INTRODUCTION TO AERODYNAMIC DERIVATIVES, EQUATIONS OF MOTION AND STABILITY

Many Items throughout the ESDU Aerodynamics Sub-series give methods for the prediction of aerodynamic derivatives. To illustrate the use of such derivatives in the study of the dynamic motion of an aircraft, this Item describes how they appear in the rigid-body equations of motion to describe the response of an aircraft to a small disturbance from steady trimmed flight. The dependence of the stability of the aircraft on the values of the aerodynamic derivatives is displayed.

1. AXIS SYSTEMS AND NOTATION

1.1 Axis Systems

A number of different axis systems are used to describe the disturbed motion of an aircraft and the forces and moments acting on it. The necessary definitions for the systems used in this Item are given below. Each is right-handed and orthogonal. It is assumed that the aircraft has a longitudinal plane of symmetry.

1.1.1 Body-axis systems, $0xyz$

These systems are fixed in the body and move with it. For simplicity, in this Item the origin 0 is taken to be at the centre of gravity position of the aircraft in a steady (datum) flight condition. The x-axis and z-axis lie in the longitudinal plane of symmetry, x positive forwards and z positive downwards, with the y-axis positive to starboard. The direction of the x-axis is fixed to define particular systems. Two are considered.

(a) **Aerodynamic-body axes** constitute a system where the undisturbed direction of the x-axis is along the projection onto the longitudinal plane of symmetry of the tangent to the flight path in the datum condition. In a datum condition of straight and symmetrical flight the direction of the x-axis is in the direction of motion of the aircraft.

(b) **Geometric-body axes** constitute a system where the fixed x-axis is defined parallel to some convenient geometric longitudinal datum.

1.1.2 Earth-axis systems, $0_0x_0y_0z_0$, $0x_0y_0z_0$

These systems have the direction of their axes fixed in space. The z_0-axis is directed vertically downwards, the x_0-axis is directed forwards in the vertical plane containing the aircraft datum direction of motion and the y_0-axis completes the right-handed system. An Earth-fixed system has an origin 0_0 that is fixed in space. An aircraft-carried system has an origin 0 that is fixed in the aircraft and moves with it, and in this Item 0 is taken to be coincident with the centre of gravity position, as in the body-axis systems.

1.2 Notation and Units

Alphabetical trios are used to denote components of force, moment, linear displacement, velocity and angular velocity. They are listed in the table in Sketch 1.1. The diagram in Sketch 1.1 shows a body axis system $0xyz$ and indicates positive directions. Forces and linear velocities are positive in the direction of the axes. Angular velocities are positive if, when looking along the appropriate axis in the positive sense, the aircraft is seen to be rotating clockwise. Moments are positive in the same sense.

The same symbols are used for dimensional quantities and the corresponding dimensionless ones represented in normalised systems (see Sections 3.4 and 4.5).

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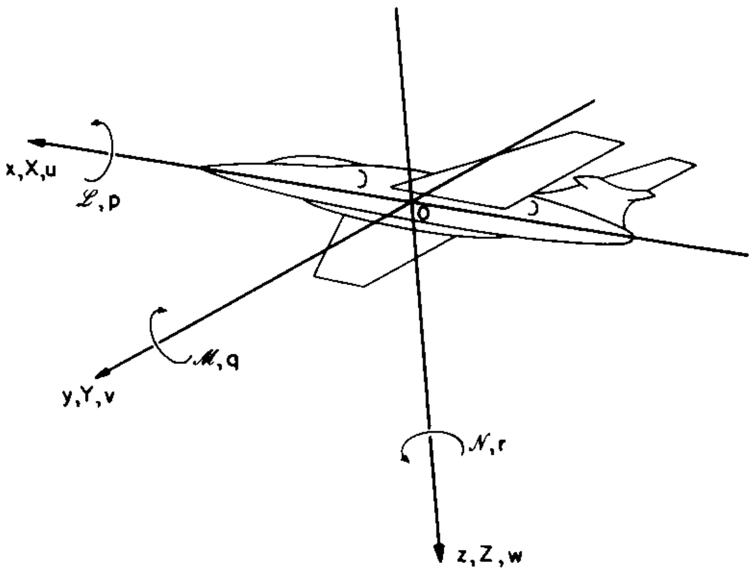
Quantity	*Components*		
Force	X	Y	Z
Moment	$\mathscr{L}$	$\mathscr{M}$	$\mathscr{N}$
Linear displacement	x	y	z
Linear velocity	u	v	w
Angular velocity	p	q	r



Sketch 1.1 System of body axes showing positive directions

		SI	*British*
a_i	constant in typical solution of equations of motion, see Equation (5.3)	m/s	ft/s
$\underline{\mathbf{A}}$	matrix in Equation (5.1)		
$\mathbf{A_{ij}}$	partition matrix from $\underline{\mathbf{A}}$		
A, B, C, D, E	constants in characteristic equations; subscripts 1, 2 denote longitudinal, lateral equations; see Section 5.3		
$\underline{\mathbf{B}}$	matrix in Equation (5.1)		
$\mathbf{B_{ij}}$	partition matrix from $\underline{\mathbf{B}}$		
B	hinge moment, positive when tending to increase positive control deflection	N m	ft lbf
ΔB	increment in hinge moment	N m	ft lbf
b	wing span	m	ft
b_x, b_y, b_z	inertia ratios $(I_z - I_y)/I_x$, $(I_x - I_z)/I_y$, $(I_y - I_x)/I_z$		

$\underline{\mathbf{C}}$	matrix in Equation (5.1)		
$\mathbf{C_{ij}}$	partition matrix from $\underline{\mathbf{C}}$		
C_B	hinge moment coefficient (B divided by $\frac{1}{2}\rho V^2$ and representative length and area of control)		
C_D	drag coefficient, $D/\frac{1}{2}\rho V^2 S$		
C_L	lift coefficient, $L/\frac{1}{2}\rho V^2 S$		
C_{LT}	lift coefficient of tailplane (tailplane lift $/\frac{1}{2}\rho V^2 S_T$)		
C_R	force coefficient, $R/\frac{1}{2}\rho V^2 S$		
C_X, C_Y, C_Z	force coefficients, $X/\frac{1}{2}\rho V^2 S$, $Y/\frac{1}{2}\rho V^2 S$, $Z/\frac{1}{2}\rho V^2 S$		
C_l, C_m, C_n	moment coefficients, $\mathscr{L}/\frac{1}{2}\rho V^2 Sl$, $\mathscr{M}/\frac{1}{2}\rho V^2 Sl$ $\mathscr{N}/\frac{1}{2}\rho V^2 Sl$; see Section 3.1		
$\bar{c}$	standard mean chord (or first mean chord)	m	ft
$\bar{\bar{c}}$	mean aerodynamic chord (or second mean chord)	m	ft
D	drag	N	lbf
D	differential operator d/dt	s^{-1}	s^{-1}
d_T	perpendicular distance from centre of gravity position to thrust vector, positive when thrust gives positive pitching moment; see Section 3.5.1	m	ft
d_x, d_y, d_z	inertia ratios, $-I_{yz}/I_x$, $-I_{yz}/I_y$, $-I_{yz}/I_z$		
e_x, e_y, e_z	inertial ratios, $-I_{zx}/I_x$, $-I_{zx}/I_y$, $-I_{zx}/I_z$		
F	free-control factor, see Section A5.1		
f_x, f_y, f_z	inertia ratios, $-I_{xy}/I_x$, $-I_{xy}/I_y$, $-I_{xy}/I_z$		
$\underline{\mathbf{g}}$	acceleration due to gravity vector	m/s^2	ft/s^2
g	magnitude of acceleration due to gravity	m/s^2	ft/s^2
g_x, g_y, g_z	components of acceleration due to gravity in x-, y-, z-direction	m/s^2	ft/s^2
g_1	$g \cos \theta_e$	m/s^2	ft/s^2
g_2	$g \sin \theta_e$	m/s^2	ft/s^2
$\underline{\mathbf{H}}$	angular momentum vector	kg m^2/s	slug ft^2/s
h_{cg}	dimensionless distance of centre of gravity position aft of reference point, see Appendix A		

h_m, h_{mf}	dimensionless distance of pitch manoeuvre point aft of reference point controls-fixed, controls-free; see Appendix A		
h_n, h_{nf}	dimensionless distance of neutral point aft of reference point controls-fixed, controls-free; see Appendix A		
$\mathbf{I_i}$	unitary $(i \times i)$ matrix		
$\mathbf{I_n}$	inertia matrix	kg m^2	slug ft^2
I_x, I_y, I_z	moments of inertia of aircraft about x-, y-, z-axis	kg m^2	slug ft^2
I_{xy}, I_{yz}, I_{zx}	products of inertia of aircraft about x- and y-axes, y- and z-axes, z- and x-axes	kg m^2	slug ft^2
i_x, i_y, i_z	inertia parameters $I_x/m_e l_0^2$, $I_y/m_e l_0^2$, $I_z/m_e l_0^2$		
$\mathscr{I}$	denotes imaginary part of		
$\underline{\mathbf{J}}$	angular momentum vector of rotating parts of aircraft or engines	kg m^2/s	slug ft^2/s
J_x, J_y, J_z	components of $\underline{\mathbf{J}}$ about x-, y-, z-axis	kg m^2/s	slug ft^2/s
k_{cg}, k_{cgf}	dimensionless centre of gravity margin (c.g. margin) controls-fixed, controls-free; see Appendix A		
k_m, k_{mf}	dimensionless manoeuvre margin controls-fixed, controls-free; see Appendix A		
k_s, k_{sf}	dimensionless static margin controls-fixed, controls-free; see Appendix A		
L	lift	N	lbf
$\mathscr{L}$, $\mathscr{M}$, $\mathscr{N}$	moments about x-, y-, z-axis (rolling moment, pitching moment, yawing moment), with associated aerodynamic derivatives L_u, M_u, M_w, N_v *etc.*; see Section 3.2	N m	lbf ft
l, m, n	concise moments, $-\mathscr{L}/I_x$, $-\mathscr{M}/I_y$, $-\mathscr{N}/I_z$	N/kg m	lbf/slug ft
l	characteristic length of aircraft	m	ft
l_0	unit of length in aero-normalised system, see Section 3.4	m	ft
l_T	distance of aerodynamic centre of tailplane aft of centre of gravity position	m	ft
$\underline{\mathbf{M}}$	moment vector	N m	lbf ft
M	Mach number		
m_e	datum value of aircraft mass, unit of mass in dynamic-normalised system; see Section 4.5	kg	slug
n	normal acceleration in units of g, see Appendix A		

P	ambient pressure of air	N/m^2	lbf/ft^2
p, q, r	angular velocity about x-, y-, z-axis, (roll rate, pitch rate, yaw rate)	rad/s	rad/s
$\underline{\mathbf{R}}$	aerodynamic force vector	N	lbf
R	magnitude of aerodynamic force vector	N	lbf
R	Reynolds number, $\rho Vl / \mu$		
R	Routh's discriminant		
$\mathscr{R}$	denotes real part of		
S	characteristic (reference) area	m^2	ft^2
S_T	tailplane planform area	m^2	ft^2
$\mathbf{S}_{\mathbf{\Phi}}, \mathbf{S}_{\mathbf{\phi}}$	axis transformation matrices, see Section 4.1		
T	period of oscillation	s	s
T	thrust	N	lbf
T_x, T_z	component of thrust in x-, z-direction	N	lbf
t	time	s	s
$t_{1/2}, t_{2/1}$	time to half amplitude, time to double amplitude	s	s
$\underline{\mathbf{u}}$	state vector of perturbation variables		
$\boldsymbol{u}_1, \boldsymbol{u}_2$	partitions of $\underline{\mathbf{u}}$		
u, v, w	components of aircraft velocity relative to air in x-, y-, z-direction	m/s	ft/s
$\underline{\mathbf{V}}$	aircraft velocity vector relative to air	m/s	ft/s
V	magnitude of velocity vector	m/s	ft/s
V_e	datum value of V, unit of speed in aero-normalised and dynamic-normalised systems	m/s	ft/s
W	aircraft weight	N	lbf
X, Y, Z	components of force in x-, y-, z-direction, with associated aerodynamic derivatives X_u, X_w, Y_v, Z_u, Z_w *etc.*; see Section 3.2	N	lbf
x, y, z	concise forces, $-X/m_e, -Y/m_e, -Z/m_e$	N/kg	lbf/slug
x, y, z	body axis co-ordinates in x-direction (positive forwards), y-direction (positive to starboard), z-direction (positive downwards)	m	ft
x_0, y_0, z_0	co-ordinates in Earth-axis system	m	ft

x_1, y_1, z_1 x_2, y_2, z_2	co-ordinates in intermediate axis systems used in defining attitude angles, Ψ, Θ, Φ; see Figure 1	m	ft
α	angle of attack, $\tan^{-1}(w/u)$	rad	rad
β	angle of sideslip, $\sin^{-1}(v/V)$	rad	rad
γ	angle of climb	rad	rad
γ	ratio of specific heat capacities of air		
$\partial\varepsilon/\partial\alpha$	downwash derivative at tailplane		
ζ, η, ξ	rudder, elevator, aileron deflection angle	rad	rad
$\bar{\eta}$	elevator angle required to trim	rad	rad
$\Delta\bar{\eta}$	increment in $\bar{\eta}$	rad	rad
$\underline{\eta}$	vector of control perturbations	rad	rad
η_1, η_2	partitions of $\underline{\eta}$	rad	rad
Φ, Θ, Ψ	bank, inclination, azimuth attitude angles	rad	rad
ϕ, θ, ψ	roll, pitch, yaw attitude-deviation angles	rad	rad
λ	variable of characteristic stability equation		
λ_i	root of characteristic equation		
μ	relative density parameter, $m_e / \frac{1}{2}\rho_e S l_0$		
μ	dynamic viscosity of air	kg/m s	slug/ft s
ρ	density of air	kg/m^3	slug/ft^3
τ	unit of time in dynamic-normalised system, $m_e / \frac{1}{2}\rho_e V_e S$ see Section 4.5	s	s
τ_η	elevator trim-tab deflection angle	rad	rad
$\bar{\tau}_\eta$	value of τ_η for trim	rad	rad
$\underline{\Omega}$	angular velocity vector	rad/s	rad/s

