PME 3201 Aula 1 03.09.2020 Turma 12

$$\boxed{\frac{dx(t)}{dt} = \alpha\, x(t)} \Longrightarrow x(t)$$

Equação Diferencial Ordinária

Linear de 1º Ordem

de um sistema autômato

Separação de variáveis:

$$\frac{dx}{dt} = \alpha x \Rightarrow \frac{1}{x}\,dx = \alpha\, dt \Rightarrow \int_{x_0}^{x} \frac{1}{x}\,dx = \int_{t_0=0}^{t} \alpha\, dt$$

$$\ln x \Big|_{x_0}^{x} = \alpha t \Rightarrow \ln\left(\frac{x}{x_0}\right) = \alpha t \Rightarrow \boxed{x(t) = x_0\, e^{\alpha t}} \mid \alpha \in \mathbb{R}$$

Solução Geral

$x(0) = x_0$

$x(t)$ uma outra possível solução:

$$\frac{dx}{dt} = \alpha \cdot x$$

$$x(t) \cdot e^{-\alpha t}$$

$$\frac{d}{dt}\left(x(t) \cdot e^{-\alpha t}\right) = \frac{dx}{dt} \cdot e^{-\alpha t} - x \cdot \alpha\, e^{-\alpha t}$$

$$= \alpha x e^{-\alpha t} - \alpha x e^{-\alpha t} = 0$$

$$\therefore \frac{d}{dt}\left(x(t) e^{-\alpha t}\right) = 0 \Rightarrow x(t)\, e^{-\alpha t} = x_0$$

$$\boxed{x(t) = x_0 e^{\alpha t}} \quad \Big| \; \alpha \in \mathbb{R}$$

Solução Geral e Única

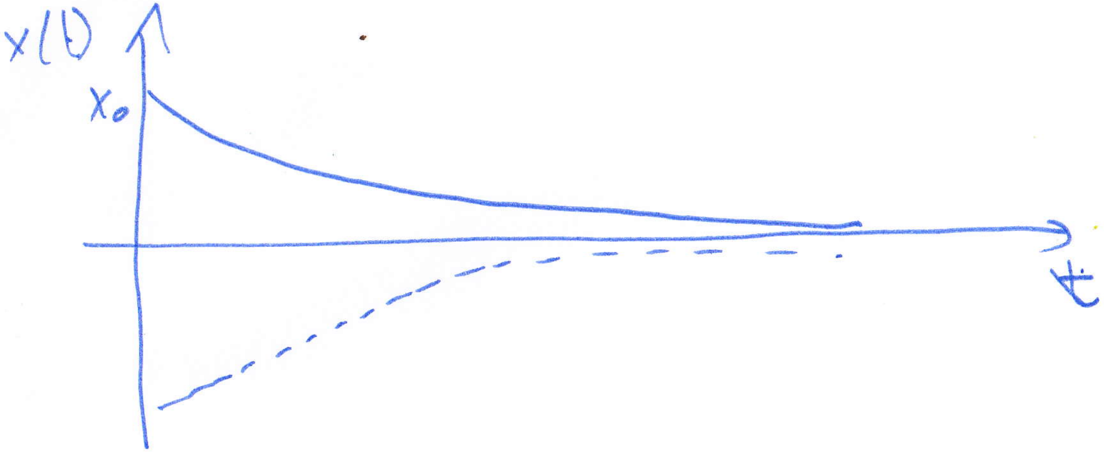
$$x(0) = x_0$$

condição inicial.

