2020-2, "FISMAT-AV", AULA 03

OBJETIVO: DISCUTIR MELHOR A DERIVADA FUNCIONAL

(CONT.) 1.3 DERIVADA FUNCIONAL

DA AULA ANTERIOR:

DERIVADA DE GÂTEAUX OU PRIMEIRA VARIAÇÃO

$$\delta J(f,h) \equiv \lim_{\varepsilon \to 0} \frac{J(f+\varepsilon h) - J(f)}{\varepsilon}$$

$x = (x^1, \dots, x^n) \in \mathbb{R}^n$

$$\overset{*}{=} \int_{\mathbb{R}^n} d^n x \, h(x) \boxed{\frac{\delta J}{\delta f(x)}}$$

DERIVADA FUNCIONAL

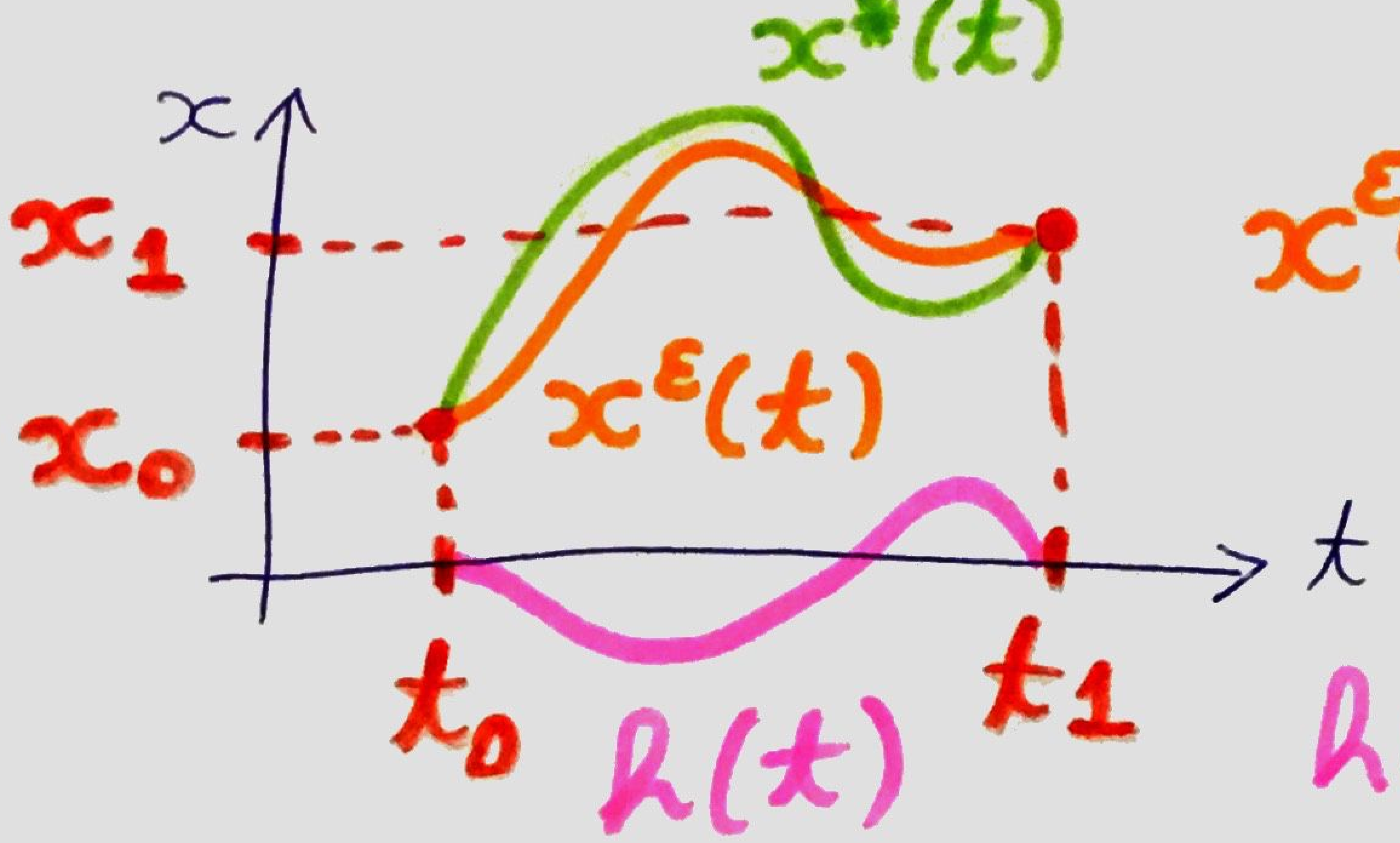
→ δJ LINEAR EM h

→ h SE ANULA NA FRONTEIRA DA REGIÃO DE INTEGRAÇÃO

∴ EXISTE A "REPRESENTAÇÃO INTEGRAL"