Dressings

Dot ($\dot{}$)	as in $\dot{w}$, denotes differentiation with respect to time
Dip ($\check{}$)	as in $\check{X}_u$, denotes an aero-normalised quantity
Cap ($\hat{}$)	as in $\hat{X}_u$, denotes a dynamic-normalised quantity
Ord ($^\circ$)	as in $\mathring{X}_u$, denotes a quantity expressed in ordinary dimensional units
Prime ($'$)	as in $u' = u - u_e$, denotes a perturbation

Subscripts

e	as in m_e, denotes a datum quantity
i, j	denote integers
$u, v, w, \dot{w}, \dot{v}$ p, q, r η, ξ, ζ	as in X_u *etc*., denote partial derivatives with respect to variable
T	denotes tail
WB	denotes wing-body

2. INTRODUCTION

2.1 General Remarks

To determine whether an aircraft is stable in a given trimmed steady flight condition it is necessary to analyse the motion of the aircraft after it has suffered a small disturbance (or perturbation) from that datum flight condition. If the disturbance dies out the aircraft is stable and if it does not then the aircraft is unstable. It is often convenient to analyse the motion of an aircraft by deriving representative equations. This requires knowledge of the aerodynamic and inertial forces acting on the aircraft.

The motion after a small disturbance can be described by linearised quasi-steady perturbation equations developed from the general rigid-body equations of motion. Although an aircraft is a flexible structure it is acceptable to study its dynamic behaviour through the equations of motion for a rigid body by modifying the aerodynamic derivatives to account for quasi-steady structural flexure under low frequency aerodynamic and inertial loads. The perturbation equations can be analysed in terms of the roots of a so-called 'characteristic' polynomial equation. The values of the roots determine whether or not the aircraft is stable. In the perturbation equations the influence of aerodynamic forces and moments is expressed through aerodynamic derivatives. The aim of the present Item is to show the connection between these aerodynamic derivatives and the roots of the characteristic equation.

The natural dynamic behaviour of the basic aircraft can be artificially changed by the inclusion in the overall aircraft system of certain automatic flight-control features of varying degrees of complexity. The aircraft is then said to be augmented. Simple 'stability augmentation' may, for instance, be provided by yaw or pitch dampers where controls operate to oppose a sensed disturbance. Item No. 83024 (Reference 26) of the Dynamics Sub-series illustrates the operation of a yaw damper. In more sophisticated cases the design features incorporated into the control system can encompass 'manoeuvre-command' systems that improve handling qualities in response to pilot commands, relaxed requirements on natural stability, protective systems to guard against mishandling of the aircraft, and the many aspects of active control technology with its incorporation of direct-force controls. All of these are discussed in Reference 28. Any such systems need to be modelled within the equations of motion. Nevertheless the dynamic behaviour of the unaugmented aircraft remains important as a sub-system even in advanced designs. For the introductory purposes of this Item it is sufficient to restrict attention to the unaugmented aircraft.

The symbols adopted for describing the dynamic motion of an aircraft have varied over the years and between different authors. For many years the system of Reference 5 (Bryant and Gates) was in widespread use in the UK. However, this and other systems had the disadvantage that there was often a loss of recognisable connection between the symbols themselves and the physical quantities they represented. To remedy this, a general system that can be recommended for universal adoption has been published by Hopkin in Reference 16. That reference contains a complete description of a scheme of nomenclature for aircraft dynamics and the associated aerodynamics. The versatility of the complete form allows it to be unambiguous in the most exacting circumstances. In less demanding circumstances considerable simplification is usually possible. The system has provided much of the background material for the notation adopted by the International Organisation for Standardisation (ISO) in Reference 18, and is used in the ESDU Dynamics Sub-series where it is described in some detail in Item Nos 67001 to 67003 (References 12 to 14).

The system of Reference 16 has also become standard in the Aerodynamics Sub-series. In Reference 16 there are tables of factors for conversion to or from the system of Reference 5.

2.2 Organisation of Item

Section 3 introduces the concepts of aerodynamic forces and moments and their derivatives. A systematic method for expressing these quantities in dimensionless form is given. Equations are provided for calculating the derivatives associated with longitudinal motion variables and reference is made to existing Items that contain methods for predicting the derivatives associated with lateral motion variables. A table is given that shows the conversion factors between derivatives expressed in the notation of this Item and in the notation most widely used in the USA.

Section 4 develops the general equations of motion for an aircraft treated as a rigid body, and shows how linearised quasi-steady perturbation equations can be formed. The equations apply rigorously only to infinitesimal disturbances but they can be used with good accuracy for quite large amplitudes provided there are no great departures from linearity.

Section 5 describes the principles underlying the analysis of the perturbation equations and shows that under certain simplifying assumptions they separate into two independent sets involving the longitudinal and the lateral motions of the aircraft. A detailed study is made of an aircraft slightly disturbed from straight and symmetrical steady flight, with controls fixed. Reference is made to the connections with the classical concepts of static margin and manoeuvre margin. Consideration is mainly confined to controls-fixed stability because the advent of power-operated controls has now rendered the problem of controls-free stability less important than in the past.

Section 6 contains a list of references. Section 7 sets out a worked example for the investigation of the controls-fixed longitudinal and lateral stability of a civil jet-transport aircraft.

Appendix A contains a traditional treatment of the problem of controls-fixed and controls-free longitudinal stability, starting from a consideration of static stability, which is concerned only with the equilibrium of the static forces and moments that develop on an aircraft in straight flight when it is disturbed slightly from a trimmed state. It then introduces the concepts of static margin and manoeuvre margin and discusses the relation between static stability and general stability in a dynamic sense. It is to be noted that the general stability of an aircraft can be studied without explicit reference to the concepts of static stability.

Appendix B provides a description in simple physical terms of the commonly occurring lateral modes of motion of an aircraft that may follow a small disturbance.

3. AERODYNAMIC FORCES AND MOMENTS AND THEIR EXPANSIONS

3.1 Form of Forces and Moments

The solution of problems in flight dynamics requires the aerodynamic forces and moments to be expressed in a form that adequately reflects the nature of the motion being considered and is convenient for the solution of the equations of motion. In order to set up a suitable form it is necessary to relate the forces and moments to the motion variables and other quantities that specify the aircraft flight condition (or state). For conditions of steady flight the forces and moments are fully determined by a reduced set of variables.

A general description of unsteady motion requires the whole set of time-varying state variables, and forces and moments that depend not only on the present values of the variables but also on the whole time history of the motion. However, the mathematical arguments developed by Tobak (see References 6 and 20 and the discussion in Reference 27), together with practical experience, show that it is permissible in most problems of aircraft stability to adopt an approach whereby the forces and moments are assumed to depend on the values of the state variables and the first derivatives with respect to time of the linear velocity components.

The aerodynamic forces depend on

(a) the overall shape of the aircraft and its skin roughness,

(b) the size of the aircraft typified by some length, l,

(c) the properties of the air, specified by the ambient pressure, P, density, ρ , and dynamic viscosity, μ ,

(d) the motion of the aircraft relative to the air, defined by the components of linear velocity, u, v, w, and of angular velocity, p, q, r,

(e) the control settings, where the elevator, aileron and rudder angles η, ξ, ζ are used as examples but other controls such as thrust vectors or direct force generators are assumed to be included,

(f) the nearness and orientation of the aircraft relative to another large object, usually the Earth (ground effect). With an Earth-fixed axis system this may be defined in terms of the co-ordinates x_0 , y_0 , z_0 and three attitude angles, Φ (bank), Θ (inclination) and Ψ (azimuth).

For an aircraft of given shape and surface condition a typical force component Z can be expressed

$$Z = Z(l, P, \rho, \mu, x_0, y_0, z_0, \Phi, \Theta, \Psi, u, v, w, p, q, r, \eta, \xi, \zeta, \dot{u}, \dot{v}, \dot{w}), \tag{3.1}$$

where although they are not independent variables $\dot{u}$, $\dot{v}$ and $\dot{w}$ have been included to emphasise the fact that they are used to represent knowledge of the history of the aircraft motion.*

A particular force or moment will usually be mainly determined by a sub-set of the variables in Equation (3.1). For the force Z and symmetrical flight a sensibly abbreviated form is

$$Z \approx Z(l, P, \rho, \mu, x_0, z_0, \Theta, u, w, q, \eta, \dot{u}, \dot{w}). \tag{3.2}$$

At this stage it is usually convenient to restrict attention to motion well away from the Earth or other large object so that x_0 , z_0 and Θ may be omitted, so that

$$Z \approx Z(l, P, \rho, \mu, u, w, q, \eta, \dot{u}, \dot{w}). \tag{3.3}$$

This can be written in a dimensionless form such as

$$\frac{Z}{\rho V^2 l^2} \approx Z\left(\frac{P}{\rho V^2}, \frac{\mu}{\rho V l}, \frac{w}{V}, \frac{ql}{V}, \eta, \frac{\dot{u}l}{V^2}, \frac{\dot{w}l}{V^2}\right) \tag{3.4}$$

where V is the magnitude of the aircraft velocity.

* See Reference 29 for a mathematically rigorous formulation.

There is the alternative and more familiar form involving Mach number, $M = V(\rho/\gamma P)^{1/2}$, Reynolds number, $R = \rho V l/\mu$, and a reference area, S,

$$\frac{Z}{\frac{1}{2}\rho V^2 S} \approx Z\left(M, R, \frac{w}{V}, \frac{ql}{V}, \eta, \frac{\dot{u}l}{V^2}, \frac{\dot{w}l}{V^2}\right). \tag{3.5}$$

For symmetrical flight at low angles of attack $\alpha \approx w/V$ and $\dot{\alpha} \approx \dot{w}/V$ and there is the form

$$\frac{Z}{\frac{1}{2}\rho V^2 S} \approx Z\left(M, R, \alpha, \frac{ql}{V}, \eta, \frac{\dot{u}l}{V^2}, \frac{\dot{\alpha}l}{V^2}\right). \tag{3.6}$$

The dimensionless quantity $Z/\frac{1}{2}\rho V^2 S$ is known as the aerodynamic force coefficient C_Z and similar relationships exist for C_X and C_Y. The aerodynamic moment coefficients C_l, C_m and C_n are similar except that the length l appears also in the divisor. Different choices of l will normally be made for the longitudinal and lateral coefficients; $\bar{\bar{c}}$ or $\bar{c}$ is usual for C_m and b or $b/2$ for C_l and C_n.

3.2 Expansion of Forces and Moments

When considering the motion of an aircraft after a small perturbation from a datum flight condition a return is made to a form such as Equation (3.1), since it is reasonable to express the aerodynamic forces and moments in linear expansions about their datum values. Before developing such expansions it is noted that as P, ρ and μ are all functions of the height above sea level, for air in equilibrium they may be replaced by this single variable. Therefore if, as is customary, it is assumed that the atmosphere is essentially uniform for small changes in height the variation of forces and moments with height can be ignored.

For an aircraft with a longitudinal plane of symmetry the usual expansion of the forces and moments with respect to a system of body-axes $Oxyz$ (defined in Section 1.1), where the x-axis and z-axis lie in the plane symmetry, is

$$X' = X - X_e = X_u u' + X_w w' + X_{\dot{w}} \dot{w}' + X_q q' + X_\eta \eta' + \ldots, \tag{3.7}$$

$$Y' = Y - Y_e = Y_v v' + Y_p p' + Y_r r' + Y_\xi \xi' + Y_\zeta \zeta' + \ldots, \tag{3.8}$$

$$Z' = Z - Z_e = Z_u u' + Z_w w' + Z_{\dot{w}} \dot{w}' + Z_q q' + Z_\eta \eta' + \ldots, \tag{3.9}$$

$$\mathscr{L}' = \mathscr{L} - \mathscr{L}_e = L_v v' + L_p p' + L_r r' + L_\xi \xi' + L_\zeta \zeta' + \ldots, \tag{3.10}$$

$$\mathscr{M}' = \mathscr{M} - \mathscr{M}_e = M_u u' + M_w w' + M_{\dot{w}} \dot{w}' + M_q q' + M_\eta \eta' + \ldots, \tag{3.11}$$

$$\mathscr{N}' = \mathscr{N} - \mathscr{N}_e = N_v v' + N_p p' + N_r r' + N_\xi \xi' + N_\zeta \zeta' + \ldots, \tag{3.12}$$

where the subscript e denotes datum conditions and primes denote perturbation from the datum, for example $u' = u - u_e$. The quantities X_u *etc.* in the expansions are commonly referred to as aerodynamic derivatives. They are evaluated at datum conditions and hence at a specified Mach number and Reynolds number. They correspond to the partial differentials $\partial X/\partial u$ *etc.* when it is valid to assume that the influence of the history of the motion on the aerodynamic forces and moments is adequately represented through present values of $\dot{u}$, $\dot{v}$ and $\dot{w}$.