$$\frac{dx}{dt} = \alpha x \implies x(t) = x_0 e^{\alpha t}$$

i) $\alpha < 0$ decrescente

$$\lim_{t \to \infty} x_0 e^{\alpha t} = 0$$



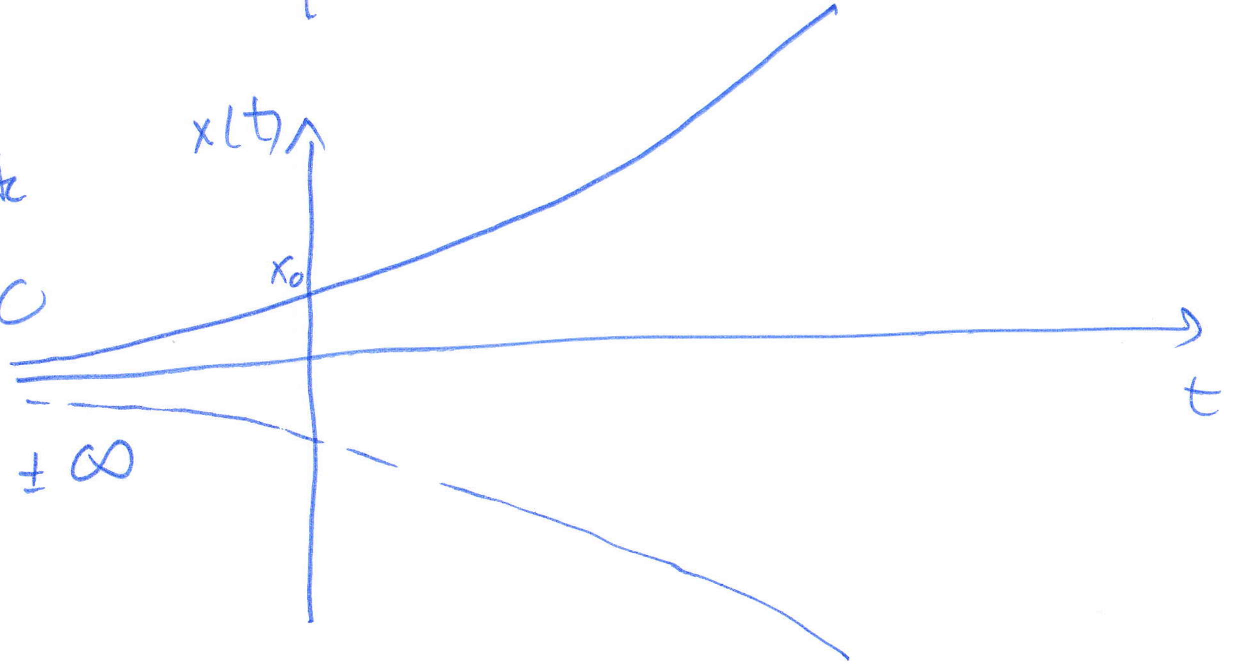
ii) $\alpha = 0$ estacionário $\frac{dx}{dt} = 0$

$$x(t) = x_0$$

iii) $\alpha > 0$ crescente

$$\lim_{t \to -\infty} x_0 e^{\alpha t} = 0$$

$$\lim_{t \to +\infty} x_0 e^{\alpha t} = \pm\infty$$



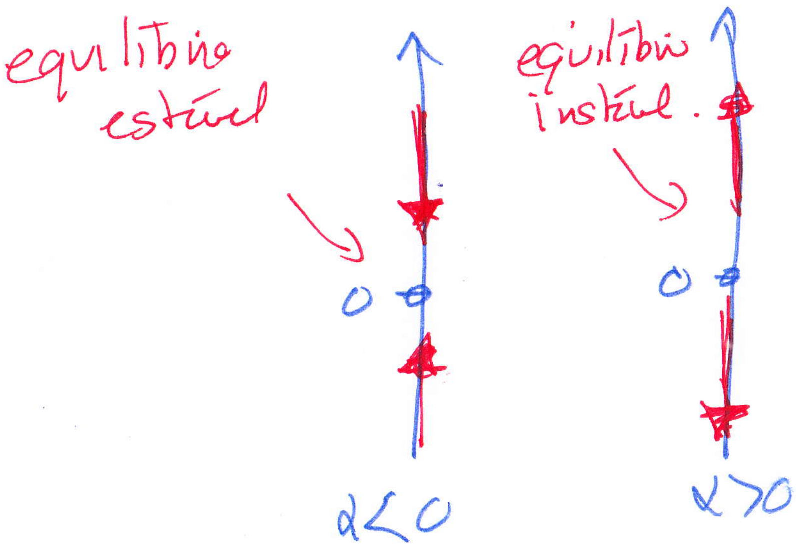
equilíbrio

$$\frac{dx}{dt} = f(x,t)$$

$$\frac{dx}{dt} = 0 \Rightarrow \alpha x = 0 \Rightarrow \underbrace{x = 0}_{\text{equilíbrio}}$$

