Lista de exercícios 6 – Equação de Bessel (do livro de Boyce e Di Prima)

Problems

In each of Problems 1 through 3, show that the given differential equation has a regular singular point at $x = 0$, and determine two solutions for $x > 0$.

1. $x^2y'' + 2xy' + xy = 0$

2. $x^2y'' + 3xy' + (1+x)y = 0$

3. $x^2y'' + xy' + 2xy = 0$

4. Find two solutions (not multiples of each other) of the Bessel equation of order $\frac{3}{2}$

$$x^2y'' + xy' + \left(x^2 - \frac{9}{4}\right)y = 0, \quad x > 0.$$

5. Show that the Bessel equation of order one-half

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0, \quad x > 0$$

can be reduced to the equation

$$v'' + v = 0$$

by the change of dependent variable $y = x^{-1/2}v(x)$. From this, conclude that $y_1(x) = x^{-1/2}\cos x$ and $y_2(x) = x^{-1/2}\sin x$ are solutions of the Bessel equation of order one-half.

6. Show directly that the series for $J_0(x)$, equation (7), converges absolutely for all x.

7. Show directly that the series for $J_1(x)$, equation (27), converges absolutely for all x and that $J_0'(x) = -J_1(x)$.

8. Consider the Bessel equation of order ν

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \quad x > 0,$$

where ν is real and positive.

a. Show that $x = 0$ is a regular singular point and that the roots of the indicial equation are ν and $-\nu$.

b. Corresponding to the larger root ν, show that one solution is

$$y_1(x) = x^{\nu}\left(1 - \frac{1}{1!(1+\nu)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!(1+\nu)(2+\nu)}\left(\frac{x}{2}\right)^4 + \sum_{m=3}^{\infty}\frac{(-1)^m}{m!(1+\nu)\cdots(m+\nu)}\left(\frac{x}{2}\right)^{2m}\right).$$

c. If 2ν is not an integer, show that a second solution is

$$y_2(x) = x^{-\nu}\left(1 - \frac{1}{1!(1-\nu)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!(1-\nu)(2-\nu)}\left(\frac{x}{2}\right)^4 + \sum_{m=3}^{\infty}\frac{(-1)^m}{m!(1-\nu)\cdots(m-\nu)}\left(\frac{x}{2}\right)^{2m}\right).$$

Note that $y_1(x) \to 0$ as $x \to 0$, and that $y_2(x)$ is unbounded as $x \to 0$.

d. Verify by direct methods that the power series in the expressions for $y_1(x)$ and $y_2(x)$ converge absolutely for all x. Also verify that y_2 is a solution, provided only that ν is not an integer.

9. In this section we showed that one solution of Bessel's equation of order zero

$$L[y] = x^2y'' + xy' + x^2y = 0$$

is J_0, where $J_0(x)$ is given by equation (7) with $a_0 = 1$. According to Theorem 5.6.1, a second solution has the form ($x > 0$)

$$y_2(x) = J_0(x)\ln x + \sum_{n=1}^{\infty} b_n x^n.$$

a. Show that

$$L[y_2](x) = \sum_{n=2}^{\infty} n(n-1)b_n x^n + \sum_{n=1}^{\infty} nb_n x^n + \sum_{n=1}^{\infty} b_n x^{n+2} + 2xJ_0'(x). \quad (34)$$

b. Substituting the series representation for $J_0(x)$ in equation (34), show that

$$b_1x + 2^2b_2x^2 + \sum_{n=3}^{\infty}(n^2b_n + b_{n-2})x^n = -2\sum_{n=1}^{\infty}\frac{(-1)^n 2nx^{2n}}{2^{2n}(n!)^2}. \quad (35)$$

c. Note that only even powers of x appear on the right-hand side of equation (35). Show that $b_1 = b_3 = b_5 = \cdots = 0$, $b_2 = \dfrac{1}{2^2(1!)^2}$, and that

$$(2n)^2b_{2n} + b_{2n-2} = -2\frac{(-1)^n(2n)}{2^{2n}(n!)^2}, \quad n = 2, 3, 4, \ldots .$$

Deduce that

$$b_4 = -\frac{1}{2^2\,4^2}\left(1 + \frac{1}{2}\right) \quad \text{and} \quad b_6 = \frac{1}{2^2\,4^2\,6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right).$$

The general solution of the recurrence relation is $b_{2n} = \dfrac{(-1)^{n+1}H_n}{2^{2n}(n!)^2}$. Substituting for b_n in the expression for $y_2(x)$, we obtain the solution given in equation (10).

10. Find a second solution of Bessel's equation of order one by computing the $c_n(r_2)$ and a of equation (24) of Section 5.6 according to the formulas (19) and (20) of that section. Some guidelines along the way of this calculation are the following. First, use equation (24) of this section to show that $a_1(-1)$ and $a_1'(-1)$ are 0. Then show that $c_1(-1) = 0$ and, from the recurrence relation, that $c_n(-1) = 0$ for $n = 3, 5, \ldots$. Finally, use equation (25) to show that

$$a_2(r) = -\frac{a_0}{(r+1)(r+3)},$$

$$a_4(r) = \frac{a_0}{(r+1)(r+3)(r+3)(r+5)},$$

and that

$$a_{2m}(r) = \frac{(-1)^m a_0}{(r+1)\cdots(r+2m-1)(r+3)\cdots(r+2m+1)}, \; m \geq 3.$$

Then show that

$$c_{2m}(-1) = \frac{(-1)^{m+1}\left(H_m + H_{m-1}\right)}{2^{2m}m!(m-1)!}, \quad m \geq 1.$$