The selected terms shown in Equations (3.7) to (3.12) are those known to be significant for a wide range of flight conditions. In certain other cases additional terms and control variables may be needed. For example, for aircraft with highly swept wings at high angles of attack the terms $L_{\dot{v}}\dot{v}$ and $N_{\dot{v}}\dot{v}$ may be important (see References 11 and 22). No cross-coupling terms such as $X_v v$, $Y_u u$ *etc*. have been included since for a symmetrical aircraft they may normally be treated as zero, although the pitching moment due to sideslip, $M_v v$, can assume significant values for some aircraft (see Reference 23). In the case of vertical-lift aircraft some further cross terms may be important.

Aerodynamic coefficients may themselves be expanded in analogous forms to those developed for the dimensional forces and moments. In such an expansion the dependence on speed and density is largely embodied within the factor $\frac{1}{2}\rho V^2$. As indicated in Reference 16 such expansions are appropriate when there is a substantially varying speed or density.

3.3 Concise Forms

To provide a compact way of writing the equations of motion, concise forms of forces and moments are introduced whereby the forces are divided by the aircraft mass and the moments by the moment of inertia about the appropriate axis. In both cases a change of sign is introduced. The concise forms are denoted by a lower case letter corresponding to the original force or moment. Concise forms of aerodynamic derivatives are produced in an identical manner. For example, the concise forms of the force Z and the moment $\mathscr{L}$ are

$$z = -Z/m_e \text{ and } l = -\mathscr{L}/I_x \,, \tag{3.13}$$

and concise forms of typical derivatives are

$$z_u = -Z_u/m_e \text{ and } l_v = -L_v/I_x \,. \tag{3.14}$$

3.4 Aero-normalised System

Thus far it has been tacitly assumed that the forces and moments and aerodynamic derivatives are quantities measured in a system of ordinary dimensional units, *e.g*. the SI system.

Just as the conversion of aerodynamic forces and moments into dimensionless coefficients is a convenient means of handling such data in that it isolates the dependence on velocity, density and size, a parallel ‘normalising’ procedure is used to express aerodynamic derivatives and motion variables in dimensionless form. The standard system adopted for aerodynamic derivatives in the Aerodynamics Sub-series is the aero-normalised system recommended in Reference 16.

The aero-normalised system uses divisors based on the appropriate combinations of three reference units of force, speed and length. The unit of force is $\frac{1}{2}\rho_e V_e^2 S$, where ρ_e and V_e are datum values of air density and aircraft speed, and can be sensibly taken as the conditions existing immediately before a disturbance. The unit of speed is V_e and the unit of length is l_0, a characteristic length for the aircraft. The devisor required for any other physical quantity can be constructed by dimensional analysis, for example the divisor for a moment is $\frac{1}{2}\rho_e V_e^2 S l_0$ and that for an angular velocity is V_e/l_0. If the equations of motion are expressed in the aero-normalised system a single value of l_0 must be used throughout.

It is a feature of the aero-normalised system that the same symbols are used to represent both dimensional and dimensionless quantities as there will seldom be confusion as to which system is being used. If it is necessary, a dressing $(^\circ)$ as in $\mathring{X}_u$ may be used to distinguish the dimensional form or a dressing $(\check{\ })$ as in $\check{X}_u$ may be used to denote the aero-normalised form. The first three columns of Table C1.1 demonstrate the normalising process for aerodynamic derivatives.

3.5 Estimation of Aerodynamic Derivatives

Numerical values of stability derivatives can be obtained from wind-tunnel tests of a particular model, from theoretical or semi-empirical prediction methods, or from an analysis of flight-test results. The values depend on the particular body-axis system chosen. Although conversion from one system to another can be made, it is convenient to adopt a system in which the derivatives are available directly.

In cases where the angle of attack is small or moderate, an aerodynamic-body axis system, defined in Section 1.1, is convenient. In the UK aerodynamic-body axes have in the past been referred to as wind-axes or wind-body axes. In the USA they are usually referred to as stability axes. They are used in many semi-empirical prediction methods.

The chief alternative to the use of aerodynamic-body axes are geometric-body axes, defined in Section 1.1, and these are well suited to the study of motion at high angles of attack. In some cases it is possible to choose the geometric-body axes to coincide with the principal axes of inertia. This has the benefit that all the cross products of inertia are eliminated from the equations of motion.

3.5.1 Longitudinal derivatives

For studies of small disturbances from straight and symmetrical steady flight the controls-fixed longitudinal derivatives about aerodynamic-body axes are often approximated by simple formulae that involve the lift, drag and thrust characteristics of the aircraft, see Reference 25. The relevant equations for the aero-normalised derivatives are

$$X_u = -\left(2C_D + V\frac{\partial C_D}{\partial V}\right)_e + \left(\frac{1}{\frac{1}{2}\rho VS}\,\frac{\mathrm{d}T}{\mathrm{d}V}\right)_e , \tag{3.15}$$

$$X_w = \left(C_L - \frac{\partial C_D}{\partial \alpha}\right)_e = \left(C_L - \frac{\partial C_D}{\partial C_L}\,\frac{\partial C_L}{\partial \alpha}\right)_e , \tag{3.16}$$

$$X_{\dot{w}} = 0 , \tag{3.17}$$

$$X_q = 0 , \tag{3.18}$$

$$Z_u = -\left(2C_L + V\frac{\partial C_L}{\partial V}\right)_e , \tag{3.19}$$

$$Z_w = -\left(C_D + \frac{\partial C_L}{\partial \alpha}\right)_e , \tag{3.20}$$

$$Z_{\dot{w}} = 0 , \tag{3.21}$$

$$Z_q = (Z_q)_{WB} - \left(\frac{\partial C_{LT}}{\partial \alpha}\right)_e \frac{S_T\, l_T}{S l_0} , \tag{3.22}$$

$$M_u = \left(V\frac{\partial C_m}{\partial V}\right)_e , \tag{3.23}$$

$$M_w = \left(\frac{\partial C_m}{\partial \alpha}\right)_e , \tag{3.24}$$

$$M_{\dot{w}} = (M_{\dot{w}})_{WB} - \left(\frac{\partial C_{LT}}{\partial \alpha}\right)_e \frac{S_T\, l_T^2}{S l_0^2}\left(\frac{\partial \varepsilon}{\partial \alpha}\right)_e , \tag{3.25}$$

$$M_q = (M_q)_{WB} - \left(\frac{\partial C_{LT}}{\partial \alpha}\right)_e \frac{S_T l_T^2}{S l_0^2} \tag{3.26}$$

The derivatives X_q, $X_{\dot{w}}$ and $Z_{\dot{w}}$ which are of negligible importance, have been set to zero. The derivatives Z_q, $M_{\dot{w}}$ and M_q are represented as the sum of a wing-body contribution and a tail contribution, with the two wing-body moment derivatives being referred to the aircraft centre of gravity position. For aircraft with long tail arms the tail contributions dominate and the wing-body contributions $(Z_q)_{WB}$, $(M_{\dot{w}})_{WB}$ and $(M_q)_{WB}$ may be omitted. The derivative $\partial C_{LT}/\partial\alpha$ is the tailplane lift-curve slope based on tailplane planform area, S_T and l_T is the distance of the aerodynamic centre of the tailplane aft of the centre of gravity position. The factor $\partial\varepsilon/\partial\alpha$ represents the downwash at the tailplane. The lift and drag coefficients and their derivatives are all evaluated at the steady datum conditions.

It is assumed in the equations that the thrust acts along the x-axis. If the thrust axis is inclined to the x-axis and its line of action passes at a perpendicular distance d_T below the centre of gravity position then the component of thrust T_x acting parallel to the x-axis replaces T in Equation (3.15). There are also small additional terms, $(\mathrm{d}T_z/\mathrm{d}V)_e / \frac{1}{2}\rho_e V_e S$ in Equation (3.19) associated with the thrust component parallel to the z-axis and $(\mathrm{d}T/\mathrm{d}V)_e d_T / \frac{1}{2}\rho_e V_e S l_0$ in Equation (3.23) associated with the thrust moment about the centre of gravity position. These two terms are usually small enough to be ignored (see Reference 22).

Equation (3.15) involves the variation of the engine thrust with speed, which will depend on the operating conditions and type of engine. For example, in the case of a piston-engined aircraft with variable-pitch propellers a reasonable approximation is that the power and efficiency of the engine are constant, *i.e.* TV = constant and $(\mathrm{d}T/\mathrm{d}V)_e = -T_e/V_e$. There is much less dependence of thrust on speed for jet engines.

The derivatives of the aerodynamic coefficients C_D, C_L and C_m with respect to V reflect changes in slipstream (see Reference 9), compressibility and Reynolds number effects, although the last of these is insignificant for full-scale flight. The general classical treatment (Reference 2), that is not confined to the assumption of a rigid body, also allows for the effect of a change in aircraft shape due to low frequency structural flexure that is in phase with the local aerodynamic loading. If such derivatives are negligible then some of the equations simplify. In particular, and for steady horizontal flight, Equation (3.24) can be replaced by

$$M_w = \left(\frac{\mathrm{d}C_m}{\mathrm{d}C_L} \frac{\partial C_L}{\partial \alpha}\right)_e = -k_s\left(\frac{\partial C_L}{\partial \alpha}\right)_e , \tag{3.27}$$

where k_s is the static margin (see Equation (A2.1) of Appendix A).

3.5.2 Lateral derivatives

Items in the Aerodynamics Sub-series may be used for predicting controls-fixed lateral stability derivatives about aerodynamic-body axes. For subcritical speeds there is a complete coverage of the derivatives due to sideslip, roll rate and yaw rate. Items for predicting the rolling moment derivatives are introduced by Item No. Aircraft 06.01.00 (Reference 8) and those for predicting yawing moment sideforce derivatives are introduced in Item No. Aircraft 07.01.00 (Reference 3).

3.5.3 Comparison of derivative definitions

In Reference 16 it is explained that various systems of stability notation are used in the USA. This Section contains information on the differences between aerodynamic derivatives expressed in a widely used American system and the aero-normalised system of Reference 16.

In this American system the basic symbols such as X, Y, Z, u, v, w, α, β *etc*. remain standard but derivatives of dimensionless coefficients based on constant values of ρ and V are used to denote aerodynamic derivatives, for example the symbol C_{xu} is used instead of X_u. Also, unlike the aero-normalised system, the American system uses different representative lengths for the normalised motion variables and the moment coefficients, respectively $\bar{\bar{c}}/2$ and $\bar{\bar{c}}$ for the longitudinal derivatives and $b/2$ and b for the lateral derivatives. When derivatives with respect to α and β are formed there is usually the implication that $\alpha \approx w/V$ and $\beta \approx v/V$ so that $C_{l\beta} = C_{lv}$ *etc*.

If it is assumed that in the American system all coefficients are evaluated at datum conditions and that the reference dimensions for the longitudinal coefficients are S and $\bar{\bar{c}}$ and for the lateral coefficients are S and b, then Table C1.2 allows for all of these differences and relates the American derivatives to the corresponding aero-normalised derivatives. In practice, symbols other than η, ξ and ζ will probably be used to denote control deflections.

All of the individual Items in the Aerodynamics Sub-series which give methods for the prediction of stability derivatives have before their Notation a note that gives the American style of the derivative, with a warning to check on reference dimensions.

4. EQUATIONS OF MOTION

This Section sets out the general equations of motion for an aircraft treated as a rigid body. It goes on to develop the linearised perturbation equations that are widely used to describe the motion following a small disturbance.

It is assumed throughout that the variation of atmospheric properties with small changes in height is insignificant, that the effects of the Earth's curvature and rotation may be neglected, and that the velocity of the surrounding air relative to the Earth is zero. The aircraft is taken to be well clear of the Earth so that there is no ground effect on the aerodynamic forces and moments.

4.1 Attitude Angles

The inertial quantities and instantaneous motion of the aircraft can be defined with respect to any set of body-axes $0xyz$, as defined in Section 1.1. The choice of particular systems can introduce simplification into the equations of motion and those brought about by aerodynamic-body axes are demonstrated in later sections.

Because the body axes move in space a complete description of the aircraft motion requires the definition of a reference axis system whose axes are in fixed directions in space. In this Item an aircraft-carried Earth system $0x_0y_0z_0$ is chosen, as defined in Section 1.1.2.

The position occupied by an aircraft in space may then be specified by the position of the axis origin 0 relative to the Earth and by the aircraft's attitude relative to the axis system $0x_0y_0z_0$. The latter may be defined by specifying three independent angular rotations, Φ (bank), Θ (inclination) and Ψ (azimuth), which if applied in a standard sequence to an axis system initially coincident with the Earth-axis system would bring it into alignment with the body-axis system.

The standard sequence of rotations, which is illustrated in Figure 1, is

(i) an initial rotation Ψ in the positive sense about the z_0-axis,

(ii) a second rotation Θ in the positive sense about the resulting (intermediate) position of the y-axis,

(iii) a third rotation Φ in the positive sense about the resulting final position of the x-axis.