$$\alpha \neq 0$$

linha de fase



estabilidade

$$\frac{d}{dx}\left(\frac{dx}{dt}\right) = \frac{d}{dx}(\alpha x) = \alpha$$

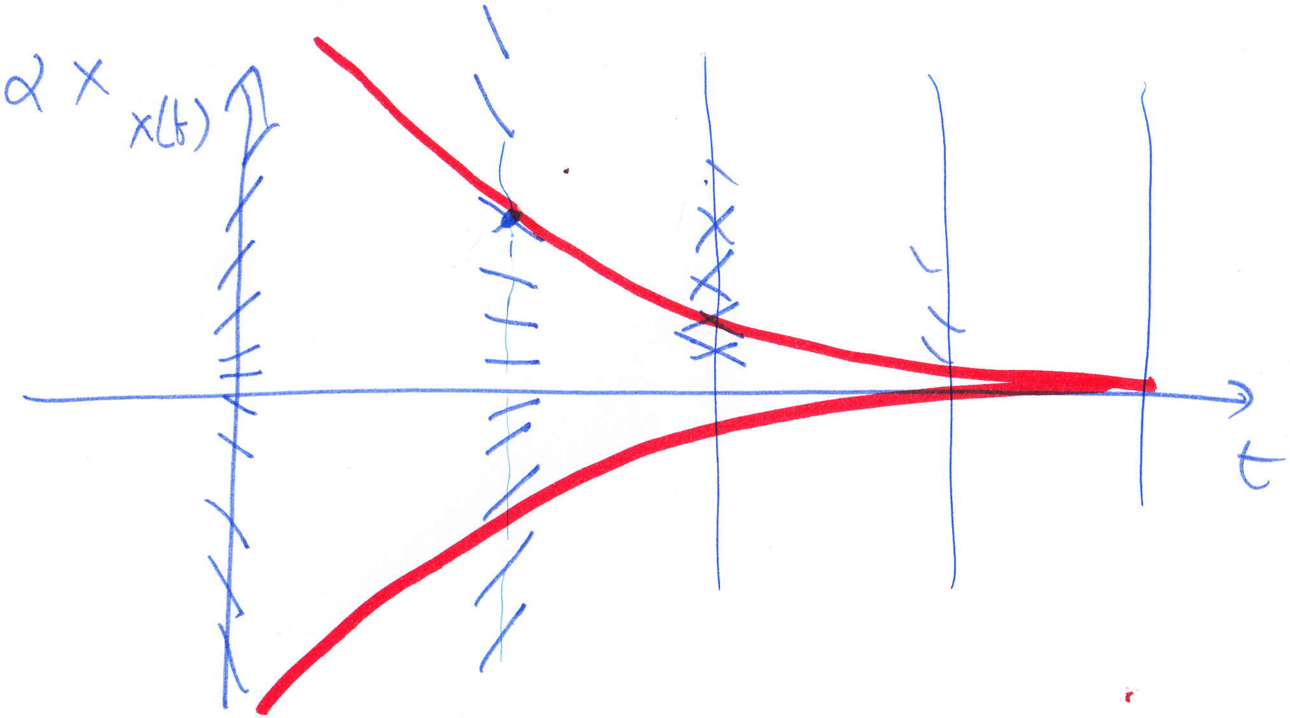
$$\alpha > 0 \Rightarrow \frac{df}{dx} > 0 \rightarrow \text{crescente}$$

$$\alpha = 0 \Rightarrow \frac{df}{dx} = 0 \rightarrow \text{estacionário}$$

$$\alpha < 0 \Rightarrow \frac{df}{dx} < 0 \rightarrow \text{decrescente}$$

$$f(x,t) = \frac{dx}{dt} = \alpha x$$

$\alpha < 0$



$\alpha > 0$

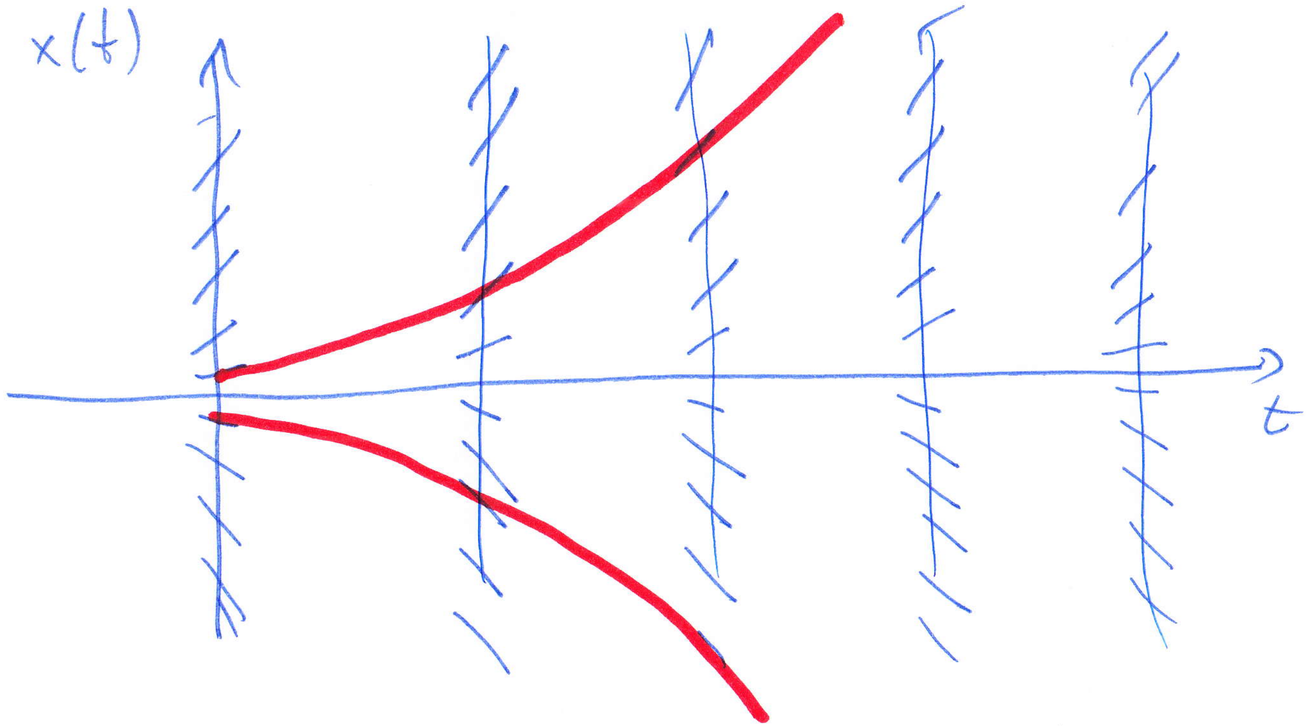
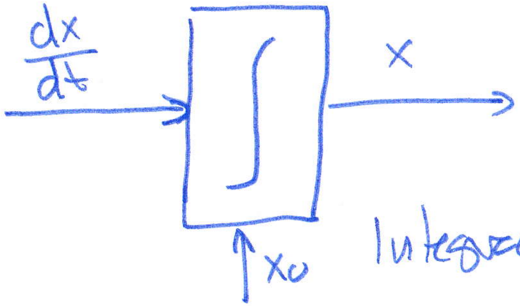