◊ EXEMPLO: EULER-LAGRANGE

MOVIMENTO 1D DE UMA PARTÍCULA



$$x^\varepsilon(t) = x^*(t) + \varepsilon . h(t)$$

$$h(t_0) = 0 = h(t_1)$$

$$S(x) = \int_{t_0}^{t_1} L(x, \dot{x}, t)\, dt$$

$$\delta S^\varepsilon \equiv S(x^\varepsilon) - S(x^*)$$

$$= \int_{t_0}^{t_1} \left[\varepsilon . h \frac{\partial L}{\partial x} + \varepsilon . \dot{h} \frac{\partial L}{\partial \dot{x}} \right] dt + \mathcal{O}(\varepsilon^2) =$$

PARTES

$$\overset{0}{=} \left[\varepsilon \frac{\partial L}{\partial \dot{x}} h(t) \right]_{t_0}^{t_1} + \int_{t_0}^{t_1} dt \left[\varepsilon . h \frac{\partial L}{\partial x} - \varepsilon h \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \right) \right] +$$

$$+ \mathcal{O}(\varepsilon^2) \Rightarrow$$

$$\Rightarrow \delta S(x, h) = \int_{t_0}^{t_1} dt . h(t) \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \right) \right]$$

$$\frac{\delta S}{\delta x(t)} \text{ ''}$$

$$\frac{\delta S}{\delta x^*(t)} = 0 \implies \boxed{\frac{\partial L}{\partial x^*} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^*}\right) = 0} \quad \lozenge$$

CONDIÇÃO DE EXTREMO

* PROPRIEDADES

SE $J_1 : A \to \mathbb{R}$ E $J_2 : A \to \mathbb{R}$ FOREM FUNCIONAIS, $c_1, c_2 \in \mathbb{R}$ E $\Phi : \mathbb{R} \to \mathbb{R}$, $\Phi \in C^1$,

LINEARIDADE

(i) $$\boxed{\frac{\delta}{\delta f(x)}(c_1 J_1 + c_2 J_2) = c_1 \frac{\delta J_1}{\delta f(x)} + c_2 \frac{\delta J_2}{\delta f(x)}}$$

(ii) $$\boxed{\frac{\delta}{\delta f(x)}(J_1 J_2) = \frac{\delta J_1}{\delta f(x)} J_2 + J_1 \frac{\delta J_2}{\delta f(x)}}$$ E

LEIBNITZ

CADEIA

(iii) $$\boxed{\frac{\delta \tilde{\Phi}}{\delta f(x)} = \frac{d\Phi}{dF} \cdot \frac{\delta F}{\delta f(x)}}$$, ONDE $F : A \to \mathbb{R}$,

$\tilde{\Phi} : A \to \mathbb{R}$ E $\tilde{\Phi} = \Phi \circ F$, $f \xmapsto{\tilde{\Phi}} \Phi(F(f))$.

◇ EXEMPLOS:

i. $J(f) = \int_{\mathbb{R}^n} d^n x \, . \, f(x)$

$\delta J^{\varepsilon} \equiv J(f+\varepsilon h) - J(f) = \int_{\mathbb{R}^n} d^n x \Big\{ \big[\cancel{f(x)} + \varepsilon h(x) \big] - \cancel{f(x)} \Big\} \Rightarrow$

$\Rightarrow \delta J(f,h) = \int_{\mathbb{R}^n} d^n x \, . \, h(x) \, . \, 1$

$$\therefore \boxed{\frac{\delta J}{\delta f(x)} = 1}.$$

ii. $J(f) = \int dx \, [f(x)]^2$

$\delta J^{\varepsilon} = \int dx \Big\{ \big[f(x) + \varepsilon h(x) \big]^2 - \big[f(x) \big]^2 \Big\}$

$= \varepsilon \int dx \, . \, h(x) \big[2 f(x) \big] + \varepsilon^2 \int dx \, h^2(x) \Rightarrow$