For small perturbation studies it is convenient to express the disturbed orientation of the aircraft relative to a datum attitude defined by Φ_e, Θ_e, Ψ_e. This is done by defining attitude-deviation angles ϕ, θ, ψ. Application of these rotations to the body-axes system in its datum condition, in the same way that the rotations Φ, Θ, Ψ are applied to the Earth-axis system, change the orientation of the body-axis system from the datum position to the disturbed position.

If the direction cosines of the Earth axes $0x_0$, $0y_0$ and $0z_0$ with respect to the body axes are l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3; respectively, a vector quantity in the Earth-axis system can be transformed into the body-axis system through multiplication by the transformation matrix.

$$\mathbf{S}_\Phi = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \tag{4.1}$$

where, by virtue of the definition of the attitude angles,

$$l_1 = \cos\Theta\cos\Psi\ , \tag{4.2}$$
$$l_2 = \cos\Theta\sin\Psi\ , \tag{4.3}$$
$$l_3 = -\sin\Theta\ , \tag{4.4}$$

$$m_1 = \sin\Phi\sin\Theta\cos\Psi - \cos\Phi\sin\Psi\ , \tag{4.5}$$
$$m_2 = \sin\Phi\sin\Theta\sin\Psi + \cos\Phi\cos\Psi\ , \tag{4.6}$$
$$m_3 = \sin\Phi\cos\Theta\ , \tag{4.7}$$

$$n_1 = \cos\Phi\sin\Theta\cos\Psi + \sin\Phi\sin\Psi\ , \tag{4.8}$$
$$n_2 = \cos\Phi\sin\Theta\sin\Psi - \sin\Phi\cos\Psi\ , \tag{4.9}$$
$$n_3 = \cos\Phi\cos\Theta\ . \tag{4.10}$$

The attitude Φ, Θ, Ψ may be considered as being the result of rotations to the datum attitude Φ_e, Θ_e, Ψ_e followed by rotations through the attitude-deviation angles ϕ, θ, ψ, so that $\mathbf{S}_\Phi$ may be written

$$\mathbf{S}_\Phi = \mathbf{S}_\phi \mathbf{S}_{\Phi e}\ , \tag{4.11}$$

where $\mathbf{S}_{\Phi e}$ and $\mathbf{S}_\phi$ are matrices of the same form as $\mathbf{S}_\Phi$. If, in addition, the attitude-deviation angles are small $\mathbf{S}_\phi$ reduces to

$$\mathbf{S}_\phi = \begin{bmatrix} 1 & \psi & -\theta \\ -\psi & 1 & \phi \\ \theta & -\phi & 1 \end{bmatrix}. \tag{4.12}$$

It is to be noted that the attitude-deviation angles are not equal to perturbations of the attitude angles. For small angles their general relationship, given in Reference 16, is

$$\phi = \Phi' - \Psi'\sin\Theta_e\ , \tag{4.13}$$
$$\theta = \theta'\cos\Phi_e + \Psi'\sin\Phi_e\cos\Theta_e\ , \tag{4.14}$$
$$\psi = -\Theta'\sin\Phi_e + \Psi'\cos\Phi_e\cos\Theta_e\ . \tag{4.15}$$

4.2 Kinematic Relationships

To set up the full set of equations governing the aircraft motion it is necessary to interrelate the body-axis angular velocity components p, q, r and the rates of change of the attitude angles Φ, Θ, Ψ or the attitude-deviation angles ϕ, θ, ψ.

The resultant angular velocity besides having components p, q, r along the body axes has the following components

$\dot{\Phi}$ along the $0x$ axis,

$\dot{\Theta}$ along the intermediate $0y$ axis,

$\dot{\Psi}$ along the $0z_0$ axis.

Hence by resolving these along the $0x$, $0y$ and $0z$ axes the following relationships result,

$$p = \dot{\Phi} - \dot{\Psi} \sin \Theta \tag{4.16}$$

$$q = \dot{\Theta} \cos \Phi + \dot{\Psi} \sin \Phi \cos \Theta \tag{4.17}$$

$$r = -\dot{\Theta} \sin \Phi + \dot{\Psi} \cos \Phi \cos \Theta . \tag{4.18}$$

The inverse relationships are given by

$$\dot{\Phi} = p + q \sin \Phi \tan \Theta + r \cos \Phi \tan \Theta \tag{4.19}$$

$$\dot{\Theta} = q \cos \Phi - r \sin \Phi \tag{4.20}$$

$$\dot{\Psi} = q \sin \Phi \sec \Theta + r \cos \Phi \sec \Theta . \tag{4.21}$$

The angular velocity components may also be considered as composed of components p_e, q_e, r_e along the datum axes together with the components

$\dot{\phi}$ along the $0x$ axis,

$\dot{\theta}$ along the intermediate $0y$ axis,

$\dot{\psi}$ along the $0z_0$ axis.

Thus

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \mathbf{S}_\phi \begin{bmatrix} p_e \\ q_e \\ r_q \end{bmatrix} + \begin{bmatrix} \dot{\phi} - \dot{\psi} \sin \theta \\ \dot{\theta} \cos \phi + \dot{\psi} \sin \phi \cos \theta \\ -\dot{\theta} \sin \phi + \dot{\psi} \cos \phi \cos \theta \end{bmatrix}, \tag{4.22}$$

so that, when p_e, q_e and r_e are all zero,

$$p = \dot{\phi} - \dot{\psi}\sin\theta = p', \tag{4.23}$$
$$q = \dot{\theta}\cos\phi + \dot{\psi}\sin\phi\cos\theta = q', \tag{4.24}$$
$$r = -\dot{\theta}\sin\phi + \dot{\psi}\cos\phi\cos\theta = r', \tag{4.25}$$

analogous relationships to Equations (4.16) to (4.18). If p_e, q_e and r_e are not zero but the attitude-deviation angles are small, the perturbation angular velocities are

$$p' = \dot{\phi} - r_e\theta + q_e\psi, \tag{4.26}$$
$$q' = \dot{\theta} + r_e\phi - p_e\psi, \tag{4.27}$$
$$r' = \dot{\psi} - q_e\phi + p_e\theta. \tag{4.28}$$

If both simplifying conditions are assumed, there is the simplest result

$$p' = \dot{\phi}, \tag{4.29}$$
$$q' = \dot{\theta}, \tag{4.30}$$
$$r' = \dot{\psi}. \tag{4.31}$$

It is also possible to form expressions for p', q' and r' in terms of perturbation of the attitude angles, Φ', Θ' and Ψ', but these are more complicated than Equations (4.23) to (4.31) and only the expressions in ϕ, θ and ψ are given in this Item.

4.3 Components of the Gravitational Force

In addition to the aerodynamic forces and moments acting upon it the aircraft weight must also be resolved into components along the body axes. For an aircraft these are denoted by $m_e g_x$, $m_e g_y$ and $m_e g_z$ and g_x, g_y and g_z are expressible in terms of the attitude angles defined previously,

$$\begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} = \mathbf{S}_\Phi \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}, \tag{4.32}$$

so $\qquad g_x = l_3 g$, $g_y = m_3 g$ and $g_z = n_3 g$.

Using Equation (4.11) it is also possible to write

$$\begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} = \mathbf{S}_\phi \mathbf{S}_{\Phi e} \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}, \tag{4.33}$$

and the form of $\mathbf{S}_\phi$ for small attitude-deviation angles implies that the increments in g_x, g_y and g_z due to a perturbation are

$$g_x' = -(g \cos\Theta_e \cos\Phi_e)\,\theta + (g\cos\Theta_e \sin\Phi_e)\,\psi$$

$$= -(g_1 \cos\Phi_e)\,\theta + (g_1 \sin\Phi_e)\,\psi, \tag{4.34}$$

$$g_y' = (g\cos\Theta_e\cos\Phi_e)\,\phi + (g\sin\Theta_e)\,\psi$$

$$= (g_1\cos\Phi_e)\,\phi + g_2\,\psi, \tag{4.35}$$

$$g_z' = -(g\cos\Theta_e\sin\Phi_e)\,\phi - (g\sin\Theta_e)\,\theta,$$

$$= -(g_1\sin\Phi_e)\,\phi - g_2\,\theta, \tag{4.36}$$

where $g_1 = g\cos\Theta_e$ and $g_2 = g\sin\Theta_e$ have been introduced for convenience.

For a datum unbanked flight condition $\Phi_e = 0$ and so

$$g_x' = -(g\cos\Theta_e)\,\theta = -g_1\,\theta, \tag{4.37}$$

$$g_y' = (g\cos\Theta_e)\,\phi + (g\sin\Theta_e)\,\psi = g_1\,\phi + g_2\,\psi, \tag{4.38}$$

$$g_z' = -(g\sin\Theta_e)\,\theta = -g_2\,\theta. \tag{4.39}$$

These relationships are valid for any body-axis system. For the particular case of aerodynamic-body axes and horizontal datum flight conditions Θ_e is also zero, so that

$$g_x' = -g\theta, \tag{4.40}$$

$$g_y' = g\phi, \tag{4.41}$$

$$g_z' = 0. \tag{4.42}$$

4.4 Equations of Motion in their General and Perturbation Forms

If, as is usual, the force and moment equations of motion based on Newton's second law are expressed in a moving body-axis system with origin at the aircraft centre of gravity position, allowance must be made for the instantaneous rotation of these axes relative to an initially coincident inertial axis system. If the vector $\underline{\mathbf{\Omega}}$ denotes the instantaneous angular velocity of the body axes, then general vector analysis (see Reference 24, for example) gives derivatives with respect to time of the velocity and angular momentum vectors, $\underline{\mathbf{V}}$ and $\underline{\mathbf{H}}$ in the inertial axis system, equal to $\dot{\underline{\mathbf{V}}} + \underline{\mathbf{\Omega}} \wedge \underline{\mathbf{V}}$ and $\dot{\underline{\mathbf{H}}} + \underline{\mathbf{\Omega}} \wedge \underline{\mathbf{H}}$ in the body axis system. Using these expressions, the vector equation describing the motion of the centre of gravity in body axes is

$$\underline{\mathbf{R}} + m_e\,\underline{\mathbf{g}} = m_e\,(\dot{\underline{\mathbf{V}}} + \underline{\mathbf{\Omega}} \wedge \underline{\mathbf{V}}), \tag{4.43}$$

and the vector equation for the rotation of the body about the centre of gravity position is

$$\underline{\mathbf{M}} = \dot{\underline{\mathbf{H}}} + \underline{\mathbf{\Omega}} \wedge \underline{\mathbf{H}}, \tag{4.44}$$

The components of $\underline{\mathbf{R}}$ and $\underline{\mathbf{M}}$ are the aerodynamic forces and moments acting on the aircraft. The components of the gravitational force $m_e \underline{\mathbf{g}}$ have been discussed previously in Section 4.3. There are no moments due to gravity because the axes origin is at the centre of gravity position.

The angular momentum vector can be expressed as

$$\underline{\mathbf{H}} = \mathbf{I_n} \underline{\Omega} + \underline{\mathbf{J}} \tag{4.45}$$

where the body-axis inertia matrix is

$$\mathbf{I_n} = \begin{bmatrix} I_x & -I_{xy} & -I_{zx} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{zx} & -I_{yz} & I_z \end{bmatrix}, \tag{4.46}$$

and $\underline{\mathbf{J}}$ represents the constant angular momentum of rotating parts of the aircraft or engines.