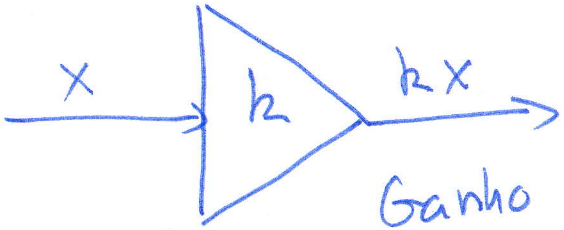
Diagrama de Blocos:

$$\boxed{\frac{dx}{dt} = \alpha x}$$



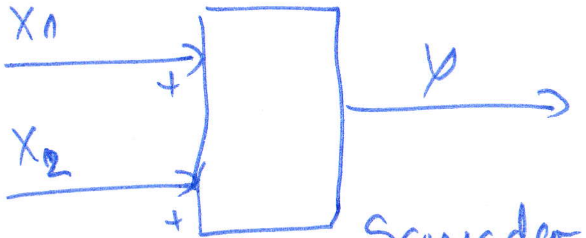
Integrador

$$x = \int \frac{dx}{dt}\, dt + x_0$$



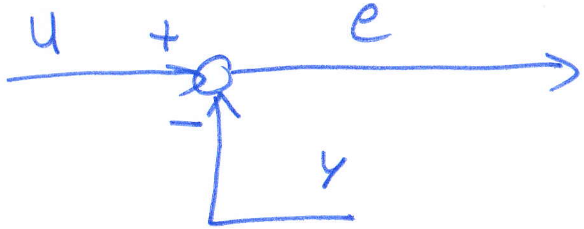
Ganho

$$y = kx$$



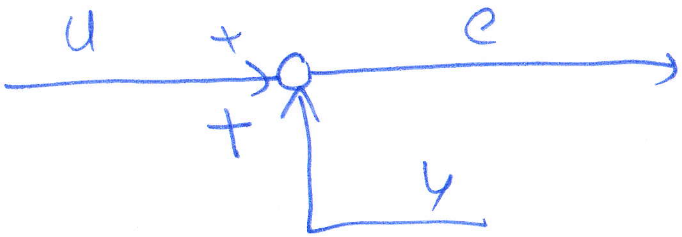
Somador

$$y = x_1 + x_2$$



$$e = u - y$$

realimentação negativa