$\Rightarrow \delta J(f,h) = \int dx \, h(x) \big[2 f(x) \big]$

$$\therefore \boxed{\frac{\delta J}{\delta f(x)} = 2 f(x)}$$

iii. $J(f)=\int dx\,[f'(x)]^2$

$$\delta J^{\varepsilon}=\int dx\left\{[f'(x)+\varepsilon h'(x)]^2-[f'(x)]^2\right\}\Rightarrow$$

$$\Rightarrow \delta J(f,h)=\int_{\Omega} dx\, h'(x)[2f'(x)]=$$

$$=[2h(x)f'(x)]_{\partial\Omega}-\int_{\Omega} dx.h(x)[2f''(x)]$$

$$h(a)=0=h(b)$$

$$\therefore \boxed{\frac{\delta J}{\delta f(x)}=-2f''(x)}$$

iv. FUNCIONAL DEPENDENTE DE UM PARÂMETRO

FUNÇÃO <u>CONHECIDA</u>

$$J_x(f)=\int_{\mathbb{R}^n} d^nx'\,K(x,x')\,f(x')\Rightarrow$$

$$\Rightarrow \delta J_x^{\varepsilon}=\int_{\mathbb{R}^n} d^nx'.K(x,x')[\cancel{f(x')}+\varepsilon h(x')-\cancel{f(x')}]$$

$$\therefore \frac{\delta J_x}{\delta f(x')}=K(x,x') \quad \text{OU} \quad \boxed{\frac{\delta J_x}{\delta f(y)}=K(x,y)}.$$

◊

* DELTA DE DIRAC

ALGUNS AUTORES RECORREM À DELTA DE DIRAC PARA ESCREVEREM UMA DEFINIÇÃO EXPLÍCITA DA DERIVADA FUNCIONAL,

$h(x) \to \delta_y(x)$

$\delta(x-y)$

$$\frac{\delta J}{\delta f(y)} = \lim_{\varepsilon \to 0} \frac{J(f+\varepsilon \delta_y) - J(f)}{\varepsilon}$$

$$= \int_{\mathbb{R}^n} d^n x \, . \, \delta(x-y) \frac{\delta J}{\delta f(x)} .$$

A DELTA DE DIRAC TAMBÉM PERMITE QUE "UMA FUNÇÃO SEJA UM FUNCIONAL DE SI MESMA":

$K(x,x')$ NO EXEMPLO (10).

$$f(x) = \int_{\mathbb{R}^n} d^n x' \, . \, \delta(x-x') \, . \, f(x')$$

$$\Downarrow$$

$$\boxed{\frac{\delta f(x)}{\delta f(y)} = \delta(x-y)}$$

EXERCÍCIOS

AULA 03

PRIMEIRO

$\boxed{03.1}$ USE A DEFINIÇÃO IMPLÍCITA DA DERIVADA FUNCIONAL E AS PROPRIEDADES DE LIMITES/DERIVADAS PARA DEMONSTRAR A PROPRIEDADE i, DA PÁGINA 03.

$\boxed{03.2}$ COMO ACIMA, PARA ii.

$\boxed{03.3}$ IDEM, PARA iii.

$\boxed{03.4}$ $J(f) = \int dx \, [f(x)]^n \, ; \; \dfrac{\delta J}{\delta f(x)} = ?$

$\boxed{03.5}$ A DERIVADA DA DELTA DE DIRAC É FORMALMENTE DEFINIDA PELA PROPRIEDADE $\displaystyle\int_{-\infty}^{+\infty} \delta'(x) . f(x) \, dx = -f'(0)$.

MOSTRE QUE $\dfrac{\delta f'(x)}{\delta f(y)} = \delta'(x-y)$.

$\boxed{03.6}$ $J(f) = \int d^n y \cdot d^n z \, K(y,z) f(y) f(z)$

MOSTRE QUE

$$\frac{\delta J}{\delta f(x)} = \int d^n y \, [K(x,y) + K(y,x)] \, f(y).$$

?

OK: $\dfrac{\delta}{\delta f(x)} \int d^n y \ldots = \int d^n y \dfrac{\delta}{\delta f(x)} \ldots$

?

$\boxed{03.7}$ USANDO A REGRA DA CADEIA E

$\dfrac{\delta}{\delta f(x)} \int_{\mathbb{R}^n} d^n y [f(y)]^n = n[f(x)]^{n-1}$, MOS-

TRE QUE A EQUAÇÃO DIFERENCIAL FUN-

CIONAL

$$\frac{\delta J}{\delta f(x)} + f(x) \cdot J = 0$$

ADMITE A SOLUÇÃO (C CONSTANTE)

$$J(f) = C \cdot \exp\left\{ -\frac{1}{2} \int_{\mathbb{R}^n} d^n x \, [f(x)]^2 \right\}.$$