Expansion of Equations (4.43) and (4.44) in scalar form gives

$$m_e(\dot{u} + qw - rv) = X + m_e g_x, \tag{4.47}$$

$$m_e(\dot{v} + ru - pw) = Y + m_e g_y, \tag{4.48}$$

$$m_e(\dot{w} + pv - qu) = Z + m_e g_z, \tag{4.49}$$

$$I_x \dot{p} - I_{yz}(q^2 - r^2) - I_{zx}(\dot{r} + pq) - I_{xy}(\dot{q} - rp) - (I_y - I_z)qr + J_z q - J_y r = \mathscr{L}, \tag{4.50}$$

$$I_y \dot{q} - I_{zx}(r^2 - p^2) - I_{xy}(\dot{p} + qr) - I_{yz}(\dot{r} - pq) - (I_z - I_x)rp + J_x r - J_z p = \mathscr{M}, \tag{4.51}$$

$$I_z \dot{r} - I_{xy}(p^2 - q^2) - I_{yz}(\dot{q} + rp) - I_{zx}(\dot{p} - qr) - (I_x - I_y)pq + J_y p - J_x q = \mathscr{N}. \tag{4.52}$$

The standard technique in the subsequent treatment of the equations for the study of small disturbances from a trimmed steady state is to seek linearised forms. To achieve this, all variables are expressed in terms of perturbations with respect to datum values appropriate to the trimmed steady state, and the substitutions $X = X_e + X'$, $u = u_e + u'$, $g_x = g_{xe} + g'_x$, *etc.* are made. Simplification is then obtained because the datum values, as a set, satisfy the equations of motion and products of perturbations, for example $r'v'$, can be neglected. With the further simplifying assumption that the angular momentum of rotating parts of the aircraft and engines is negligible, so that $J_x = J_y = J_z = 0$, the linearised equations of motion are

$$\dot{u}' + q_e w' + w_e q' - r_e v' - v_e r' - g'_x = X'/m_e = -x', \tag{4.53}$$

$$\dot{v}' + r_e u' + u_e r' - p_e w' - w_e p' - g'_y = Y'/m_e = -y', \tag{4.54}$$

$$\dot{w}' + p_e v' + v_e p' - q_e u' - u_e q' - g'_z = Z'/m_e = -z', \tag{4.55}$$

$$\dot{p}' + 2d_x(q_e q' - r_e r') + e_x(\dot{r}' + p_e q' + q_e p') + f_x(\dot{q}' - r_e p' - p_e r') + b_x(q_e r' + r_e q') = \mathscr{L}'/I_x = -l', \tag{4.56}$$

$$\dot{q}' + 2e_y(r_e r' - p_e p') + f_y(\dot{p}' + q_e r' + r_e q') + d_y(\dot{r}' - p_e q' - q_e p') + b_y(r_e p' + p_e r') = \mathcal{M}'/I_y = -m', \tag{4.57}$$

$$\dot{r}' + 2f_z(p_e p' - q_e q') + d_z(\dot{q}' + r_e p' + p_e r') + e_z(\dot{p}' - q_e r' - r_e q') + b_z(p_e q' + q_e p') = \mathcal{N}'/I_z = -n', \tag{4.58}$$

where, for convenience, the inertia ratios

$$(d_x, d_y, d_z) = (-I_{yz}/I_x, -I_{yz}/I_y, -I_{yz}/I_z), \tag{4.59}$$

$$(e_x, e_y, e_z) = (-I_{zx}/I_x, -I_{zx}/I_y, -I_{zx}/I_z), \tag{4.60}$$

$$(f_x, f_y, f_z) = (-I_{xy}/I_x, -I_{xy}/I_y, -I_{xy}/I_z), \tag{4.61}$$

and

$$(b_x, b_y, b_z) = ((I_z - I_y)/I_x, (I_x - I_z)/I_y, (I_y - I_x)/I_z) \tag{4.62}$$

have been introduced. The forces and moments are written in the concise forms x', y', z', l', m', n', defined in Section 3.3. For their solution Equations (4.53) to (4.58) must be associated with the kinematic relationships of Equations (4.26) to (4.28) of Section 4.2 relating angular velocity perturbations to rates of change of attitude-deviation angles. Nine simultaneous equations for u', v', w', p', q', r', ϕ, θ and ψ are therefore involved.

It is to be noted that for an aircraft with mass symmetry about the xz plane $d_x = d_y = d_z = f_x = f_y = f_z = 0$ and there is a considerable simplification of Equations (4.56) to (4.58),

$$\dot{p}' + e_x(\dot{r}' + p_e q' + q_e p') + b_x(q_e r' + r_e q') = -l', \tag{4.63}$$

$$\dot{q}' + 2e_y(r_e r - p_e p') + b_y(r_e p' + p_e r') = -m', \tag{4.64}$$

$$\dot{r}' + e_z(\dot{p}' - q_e r' - r_e q') + b_z(p_e q' + q_e p') = -n'. \tag{4.65}$$

4.5 Dynamic-normalised System

So far the equations of motion have been developed in dimensional form. More generality is provided by a consistent set of dimensionless units.

Although the aero-normalised system (see Section 3.4) is convenient from an aerodynamic viewpoint, it is not well suited to the discussion of aircraft dynamics. It has two main drawbacks, the first of these being the smallness of the unit of time, l_0/V_e, and the second being the fact that it is unrelated to the inertial properties of the aircraft. To remedy these drawbacks an alternative normalising system is required for forming dimensionless quantities in dynamic studies, and the dynamic-normalised system recommended in Reference 16 is adopted.

In the dynamic-normalised system the basic units are those of force, speed and mass. As in the aero-normalised system the unit of force is $\frac{1}{2}\rho_e V_e^2 S$ and the unit of speed is V_e. The unit of mass is that of the aircraft, strictly at the datum condition, m_e. The same symbols are used to denote quantities in dimensionless or dynamic-normalised form. There is seldom any confusion as to which system is being used but, if necessary, a dressing ($^\circ$) for the former or a dressing ($\hat{}$) for the latter may be adopted, *e.g.* $\mathring{z}$ or $\hat{z}$.

By dimensional analysis the normalising factors for any quantity can be found by an appropriate combination of the basic units. This process, which is also a feature of the aero-normalised system, ensures that all units are mutually consistent. In consequence the equations of motion assume the same form whether in ordinary, aero-normalised or dynamic-normalised units and need carry no distinguishing marks.

Tables C1.3 and C1.4 give the normalising factors required to convert the most commonly needed quantities from dimensional to dynamic-normalised form. These involve V_e and the dynamic-normalised unit of time,

$$\tau = \frac{m_e}{\tfrac{1}{2}\rho_e V_e S} . \tag{4.66}$$

The ratio of the dynamic-normalised unit of mass to the aero-normalised unit of mass is expressed as the relative density parameter

$$\mu = \frac{m_e}{\tfrac{1}{2}\rho_e S l_0} . \tag{4.67}$$

The ratios of the units of length and time are also equal to μ and this parameter features prominently in the conversion of aero-normalised data to dynamic-normalised form. Aerodynamic derivatives will in most cases be provided in aero-normalised form, and so it is convenient to convert them directly into dynamic-normalised concise derivatives for substitution in dynamic-normalised equations of motion. The last three columns of Table C1.1 give the simple relationships by which this is achieved. The conversion factors for the force derivatives are -1 or $-1/\mu$. Those for the moment derivatives are -1 or $-\mu$ with an inertia parameter divisor i_x, i_y, i_z, as appropriate, where

$$(i_x , i_y , i_z) = \frac{1}{m_e l_0^2} (I_x , I_y , I_z) . \tag{4.68}$$

The basic units of the dynamic-normalised system do not involve a choice of length. Therefore the length l_0 selected for forming the aero-normalised derivatives is immaterial so long as the conversion to dynamic-normalised concise derivatives is carried out with consistent substitution. For example, if the aero-normalised value of L_v is based on $\tfrac{1}{2}\rho_e V_e Sb$ it is essential to associate this with μ defined as $m_e / \tfrac{1}{2}\rho_e Sb$ and i_x as $I_x / m_e b^2$ so that the dynamic-normalised value of l_v is independent of b. Because of this, separate sets of longitudinal and lateral aero-normalised derivatives based on two different representative lengths, for example $l_0 = \bar{\bar{c}}$ or $\bar{c}$ for the former and b or $b/2$ for the latter, will provide a single consistent set of concise dynamic-normalised derivatives suitable for studying coupled motion, provided that they are associated with longitudinal and lateral values of μ and inertia parameters that are in each case based on the relevant reference length.

5. ANALYSIS OF EQUATIONS

5.1 State-space Formulation

The ready availability of computerised matrix handling techniques allows numerical solutions for the nine simultaneous equations describing small perturbation motion to be achieved rapidly. This is accomplished most simply if the complete set of equations is arranged in a state-space form where the time derivation of each state variable is expressed as a function of the current values of the state variables and any forcing inputs.

Throughout Section 5 only perturbation motions are considered and therefore the primes that are used formally to identify perturbation variables are dropped from the notation.

Thus if $\underline{\mathbf{u}}$ is the state vector of perturbation variables, and $\underline{\eta}$ is the vector of control perturbations the equations of motion may be expressed in the form

$$\underline{\mathbf{A}}\ \dot{\underline{\mathbf{u}}} + \underline{\mathbf{B}}\,\underline{\mathbf{u}} + \underline{\mathbf{C}}\,\underline{\eta} = 0 \tag{5.1}$$

where $\underline{\mathbf{A}}$, $\underline{\mathbf{B}}$ and $\underline{\mathbf{C}}$ are matrices.

For controls-fixed motion $\underline{\eta} = 0$ and the nine roots (or eigenvalues) of the determinant $\left|\boldsymbol{\lambda}\,\mathbf{I_9} + \underline{\mathbf{A}}^{-1}\,\underline{\mathbf{B}}\right|$ determine the stability of the aircraft.

The roots satisfy a ninth-order polynomial in λ given by

$$\det\left|\lambda\,\mathbf{I_9} + \underline{\mathbf{A}}^{-1}\,\underline{\mathbf{B}}\right| = 0 \tag{5.2}$$

which is known as the characteristic equation. This can be extracted either numerically or algebraically, although the latter approach is only really practicable in simple cases.

Each variable will have a solution involving the nine roots which, taking u as an example, will be of the form

$$u = \sum_{i=1}^{9} a_i \exp\,(\lambda_i t)\,, \tag{5.3}$$

where a_i is constant.

A study of the roots of the characteristic equation shows up certain critical features of the disturbed motion which may be examined independently of the full solution of the equations of motion. Stability depends on the nature of all the roots and the aircraft is only stable if every root corresponds to a disturbance that ultimately dies out.

The motions associated with individual roots of different type are as follows,

	λ	zero corresponds to an undamped persistence of the original disturbance (neutral stability),
	λ	real and positive corresponds to an increasing disturbance (divergence) with time to double amplitude, $t_{2/1} = (\log_e 2)/\lambda$,
	λ	real and negative corresponds to a decreasing disturbance (subsidence) with time to half amplitude, $t_{1/2} = (\log_e 2)/(-\lambda)$,
and	λ	complex corresponds to an oscillation, increasing in magnitude if the real part is positive or decreasing in magnitude if it is negative. The time to double or half amplitude to $(\log_e 2) / \lvert \mathscr{R}(\lambda) \rvert$, and the period of oscillation is $T = 2\pi / \lvert \mathscr{J}(\lambda) \rvert$.

Thus the aircraft is stable if there is no root with a positive real part. The parameters, $t_{2/1}, t_{1/2}$ and T are measured in the time unit of the system of units being used. For a dimensional system they are expressed in seconds, in a dynamic-normalised system in units of τ *etc*. Other parameters can be used to categorise the damping of the motions and these are discussed in Item No. 67038 (Reference 15) of the Dynamics Sub-series.

The dependence of the stability of an aircraft on some particular parameter is best presented graphically. For example, a root locus can be plotted on an Argand diagram as the parameter is varied. A description of this and other types of graphical display may be found in Item No. 74020 (Reference 19) of the Dynamics Sub-series.

5.2 Separation of Equations into Longitudinal and Lateral Groups

In some cases it is possible to separate the equations of motion into a ‘longitudinal’ group involving only the variables u, w, q, θ and η and a ‘lateral’ group involving v, p, r, ϕ, ψ, ξ and ζ. In such a case the longitudinal and lateral motions are fully independent and exist with no coupling.

The first requirement is to impose a datum condition of straight (rectilinear) motion, $p_e = q_e = r_e = 0$. This removes cross-coupling from the kinematic equations and allows Equations (4.29) to (4.31) to be used. It also reduces, but does not eliminate, the cross-coupling of linear and angular velocities in Equations (4.53) to (4.55), $q_e w' = 0$ *etc*., and removes some, but not all, of the inertia related cross-coupling in Equations (4.56) to (4.58), *i.e*. the terms involving d_x, b_x, e_y, b_y, f_z and b_z disappear.

The concise forms of the aerodynamic forces and moments are now expanded in derivative form. In addition to the derivatives such as x_u, y_v *etc*. that correspond to the concise forms of the aerodynamic derivatives appearing explicitly in the expansions of Equations (3.7) to (3.12), for illustration purposes velocity derivatives such as x_v, y_u *etc*. are included as typical cross-coupling derivatives. With some rearrangement to anticipate separation the simplified equations of motion may now be written in the state-space form of

Equation (5.1) with

$$\underline{\mathbf{A}} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$$

$$= \left(\begin{array}{cccc|ccccc} 1 & x_{\dot{w}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1+z_{\dot{w}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_{\dot{w}} & 1 & 0 & 0 & f_y & d_y & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & f_x & 0 & 0 & 1 & e_x & 0 & 0 \\ 0 & 0 & d_z & 0 & 0 & e_z & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \tag{5.4}$$

$$\underline{\mathbf{B}} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{pmatrix}$$

$$= \left(\begin{array}{cccc|ccccc} x_u & x_w & x_q+w_e & g_1\cos\Phi_e & x_v & 0 & -v_e & 0 & -g_1\sin\Phi_e \\ z_u & z_w & z_q-u_e & g_2 & z_v & v_e & 0 & g_1\sin\Phi_e & 0 \\ m_u & m_w & m_q & 0 & m_v & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline y_u & y_w & 0 & 0 & y_v & y_p-w_e & y_r+u_e & -g_1\cos\Phi_e & -g_2 \\ l_u & l_w & 0 & 0 & l_v & l_p & l_r & 0 & 0 \\ n_u & n_w & 0 & 0 & n_v & n_p & n_r & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{array}\right) \tag{5.5}$$

$$\underline{\mathbf{C}} = \begin{pmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{pmatrix}$$

$$= \left(\begin{array}{c:cc} x_\eta & 0 & 0 \\ z_\eta & 0 & 0 \\ m_\eta & 0 & 0 \\ 0 & 0 & 0 \\ \hdashline 0 & y_\xi & y_\zeta \\ 0 & l_\xi & l_\zeta \\ 0 & n_\xi & n_\zeta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \tag{5.6}$$

$$\underline{\mathbf{u}} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} = \left(\begin{array}{c} u \\ w \\ q \\ \theta \\ \hdashline v \\ p \\ r \\ \phi \\ \psi \end{array}\right) \tag{5.7}$$

and

$$\underline{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \left(\begin{array}{c} \eta \\ \hdashline \xi \\ \zeta \end{array}\right) . \tag{5.8}$$

In partitioned form Equation (5.1) is

$$\begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}\begin{pmatrix} \dot{\mathbf{u}}_1 \\ \dot{\mathbf{u}}_2 \end{pmatrix} + \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{pmatrix}\begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} + \begin{pmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = 0 . \tag{5.9}$$

For the longitudinal and lateral motions to be separable it is necessary that $\mathbf{A}_{12} = \mathbf{A}_{21} = \mathbf{B}_{12} = \mathbf{B}_{21} = \mathbf{C}_{12} = \mathbf{C}_{21} = 0$.