$$e = u + y$$

realimentação positiva

$$\frac{dx}{dt} = f(x,t) = \alpha x$$

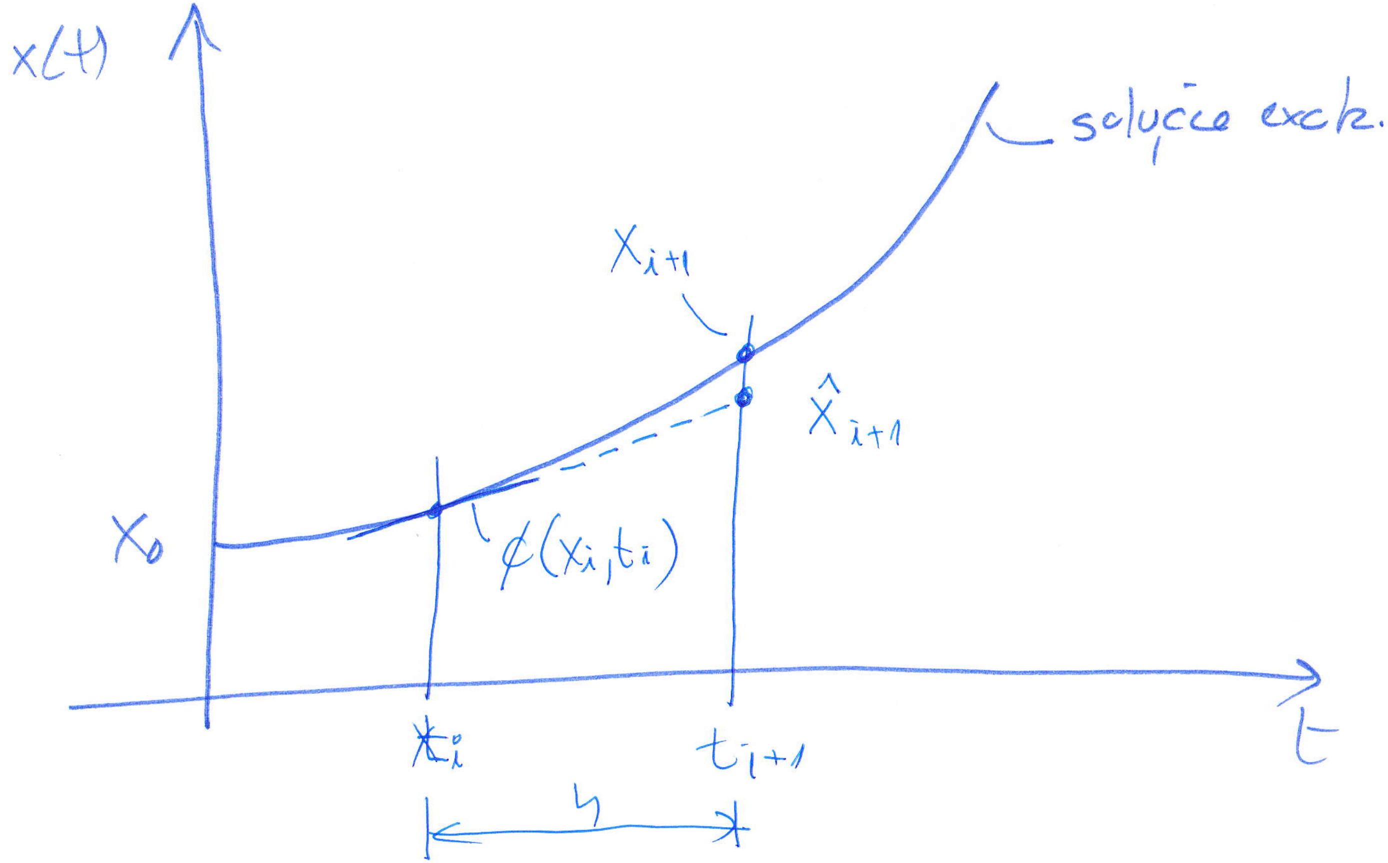
- escrever a função $f(x,t) = xdot$

- submeter ao integrador numérico.

Metodo di Euler:

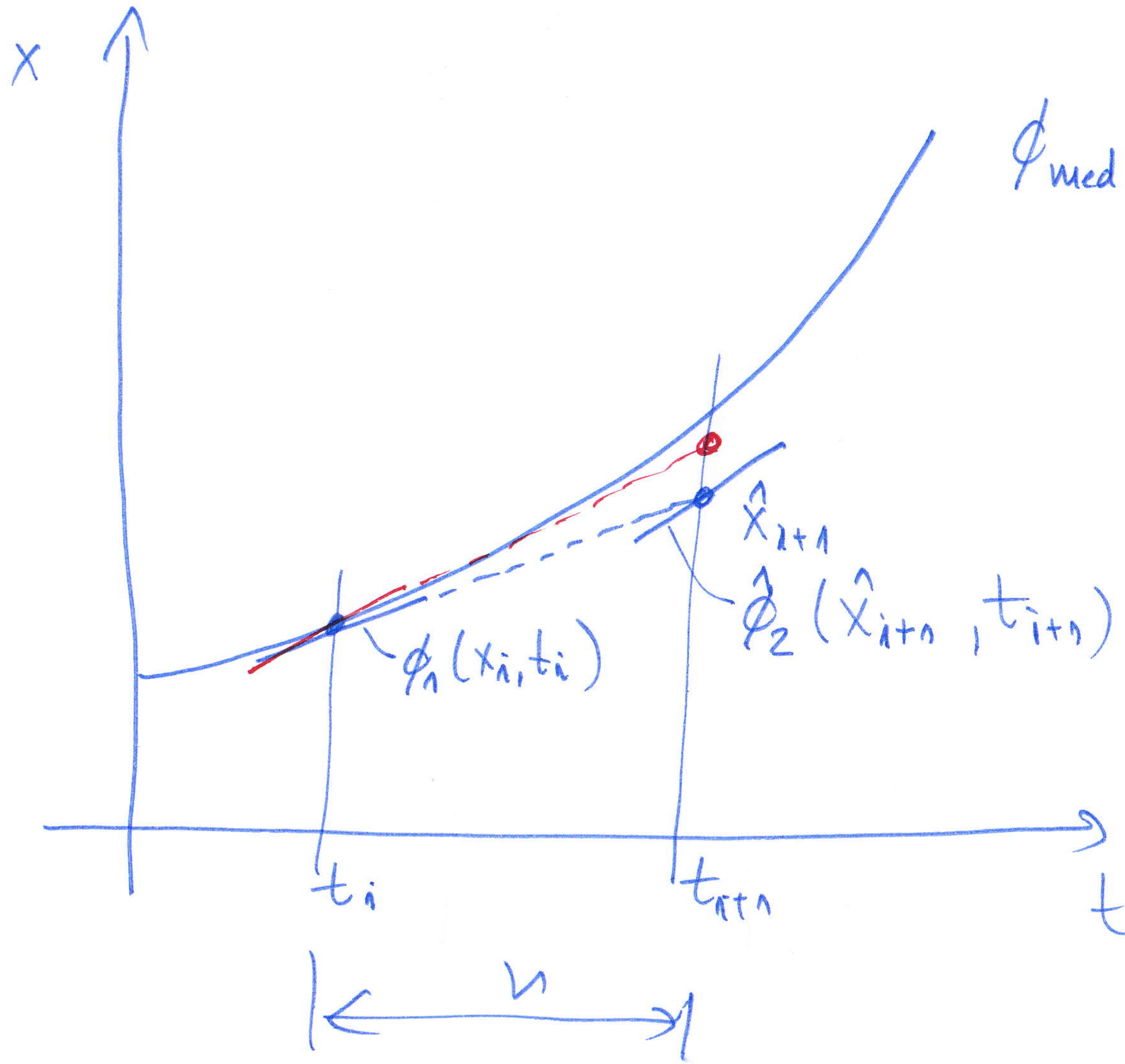
$$\frac{dx}{dt} = f(x,t) \qquad f(x,t) = \alpha x = \phi \qquad \Delta t = h$$ passo di integração

