a repeated root of

$$r(r-1) + \alpha r + \beta = 0,$$

then x^{r_1} and $x^{r_1}\ln x$ are solutions of $x^2y''+\alpha xy'+\beta y = 0$ for $x > 0$.

29. Verify that $W[x^\lambda\cos(\mu\ln x), x^\lambda\sin(\mu\ln x)] = \mu x^{2\lambda-1}$.

In each of Problems 30 and 31, show that the point $x = 0$ is a regular singular point. In each problem try to find solutions of the form $\sum_{n=0}^{\infty} a_nx^n$. Show that (except for constant multiples) there is only one nonzero solution of this form in Problem 30 and that there are no nonzero solutions of this form in Problem 31. Thus in neither case can the general solution be found in this manner. This is typical of equations with singular points.

30. $2xy'' + 3y' + xy = 0$

31. $2x^2y'' + 3xy' - (1+x)y = 0$

32. **Singularities at Infinity.** The definitions of an ordinary point and a regular singular point given in the preceding sections apply only if the point x_0 is finite. In more advanced work in differential equations, it is often necessary to consider the point at infinity. This is done by making the change of variable $\xi = 1/x$ and studying the resulting equation at $\xi = 0$. Show that, for the differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

the point at infinity is an ordinary point if

$$\frac{1}{P(1/\xi)}\left(\frac{2P(1/\xi)}{\xi} - \frac{Q(1/\xi)}{\xi^2}\right) \quad \text{and} \quad \frac{R(1/\xi)}{\xi^4P(1/\xi)}$$

have Taylor series expansions about $\xi = 0$. Show also that the point at infinity is a regular singular point if at least one of the above functions does not have a Taylor series expansion, but both

$$\frac{\xi}{P(1/\xi)}\left(\frac{2P(1/\xi)}{\xi} - \frac{Q(1/\xi)}{\xi^2}\right) \quad \text{and} \quad \frac{R(1/\xi)}{\xi^2P(1/\xi)}$$

do have such expansions.

In each of Problems 33 through 37, use the results of Problem 32 to determine whether the point at infinity is an ordinary point, a regular singular point, or an irregular singular point of the given differential equation.

33. $y'' + y = 0$

34. $x^2y'' + xy' - 4y = 0$

35. $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0 \quad$ (Legendre equation)

36. $y'' - 2xy' + \lambda y = 0 \quad$ (Hermite equation)

37. $y'' - xy = 0 \quad$ (Airy equation)