For the case shown, where control input is provided through η, ξ and ζ, $\mathbf{C}_{12}$ and $\mathbf{C}_{21}$ are both zero and the controls provide no coupling. For more general control systems this may not be so. But attention will now be formally restricted to motion with fixed controls and only matrices $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ will be considered.

It can be seen that the imposition of straight flight is not in itself sufficient for separation of the longitudinal and lateral motions. For the matrix $\underline{\mathbf{A}}$, $\mathbf{A}_{12}$ and $\mathbf{A}_{21}$ are zero only if f_x, f_y, d_y and d_z are all zero. This requires $I_{xy} = I_{yz} = 0$, a condition that will be satisfied if the aircraft has mass symmetry about the xz plane. If this is so then separation depends only on requirements imposed on the matrix $\underline{\mathbf{B}}$.

The conditions on matrix $\underline{\mathbf{B}}$ for separating the longitudinal equations are more stringent than those for the lateral equations. The lateral equations can be treated separately provided that any lateral cross-coupling derivatives, y_u *etc*. appearing in $\mathbf{B}_{21}$ are zero. The longitudinal equations can always be considered separately if the lateral perturbations v, p, r, ϕ, ψ are zero. However, if lateral perturbation is permitted, the longitudinal equations separate only if the datum flight condition is symmetrical so that there is no sideslip velocity, $v_e = 0$, and no bank angle, $\Phi_e = 0$, and if any longitudinal cross-coupling derivatives, x_v *etc*. are also zero, so that every element of $\mathbf{B}_{12}$ is zero. Reference 16 contains a more general discussion on the conditions for separation of the equations, and gives the additional requirements if the aircraft is in ground effect or there is a wind.

With the necessary conditions for separation satisfied the longitudinal motion in straight symmetrical flight with controls fixed is governed by the matrix equation.

$$\mathbf{A}_{11}\,\dot{\mathbf{u}}_1 + \mathbf{B}_{11}\,\mathbf{u}_1 = 0\,, \tag{5.10}$$

with characteristic roots given by the quartic equation

$$\mathbf{det}\left|\lambda\,\mathbf{I}_4 + \mathbf{A}_{11}^{-1}\,\mathbf{B}_{11}\right| = 0\,. \tag{5.11}$$

Similarly, the separated equations for lateral motion are

$$\mathbf{A}_{22}\,\dot{\mathbf{u}}_2 + \mathbf{B}_{22}\,\mathbf{u}_2 = 0 \tag{5.12}$$

with characteristic roots given by the quintic equation

$$\mathbf{det}\left|\lambda\,\mathbf{I}_5 + \mathbf{A}_{22}^{-1}\,\mathbf{B}_{22}\right| = 0\,. \tag{5.13}$$

Thus of the nine roots of the general solution of controls-fixed motion four are now associated with the longitudinal motion and five with the lateral motion.

From Equations (5.4), (5.5) and (5.10), the longitudinal equations are expanded and slightly rearranged to give

$$(\mathrm{D} + x_u)\,u + (x_{\dot{w}}\mathrm{D} + x_w)\,w + (x_q + w_e)\,q + g_1\theta = 0\,, \tag{5.14}$$

$$z_u\,u + [(1 + z_{\dot{w}})\,\mathrm{D} + z_w]\,w + (z_q - u_e)\,q + g_2\theta = 0\,, \tag{5.15}$$

$$m_u\,u + (m_{\dot{w}}\mathrm{D} + m_w)\,w + (\mathrm{D} + m_q)\,q = 0\,, \tag{5.16}$$

$$-\,q + \mathrm{D}\theta = 0\,, \tag{5.17}$$

where D is the differential operator d/dt.

Similarly, from Equations (5.4), (5.5) and (5.12), the group of lateral equations is

$$(\mathrm{D}+y_v)\,v+(y_p-w_e)\,p+(y_r+u_e)\,r-g_1\phi-g_2\psi = 0\,, \tag{5.18}$$

$$l_v v+(\mathrm{D}+l_p)\,p+(e_x\mathrm{D}+l_r)\,r = 0\,, \tag{5.19}$$

$$n_v v+(e_z\mathrm{D}+n_p)\,p+(\mathrm{D}+n_r)\,r = 0\,, \tag{5.20}$$

$$-p+\mathrm{D}\phi = 0\,, \tag{5.21}$$

$$-r+\mathrm{D}\psi = 0\,. \tag{5.22}$$

These equations hold for any body-axis system. If aerodynamic body-axes are specified there is the simplification $u_e = V_e$ and $w_e = 0$, and Θ_e is equal to the angle of climb, γ_e. Although the equations are of equal validity in dimensional, aero-normalised or dynamic-normalised form, the last of these is customary and permits the substitution $V_e = 1$. With the assumptions of aerodynamic-body axes and dynamic normalised form the gravitational factors become proportional to the lift coefficient in the datum condition with

$$g_1 = g\cos\Theta_e = C_{Le} \tag{5.23}$$

and

$$g_2 = g\sin\Theta_e = C_{Le}\tan\Theta_e = C_{Le}\tan\gamma_e\,. \tag{5.24}$$

5.3 Analysis of Separated Equations

A simultaneous solution of all nine roots of the controls-fixed motion can be achieved by computer directly from Equation (5.2) with the elements of the matrices $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ satisfying the conditions necessary for separation of the longitudinal and lateral motions. Alternatively, and more simply, the four roots of the longitudinal motion can be found by solving Equation (5.11) and the five roots of the lateral motion by solving Equation (5.13). Numerical solutions are now customary and provide a direct test of stability.

Nevertheless, it is constructive to consider the characteristic equations of the separated longitudinal and lateral motions in their algebraic forms so that the importance of the various individual aerodynamic derivatives is clear. For the longitudinal motion the connections with the classical concepts of static margin and manoeuvre margin, as introduced in Reference 2, can also be demonstrated.

With the assumption of aerodynamic-body axes and a dynamic-normalised system the algebraic form of the characteristic equation for the longitudinal motion is obtained through Equation (5.10) as

$$\mathbf{det}\,\left|\lambda\,\mathbf{A_{11}}+\,\mathbf{B_{11}}\right| = \mathbf{det}\begin{vmatrix} \lambda+x_u & x_{\dot{w}}\lambda+x_w & x_q & g_1 \\ z_u & (1+z_{\dot{w}})\lambda+z_w & z_q-1 & g_2 \\ m_u & m_{\dot{w}}\lambda+m_w & \lambda+m_q & 0 \\ 0 & 0 & -1 & \lambda \end{vmatrix} = 0 \tag{5.25}$$

which is equivalent to Equation (5.11).

This can be fully expanded as the quartic equation

$$A_1\lambda^4 + B_1\lambda^3 + C_1\lambda^2 + D_1\lambda + E_1 = 0\,, \tag{5.26}$$

where
$$A_1 = 1 + z_{\dot{w}}\,, \tag{5.27}$$
$$B_1 = x_u(1+z_{\dot{w}}) + z_w - x_{\dot{w}}z_u + (1+z_{\dot{w}})\,m_q + (1-z_q)\,m_{\dot{w}}\,, \tag{5.28}$$
$$\begin{aligned} C_1 = {} & x_u z_w - x_w z_u + [x_u(1+z_{\dot{w}}) + z_w - x_{\dot{w}}z_u]\,m_q \\ & + [x_u(1-z_q) + x_q z_u - g_2]\,m_{\dot{w}} + (1-z_q)\,m_w \\ & - [x_{\dot{w}}(1-z_q) + x_q(1+z_{\dot{w}})]\,m_u\,, \end{aligned} \tag{5.29}$$
$$\begin{aligned} D_1 = {} & (x_u z_w - x_w z_u)\,m_q + (g_1 z_u - g_2 x_u)\,m_{\dot{w}} \\ & + [x_u(1-z_q) + x_q z_u - g_2]\,m_w \\ & - [x_w(1-z_q) + x_q z_w + g_1(1+z_{\dot{w}}) - g_2 x_{\dot{w}}]\,m_u\,, \end{aligned} \tag{5.30}$$
and
$$E_1 = (g_1 z_u - g_2 x_u)\,m_w - (g_1 z_w - g_2 x_w)\,m_u\,. \tag{5.31}$$

Similarly, the characteristic equation for the lateral group is obtained via Equation (5.12)

$$\mathbf{det}\left|\lambda\,\mathbf{A}_{22} + \mathbf{B}_{22}\right| = \mathbf{det}\begin{vmatrix} \lambda + y_v & y_p & y_r + 1 & -g_1 & -g_2 \\ l_v & \lambda + l_p & e_x\lambda + l_r & 0 & 0 \\ n_v & e_z\lambda + n_p & \lambda + n_r & 0 & 0 \\ 0 & -1 & 0 & \lambda & 0 \\ 0 & 0 & -1 & 0 & \lambda \end{vmatrix} = 0 \tag{5.32}$$

which is equivalent to Equation (5.13).

In expanded form this is a quintic equation

$$\lambda(A_2\lambda^4 + B_2\lambda^3 + C_2\lambda^2 + D_2\lambda + E_2) = 0\,, \tag{5.33}$$

where
$$A_2 = 1 - e_x e_z\,, \tag{5.34}$$
$$B_2 = l_p + n_r - e_x n_p - e_z l_r + (1 - e_x e_z)\,y_v\,, \tag{5.35}$$
$$\begin{aligned} C_2 = {} & l_p n_r - l_r n_p + (l_p + n_r - e_x n_p - e_z l_r)\,y_v \\ & + [e_z(1+y_r) - y_p]\,l_v - [1 + y_r - e_x y_p]\,n_v\,, \end{aligned} \tag{5.36}$$
$$\begin{aligned} D_2 = {} & (l_p n_r - l_r n_p)\,y_v + [n_p(1+y_r) - n_r y_p + g_1 - e_z g_2]\,l_v \\ & - [l_p(1+y_r) - l_r y_p - g_2 + e_x g_1]\,n_v\,, \end{aligned} \tag{5.37}$$
and
$$E_2 = (g_1 n_r - g_2 n_p)\,l_v - (g_1 l_r - g_2 l_p)\,n_v\,. \tag{5.38}$$

It can be seen that Equation (5.33) has one trivial root $\lambda = 0$. This reflects the fact that the aircraft heading (azimuth) is immaterial so far as stability is concerned (see also Reference 25).* Thus the longitudinal motion and, in essence, the lateral motion are both governed by characteristic quartic equations. Although

* And a stable aircraft will not necessarily resume its original heading after a disturbance.

it is nowadays normally rendered superfluous by direct numerical extraction of the roots, there is a traditional test that can be applied to a quartic equation to verify that there is no root with a positive real part, implying overall stability. If the equation is of the form

$$A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0\,, \tag{5.39}$$

then the necessary and sufficient condition is that all of the coefficients must be positive and Routh's discriminant

$$R = D(BC - AD) - B^2 E \tag{5.40}$$

must be positive.

5.3.1 Longitudinal motion

(i) Static stability and manoeuvrability

For an angle of climb $\Theta_e = \gamma_e$ Equation (5.31) can be written

$$E_1 = g[(z_u m_w - z_w m_u)\cos\gamma_e + (x_w m_u - x_u m_w)\sin\gamma_e]\,. \tag{5.41}$$

References 2 and 9 show that the coefficient E_1 is proportional to and has the same sign as the derivative $-\,\mathrm{d}C_m/\mathrm{d}C_L$, where the differentiation is performed subject to the imposed condition

$$C_R V^2 = \frac{W}{\frac{1}{2}\rho S} \text{ (= constant for constant } \rho\text{).} \tag{5.42}$$

In the general case, when the coefficient derivatives $(\partial C_m/\partial V)_e$ and $(\partial C_R/\partial V)_e$ are not zero, the static margin k_s is written

$$k_s = -\frac{\mathrm{d}C_m}{\mathrm{d}C_R} = \frac{1}{2C_R(\partial C_R/\partial\alpha)}\frac{i_y}{\mu}[(z_u m_w - z_w m_u)\cos\gamma_e + (x_w m_u - x_u m_w)\sin\gamma_e]\,. \tag{5.43}$$

Positive static stability, $E_1 > 0$, is a necessary but not sufficient condition for stability in dynamic motion.