$$\frac{\Delta x}{\Delta t} = f(x,t) \Rightarrow \boxed{x_{i+1} = x_i + \phi(x_i, t_i) \cdot h}$$



Método de Heun

$$x_{i+1} = x_i + \phi_{med}(\ \ ) \cdot h$$



$$\phi_{med} = \frac{\phi_1 + \phi_2}{2}$$

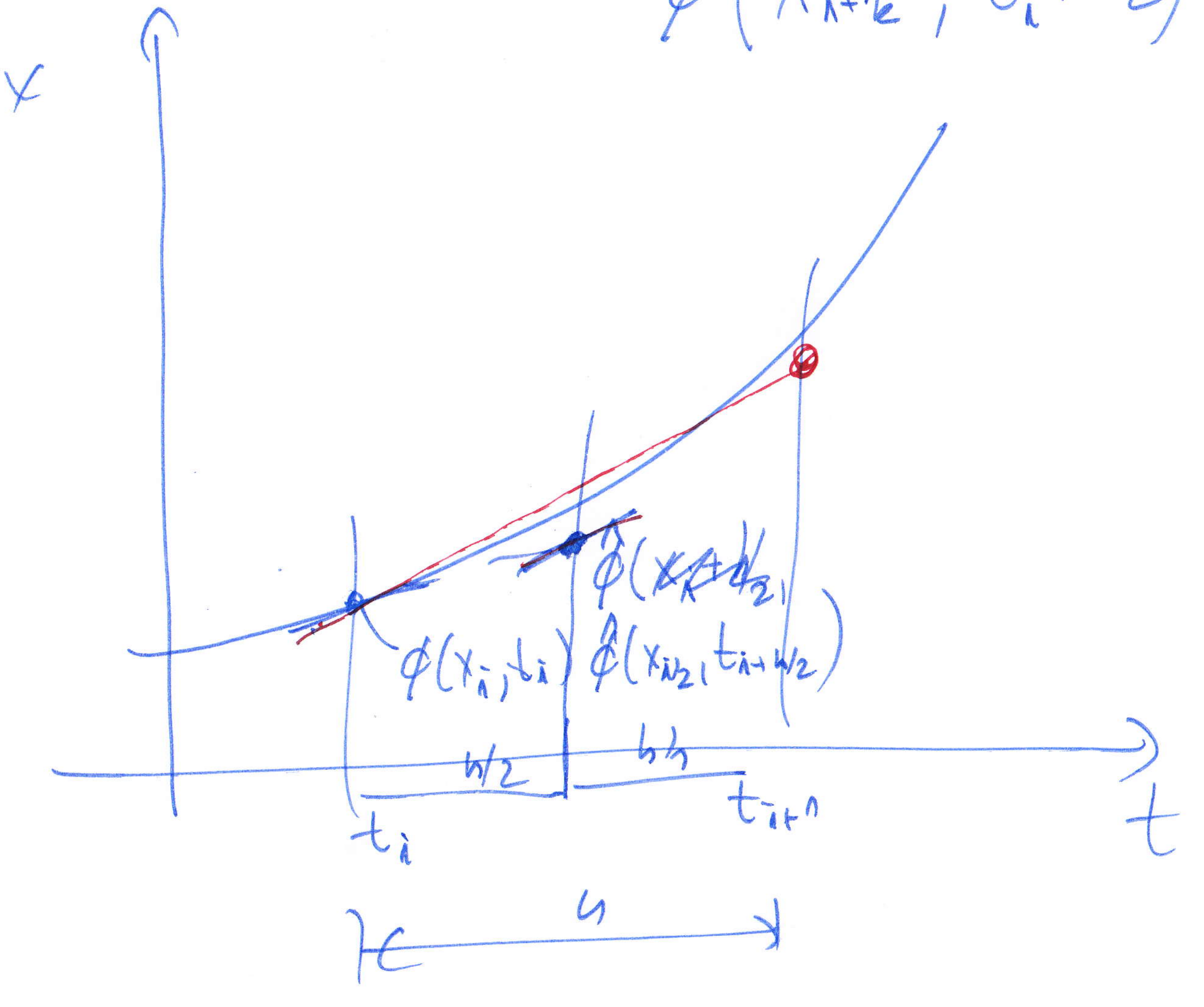
preditor

corretor

Metodo de Valor Médio

$$x_{i+1} = x_i + \phi \cdot h$$

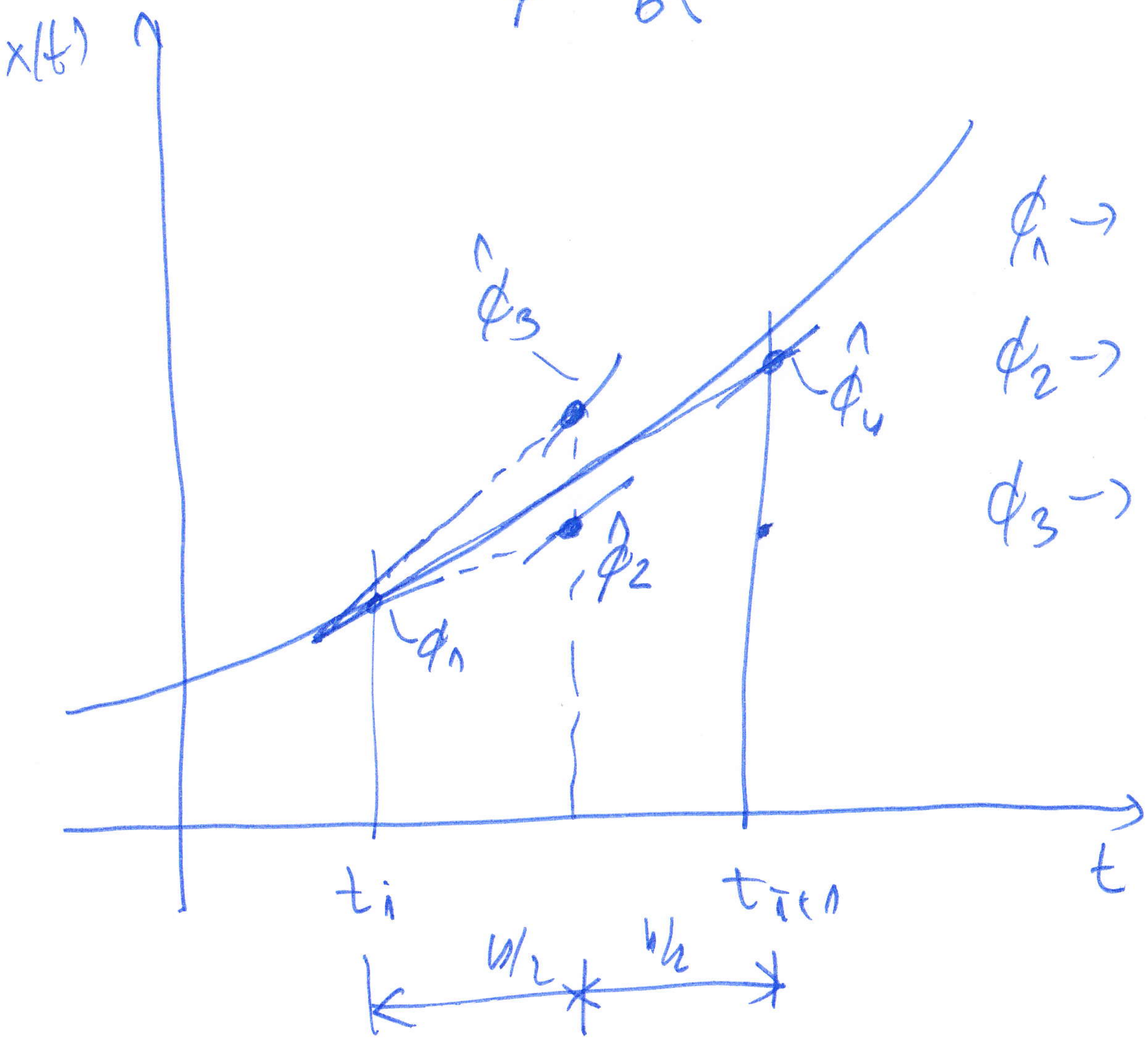
$$\hat{\phi}(\hat{x}_{i+1/2}, t_i + h/2)$$



Método de Runge-Kutta: 4º orden:

$$X_{i+1} = X_i + \hat{\phi} \cdot h$$

$$\hat{\phi} = \frac{1}{6}\left(\phi_1 + 2\hat{\phi}_2 + 2\hat{\phi}_3 + \hat{\phi}_4\right)$$