In the basic case, when $(\partial C_m/\partial V)_e = (\partial C_R/\partial V)_e = 0$, and for horizontal flight, $\gamma_e = 0$, there are simplified forms of Equations (5.41) and (5.43),

$$E_1 = g z_u m_w \tag{5.44}$$

and

$$k_s = \frac{i_y}{\mu}\frac{m_w}{(\partial C_L/\partial\alpha)}\,. \tag{5.45}$$

Attention is now restricted to a horizontal flight condition under the assumption that speed does not change, that is $u = 0$. With respect to aerodynamic-body axes the disturbed motion in this case is described by the equations

$$(\mathrm{D}+z_w)\,w-q=0\,, \tag{5.46}$$

and $$(m_{\dot{w}}\mathrm{D}+m_w)\,w+(\mathrm{D}+m_q)\,q=0\,, \tag{5.47}$$

where the small terms in z_q and $z_{\dot{w}}$ have been neglected. Their characteristic equation is

$$\lambda^2+(m_q+m_{\dot{w}}+z_w)\,\lambda+(m_w+m_q z_w)=0\,. \tag{5.48}$$

The conditions which lead to Equation (5.48) are precisely those underlying the derivation of the manoeuvre margin (see Appendix A). There is therefore a close link between the manoeuvre margin and the constant term $(m_w+m_q z_w)$. In fact, with the assumption $z_w=\partial C_L/\partial\alpha+C_D\approx\partial C_L/\partial\alpha$, the manoeuvre margin is

$$k_m=\frac{i_y}{\mu}\left(\frac{m_w}{z_w}+m_q\right)=\frac{i_y}{\mu z_w}\,(m_w+m_q z_w)\,. \tag{5.49}$$

In general, see References 2 and 9, it can be shown that when a steady-state (or trimmed) flight condition is changed to a neighbouring one the increment in elevator angle, $\Delta\bar{\eta}$, required to maintain this new state is proportional to the constant term of the appropriate characteristic equation. In this way the physical significance of the two concepts is that, as indicated in Appendix A,

(i) the controls-fixed static margin is proportional to $d\bar{\eta}/dC_R$,

(ii) the controls-fixed manoeuvre margin is proportional to $\Delta\bar{\eta}$ per g.

(ii) Nature of the motion

To place the preceding discussion within the context of the full longitudinal motion it is necessary to return to the general quartic form of Equation (5.26). Although other possibilities exist, the characteristic equation will almost always separate into two quadratic factors, each with a pair of complex conjugate roots.

For aircraft for which $D_1 \ll C_1$ and $E_1 \ll C_1^2$, a good approximation to the exact factorisation is provided by

$$(A_1\lambda^2+B_1\lambda+C_1)\left(\lambda^2+\frac{(C_1D_1-B_1E_1)}{C_1^2}\,\lambda+\frac{E_1}{C_1}\right)=0\,. \tag{5.50}$$

The first factor in Equation (5.50) is akin to the characteristic equation of the constant speed motion, Equation (5.48). It usually corresponds to a well-damped short-period oscillation that, in the general case, mainly involves changes in pitch rate, q, and angle of attack with very little change in forward speed.

The second factor usually corresponds to a lightly-damped long-period oscillation, the so-called phugoid. In this mode of motion the main changes occur in forward speed and pitch angle, θ, (or height) with very little change in angle of attack.

Thus in any motion initiated by changes in pitch rate or angle of attack the short-period mode dominates the initial stages whilst the phugoid persists into and dominates the closing stages of the disturbed motion.

5.3.2 Lateral motion

The four non-trivial roots of Equation (5.33) embody the nature of the lateral motion. In contrast to the longitudinal motions the lateral roots are more dependent on the geometry and inertia constants of the aircraft through the ratios e_x and e_z.

For a wide range of aircraft with wings of large aspect ratio and of modest sweep there are two real roots corresponding to two exponential modes, the rolling subsidence and the spiral mode, and a pair of complex conjugate roots corresponding to an oscillation, the so-called Dutch roll. A physical description of the motion in the two exponential modes and the Dutch roll (each in isolation) is given in Appendix B, but the salient points in relation to the roots of the quartic factor of Equation (5.33) are made below.

General ways of solving the characteristic equation are described in Item No. 83024 (Reference 26) of the Dynamics Sub-series, but for aircraft for which $A_2 \approx 1$, $B_2 \approx (l_p \gg 1)$ and $|E_2| \ll D_2$ there are two simple approximations to the roots for the exponential modes. There is the large root

$$\lambda \approx -B_2/A_2 \approx -l_p, \tag{5.51}$$

and the small root

$$\lambda \approx -E_2/D_2. \tag{5.52}$$

The first root corresponds to the rolling subsidence. The second root corresponds to the spiral motion. For small disturbances from horizontal flight $E_2 = g_1(n_r l_v - l_r n_v)$ and its sign depends on the relative magnitude of $n_r l_v$ and $l_r n_v$. For many aircraft D_2 is positive and the sign of E_2 determines the sign of the root, and if $E_2 > 0$ the spiral motion subsides and if $E_2 < 0$ it diverges.

The lateral oscillation is determined by the remaining pair of complex roots and increases or decreases according to the sign of their real part. The period and damping of the motion depend in a complicated way on the aerodynamic derivatives and inertia characteristics of the aircraft but once the two real roots have been determined accurately the quadratic factor containing the roots of the Dutch roll is easily isolated numerically, see the worked example in Section 7.

Where the geometrical and inertial characteristics depart markedly from those described above, for example for aircraft with highly swept wings of small aspect ratio, the above approximations become increasingly less reliable (see References 7 and 11).

Eventually it is possible for aircraft that have highly swept wings and are inertially slender to exhibit different modes of lateral motion, namely two oscillatory modes. The Dutch roll exists as before, but has a much increased roll to yaw ratio, and a long period oscillation replaces the roll subsidence and spiral modes.

Alternatively, and rarely, it is possible for the Dutch roll to break down into two exponential modes (see Reference 21).

In all the cases mentioned the more general approximate solutions given in Item No. 83024 provide satisfactory results.

5.4 Controls-free Stability

The advent of power-operated controls has reduced the former significance of controls-free stability, although it is still of importance for aircraft for which there is a simple mechanical link between the cockpit controls and the control surfaces.

To study the controls-free stability, the equations describing the motion of the aircraft must now include control force and moment terms and be solved simultaneously with the equations that describe the motion of the controls about their hinge line. This increases the order of the characteristic equation. For example, with a free elevator the longitudinal motion (assumed to be separable) has a characteristic equation that is a sextic, although simplifying assumptions are available to reduce it to a quartic, see Reference 9. Similar considerations apply for lateral motion with a free rudder, see Reference 4.

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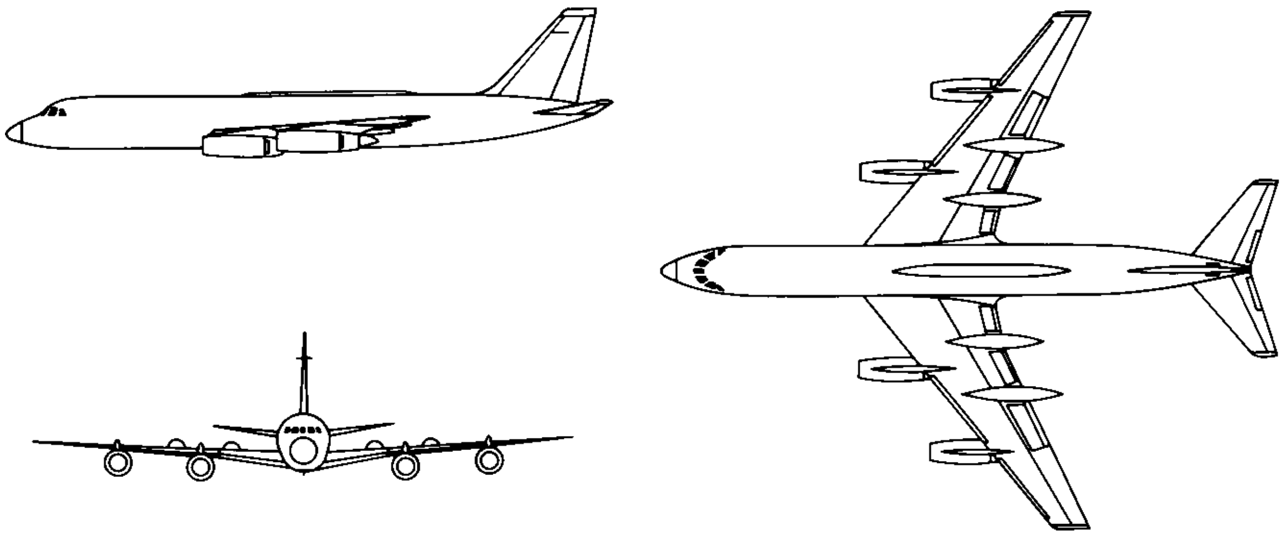
7. EXAMPLE

Find the roots of the characteristic equations for the controls-fixed longitudinal and lateral stability of the civil jet transport aircraft shown in Sketch 7.1 when it is slightly disturbed from straight and symmetrical steady level flight.

The aircraft properties and flight conditions are given in the table. The values are based on information given in Reference 17.



Sketch 7.1

$m_e = 75600$ kg	$C_L = 0.700$
$S = 180$ m^2	$C_D = 0.020 + 0.050 C_L^2$
$S_T = 36$ m^2	$\partial C_L/\partial\alpha = 4.50$ rad^{-1}
$l_T = 18$ m	$\partial C_{LT}/\partial\alpha = 3.50$ rad^{-1} (based on S_T)
	$\partial C_m/\partial\alpha = -0.675$ rad^{-1}
l_0 (longitudinal) = $\bar{\bar{c}} = 6$ m	$\partial\varepsilon/\partial\alpha = 0.5$
l_0 (lateral) = $b = 36$ m	$V_e = 120$ m/s
	$\rho_e = 0.700$ kg/m^3
$I_x = 2.5 \times 10^6$ kg m^2	$M_e = 0.4$
$I_y = 4.8 \times 10^6$ kg m^2	$\mathrm{d}T/\mathrm{d}V = 0$
$I_z = 7.3 \times 10^6$ kg m^2	$g = 9.81$ m/s^2
$I_{zx} = -0.47 \times 10^6$ kg m^2	
$I_{xy} = I_{yz} = 0$	

The datum conditions of straight and symmetrical level flight $(p_e = q_e = r_e = v_e = \Phi_e = 0)$ and the fact that the aircraft has mass symmetry about the *xz*-plane $(I_{xy} = I_{yz} = 0)$ satisfy the conditions for the longitudinal and lateral motions of the aircraft to be considered separately (see Section 5.2). The equations of motion will be considered in dynamic-normalised form and for a system of aerodynamic-body axes (so $u_e = 1$ and $w_e = 0$).

Longitudinal stability

Equations (3.15) to (3.26) may be used to calculate the longitudinal aero-normalised aerodynamic derivatives. It is assumed that the derivatives $\partial C_L/\partial V$, $\partial C_D/\partial V$ and $\partial C_m/\partial V$ can be set to zero. Using the aerodynamic data from the table, assuming that the wing-body contributions $(Z_q)_{WB}$, $(M_{\dot{w}})_{WB}$ and $(M_q)_{WB}$ can be neglected in comparison with the tailplane contributions, and noting that $\mathrm{d}T/\mathrm{d}V = 0$, it follows that

$$\begin{aligned}
X_u &= -2\,(0.020 + 0.050 \times 0.700^2) - 0 + 0 = -0.0890,\\
X_w &= 0.700 - (0.1 \times 0.700 \times 4.5) = 0.385,\\
X_{\dot{w}} &= 0,\\
X_q &= 0,
\end{aligned}$$

$$\begin{aligned}
Z_u &= -2 \times 0.700 + 0 = -1.400,\\
Z_w &= -(0.0445 + 4.5) = -4.545,\\
Z_{\dot{w}} &= 0,\\
Z_q &= 0 - (3.50 \times 36 \times 18)/(180 \times 6) = -2.10,
\end{aligned}$$

$$\begin{aligned}
M_u &= 0,\\
M_w &= -0.675,\\
M_{\dot{w}} &= 0 - (3.50 \times 36 \times 18^2)/(180 \times 6^2) \times 0.5 = -3.15,\\
M_q &= 0 - (3.50 \times 36 \times 18^2)/(180 \times 6^2) = -6.30.
\end{aligned}$$

The relative density parameter is

$$\mu = \frac{m_e}{\frac{1}{2}\rho_e S l_0} = \frac{75600}{\frac{1}{2} \times 0.700 \times 180 \times 6} = 200\,,$$

the dynamic-normalised unit of time is

$$\tau = \frac{m_e}{\frac{1}{2}\rho_e V_e S} = \frac{75600}{\frac{1}{2} \times 0.700 \times 120 \times 180} = 10.0\ \text{s},$$

and the inertia parameter

$$i_y = \frac{I_y}{m_e l_0^2} = \frac{4.8 \times 10^6}{75600 \times 6^2} = 1.764\,.$$

From the relationships set out in columns three to five of Table C1.1 the dynamic-normalised concise forms of the derivatives are

$$
\begin{aligned}
x_u &= -X_u &&= 0.0890,\\
x_w &= -X_w &&= -0.385,\\
x_{\dot w} &= -X_{\dot w}/\mu &&= 0,\\
x_q &= -X_q/\mu &&= 0,
\end{aligned}
$$

$$
\begin{aligned}
z_u &= -Z_u &&= 1.400,\\
z_w &= -Z_w &&= 4.545,\\
z_{\dot w} &= -Z_{\dot w}/\mu &&= 0,\\
z_q &= -Z_q/\mu &&= 0.0105,
\end{aligned}
$$

$$
\begin{aligned}
m_u &= -\mu M_u/i_y &&= 0,\\
m_w &= -\mu M_w/i_y &&= 76.531,\\
m_{\dot w} &= -M_{\dot w}/i_y &&= 1.786,\\
m_q &= -M_q/i_y &&= 3.571.
\end{aligned}
$$