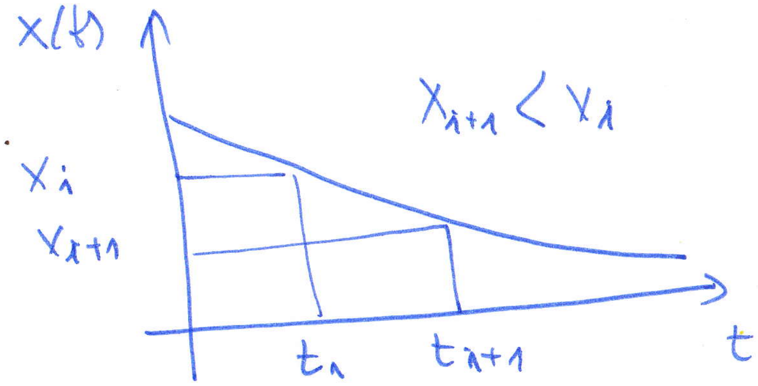
$\phi_1 = \phi(x_i, t_i)$

$\hat{\phi}_1 \rightarrow \phi_2 = \phi(x_{i+h/2}, t_i + h/2)$

$\phi_2 \rightarrow \phi_3 = \phi(x_{i+h/2}, t_i + h/2)$

$\phi_3 \rightarrow \phi_4 = \phi(x_{i+1}, t_{i+1})$

$$\frac{dx}{dt} = f(x,t) = -\alpha x \quad \Big| \ \alpha > 0$$



- Euler explícito: $f(x_i, t_i)$

$$x_{i+1} = x_i - (\alpha x_i)\, h$$

$$x_{i+1} = (1 - \alpha h)\, x_i$$

como $x_{i+1} < x_i \Rightarrow (1-\alpha h) < 1 \Rightarrow \boxed{h < \frac{2}{\alpha}}$

p/ garantir a estabilidade!
senão a solução diverge.

- Euler implícito: $f(x_{i+1}, t_{i+1})$

$$x_{i+1} = x_i - (\alpha x_{i+1})\, h$$

$$(1+\alpha h)\, x_{i+1} = x_i$$

$$x_{i+1} = \frac{1}{1+\alpha h}\, x_i$$

como $x_{i+1} < x_i \Rightarrow \forall h \Rightarrow \frac{1}{1+\alpha h} < 1 \therefore \forall h \in \mathbb{R} \mid h > 0$

haverá convergência da solução.

$$\frac{dx}{dt} = \alpha x \left(1 - \frac{x}{N}\right)$$

$$x' = \frac{x}{N} \Rightarrow \frac{dx'}{dt} = \frac{1}{N} \cdot \frac{dx}{dt} = \frac{1}{\cancel{N}} \alpha \cancel{N} x' (1 - x')$$

$$\boxed{\frac{dx'}{dt} = \alpha x' (1 - x')} \Rightarrow x'(t) \Rightarrow x(t) = N x'(t)$$

$$\frac{dx}{dt} = \alpha x(1-x)$$

$$\frac{1}{x(1-x)}\,dx = \alpha\,dt$$

$$\frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x} = \frac{A(1-x)+B\cdot x}{x(1-x)}$$

$$A(1-x)+Bx = 1 \Rightarrow A + (B-A)x = 1$$

$$A = 1 \Rightarrow A = 1$$

$$B-A = 0 \Rightarrow B = 1 \,//$$

$$\int \frac{1}{x}\,dx + \int \frac{1}{1-x}\,dx = \alpha t$$

$$\dots$$

equilíbrio:

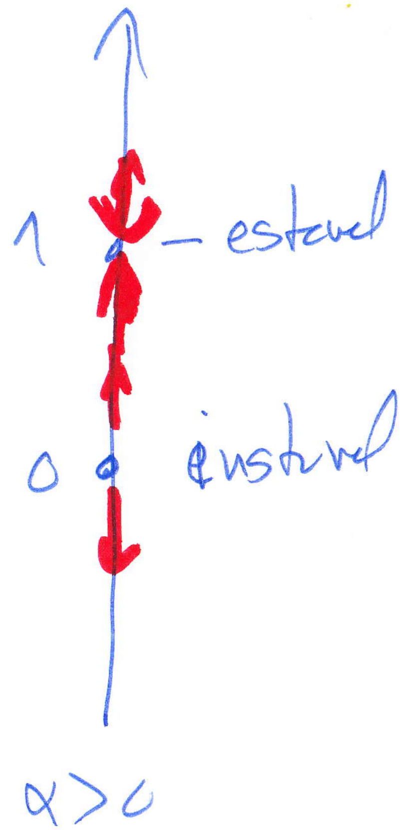
$$\frac{dx}{dt}=0 \qquad \alpha x(1-x)=0 \Rightarrow \begin{cases} x=0 \\ x=1 \end{cases}$$

estabilidade:

$$\frac{d}{dx}\left(\frac{dx}{dt}\right)\Big|_{x=0} = \alpha(1-2x)\Big|_{x=0} = \alpha$$

$$\frac{d}{dx}\left(\frac{dx}{dt}\right)\Big|_{x=1} = \alpha(1-2x)\Big|_{x=1} = -\alpha$$

$\alpha>0$



$\alpha > 0$

$$f(x,t) = \alpha \cdot x \cdot (1-x)$$