From Table C1.4 the dynamic-normalised value of g is $\overset{\circ}{g}\tau/V_e$. A datum condition of level flight implies $\Theta_e = 0$ so the dynamic-normalised values of g_1 and g_2 are

$$g_1 = g\cos\Theta_e = 9.81 \times \frac{10}{120} \times 1 = 0.818$$

and $$g_2 = g\sin\Theta_e = 0\,.$$

Using computerised matrix handling techniques the characteristic equation and its roots can be found for the longitudinal motion by substituting the appropriate values of the concise derivatives, gravitational terms and datum conditions $(u_e = 1, v_e = w_e = \Phi_e = 0)$ into Equation (5.11),

$$\mathbf{det}\left|\lambda\,\mathbf{I_4} + \mathbf{A_{11}^{-1}}\,\mathbf{B_{11}}\right| = 0\,.$$

From Equations (5.4) and (5.5)

$$\mathbf{A_{11}} = \begin{pmatrix} 1 & x_{\dot w} & 0 & 0\\ 0 & 1+z_{\dot w} & 0 & 0\\ 0 & m_{\dot w} & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 1.786 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $$\mathbf{B_{11}} = \begin{pmatrix} x_u & x_w & x_q + w_e & g_1\cos\Phi_e\\ z_u & z_w & z_q - u_e & g_2\\ m_u & m_w & m_q & 0\\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0.0890 & -0.385 & 0 & 0.818\\ 1.400 & 4.545 & -0.990 & 0\\ 0 & 76.531 & 3.571 & 0\\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Hence the characteristic equation is

$$\lambda^4 + 9.973\,\lambda^3 + 93.415\,\lambda^2 + 12.158\,\lambda + 87.643 = 0\,,$$

which has the two pairs of complex conjugate roots

(i) $\lambda = -4.97 \pm 8.21\mathrm{i}$

and (ii) $\lambda = -0.0147 \pm 0.975\mathrm{i}\,.$

From the expressions in Section 5.1, the period of the first (short-period) oscillation is

$$T = \frac{2\pi}{8.21} = 0.765 \text{ in dynamic-normalised units}$$
$$= 0.765 \times \tau = 7.65 \text{ s in ordinary units,}$$

and the time to half amplitude is

$$t_{1/2} = \frac{\log_e 2}{4.97} = 0.139 \text{ in dynamic-normalised units}$$
$$= 0.139 \times \tau = 1.39 \text{ s in ordinary units.}$$

Similarly, for the second oscillation (the phugoid), in ordinary units,

$$T = 6.44 \times \tau = 64.4 \text{ s}$$

and $$t_{1/2} = 47.2 \times \tau = 472 \text{ s.}$$

Alternatively the constants of the characteristic equation can be found from the algebraic forms of Equations (5.27) to (5.31) as

$$A_1 = 1,$$
$$B_1 = 9.973,$$
$$C_1 = 93.415,$$
$$D_1 = 12.158,$$
and $$E_1 = 87.643\,.$$

Since $D_1 \ll C_1$ and $E_1 \ll C_1^2$, estimates of the roots can be obtained by writing the characteristic equation in the approximate form of Equation (5.50).

$$\left(A_1\lambda^2 + B_1\lambda + C_1\right)\left(\lambda^2 + \frac{(C_1D_1 - B_1E_1)}{C_1^2}\lambda + \frac{E_1}{C_1}\right) = 0\,,$$

$$(\lambda^2 + 9.973\lambda + 93.415)(\lambda^2 + 0.030\lambda + 0.938) = 0\,.$$

In this case the first quadratic factor gives the roots of the short-period oscillation as

$$\lambda = -\ 4.99\ \pm\ 8.28\mathrm{i}\,.$$

so, in ordinary units,

$$T = 0.759 \times \tau = 7.59\ \ \mathrm{s}$$

and $$t_{1/2} = 0.139 \times \tau = 1.39\ \ \mathrm{s}.$$

From the second quadratic factor the roots of the phugoid are

$$\lambda = -\ 0.0150\ \pm\ 0.968\mathrm{i}\,,$$

so, in ordinary units,

$$T = 6.49 \times \tau = 64.9\ \ \mathrm{s}$$

and $$t_{1/2} = 46.2 \times \tau = 462\ \ \mathrm{s}.$$

The exact and approximate results are summarised in the following table.

Mode	*Exact values*			*Approximate values*		
	λ	T (s)	$t_{1/2}$ (s)	λ	T (s)	$t_{1/2}$ (s)
Short-period oscillation	$-4.97 \pm 8.21\mathrm{i}$	7.65	1.39	$-4.99 \pm 8.28\mathrm{i}$	7.59	1.39
Phugoid	$-0.0147 \pm 0.975\mathrm{i}$	64.4	472	$-0.0150 \pm 0.968\mathrm{i}$	64.9	462

The approximate values are thus very close to the exact values.

Lateral stability

The following set of lateral aero-normalised aerodynamic derivatives may be inferred from the information given for the subject aircraft in Reference 17,

$$\begin{array}{llllll}
L_v & = -\ 0.242 & N_v & = \ \ 0.147 & Y_v & = -\ 0.603 \\
L_r & = \ \ 0.0829 & N_r & = -\ 0.0867 & Y_r & = \ \ 0 \\
L_p & = -\ 0.192 & N_p & = -\ 0.0261 & Y_p & = \ \ 0
\end{array}$$

The relative density parameter is

$$\mu = \frac{m_e}{\frac{1}{2}\rho_e S l_0} = \frac{75600}{\frac{1}{2}\times 0.700\times 180\times 36} = 33.3\,,$$

and, as before,

$$\tau = 10.0\ \text{s}.$$

The inertia parameters are

$$i_x = \frac{I_x}{m_e l_0^2} = \frac{2.5\times 10^6}{75600\times 36^2} = 0.0255\ ,$$

$$i_z = \frac{I_z}{m_e l_0^2} = \frac{7.3\times 10^6}{75600\times 36^2} = 0.0745$$

and

$$e_x = -I_{zx}/I_x = \frac{-0.47\times 10^6}{2.5\times 10^6} = 0.188\,,$$

$$e_z = -I_{zx}/I_z = \frac{-0.47\times 10^6}{7.3\times 10^6} = 0.0644\,.$$

As before, for level flight,

$$g_1 = g\cos\Theta_e = 0.818$$

and

$$g_2 = g\sin\Theta_e = 0\,.$$

From the relationships set out in Table C1.1 the dynamic-normalised concise forms of the derivatives are

$$\begin{aligned}
l_v &= -\ \mu L_v/i_x = 316.024,\\
l_r &= -\ L_r/i_x = -3.251,\\
l_p &= -\ L_p/i_x = 7.529,\\
\\
n_v &= -\ \mu N_v/i_z = -65.706,\\
n_r &= -\ N_r/i_z = 1.164,\\
n_p &= -\ N_p/i_z = 0.350,\\
\\
y_v &= -\ Y_v = 0.603,\\
y_r &= -\ Y_r/\mu = 0,\\
y_p &= -\ Y_p/\mu = 0.
\end{aligned}$$

Using computerised matrix handling techniques the characteristic equation and its roots can be found for the lateral motion by substituting the appropriate values of the concise derivatives, gravitational and inertial terms and datum conditions $(u_e = 1, w_e = 0, \Phi_e = 0)$ into Equation (5.13),

$$\mathbf{det}\left|\lambda\,\mathbf{I_5} + \mathbf{A_{22}^{-1}}\,\mathbf{B_{22}}\right| = 0\,.$$

From Equations (5.4) and (5.5)

$$\mathbf{A_{22}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & e_x & 0 & 0 \\ 0 & e_z & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0.188 & 0 & 0 \\ 0 & 0.0644 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\mathbf{B_{22}} = \begin{pmatrix} y_v & y_p - w_e & y_r + u_e & -g_1 \cos\Phi_e & -g_2 \\ l_v & l_p & l_r & 0 & 0 \\ n_v & n_p & n_r & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0.603 & 0 & 1 & -0.818 & 0 \\ 316.024 & 7.529 & -3.251 & 0 & 0 \\ -65.706 & 0.350 & 1.164 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

and the characteristic equation is

$$\lambda(0.988\,\lambda^4 + 9.432\,\lambda^3 + 101.288\,\lambda^2 + 879.892\,\lambda + 126.170) = 0\,.$$

The five exact numerical roots of the characteristic equation are

(i) $\lambda = 0$, the trivial solution

(ii) $\lambda = -8.99$, the rolling subsidence root

(iii) $\lambda = -0.146$, the spiral motion root

(iv) $\lambda = -0.207 \pm 9.87i$ the roots of the Dutch roll.

The expressions in Section 5.1 give, in ordinary units,

$$t_{1/2} = \frac{\log_e 2}{8.99} \times \tau = 0.771\ \text{s} \quad \text{for the rolling subsidence,}$$

and $$t_{1/2} = \frac{\log_e 2}{0.146} \times \tau = 47.5\ \text{s} \quad \text{for the spiral motion,}$$

with $$T = \frac{2\pi}{9.87} \times \tau = 6.37\ \text{s}$$

and $$t_{1/2} = \frac{\log_e 2}{0.207} \times \tau = 33.5\ \text{s} \quad \text{for the Dutch roll.}$$

Alternatively the constants in the characteristic equation may be found from the algebraic forms of Equations (5.34) to (5.38) as

$$A_2 = 0.988,$$
$$B_2 = 9.432,$$
$$C_2 = 101.288,$$
$$D_2 = 879.892,$$
and
$$E_2 = 126.170.$$

The two real roots can then be approximated by Equations (5.51) and (5.52), giving

$$\lambda = -B_2/A_2 = -9.55 \quad \text{for the rolling subsidence}$$
and
$$\lambda = -E_2/D_2 = -0.143 \quad \text{for the spiral motion.}$$

Dividing $\lambda(\lambda+9.55)(\lambda+0.143)$ out of the characteristic equation leaves an approximate quadratic factor

$$(0.988\ \lambda^2 + 0.145\ \lambda + 101.343)\,,$$

giving the pair of complex conjugate roots

$$\lambda = +\ 0.0734 \pm 10.13\mathrm{i} \quad \text{suggesting a divergent Dutch roll.}$$

It is however possible to improve on these initial estimates by making trial-and-error substitutions in the characteristic equation of values near to the initial estimates of the two real roots. This quickly leads to the two exact roots

$$\lambda = -8.99$$
and
$$\lambda = -0.146\,.$$

The approximate quadratic factor then becomes

$$(0.988\ \lambda^2 + 0.406\ \lambda + 96.282)$$

giving the roots

$$\lambda = -\ 0.205 \pm 9.87\mathrm{i} \quad \text{for the Dutch roll.}$$

The table below summarises the exact and approximate results. Estimates made using the solution method of Item No. 83024 (Reference 26), see Section 5.3.2, are included and for the magnitude of μ concerned involve Equations (3.7) and (3.10) of that Item.

Mode	Exact Values			Approximate Values			Estimates using Item No. 83024		
	λ	T (s)	$t_{1/2}$ (s)	λ	T (s)	$t_{1/2}$ (s)	λ	T (s)	$t_{1/2}$ (s)
Rolling subsidence	– 8.99	–	0.771	– 9.55 1st estimate – 8.99 final estimate	– –	0.736 0.771	– 9.01	–	0.769
Spiral motion	– 0.146	–	47.5	– 0.143 1st estimate – 0.146 final estimate	– –	48.5 47.5	– 0.145	–	47.8
Dutch roll	– 0.207 ± 9.87i	6.37	33.5	0.0734 ± 10.13i 1st estimate – 0.205 ± 9.87i final estimate	6.20 6.37	($t_{2/1}$ = 94.4) 33.8	– 0.197 ± 9.88i	6.36	35.2

The first estimates of the approximate values of the roots of the rolling subsidence and spiral motion give reasonable estimates of $t_{1/2}$ for these modes, but they are not good enough to allow a satisfactory estimate of the roots of the Dutch roll. Their removal from the characteristic equation, together with the trivial root, leaves a quadratic factor that has a complex conjugate pair of roots with positive real part, implying an unstable oscillation. Improved estimates of the real roots lead to good accuracy in all cases, as indeed does the method of Item No. 83024.

APPENDIX A LONGITUDINAL STABILITY AND CONTROL: CLASSICAL CRITERIA

This Appendix examines the classical concepts of static margin and manoeuvre margin. It demonstrates how the 'static margin' may be derived from a consideration of only the static moments which arise when an aircraft is disturbed from a trimmed state in straight and level flight, and the 'manoeuvre margin' from a consideration of the incremental pitching moments that arise in the transition to or from horizontal flight and circling flight at a constant normal acceleration.

These concepts are related to the constant terms of the approximate characteristic equation of the dynamic motions discussed in Section 5.3.1 of the main text. In the past, when in the main only static wind-tunnel data were readily available and methods of estimating the aerodynamic derivatives generally were crude or non-existent, these concepts played an important part in aircraft design. Developments, both theoretical and experimental, together with the ease with which the roots of the characteristic equation can be evaluated, have much reduced that importance.

As power-operated controls have become an increasingly common feature of present-day aircraft the problem of controls-free stability has become less and less important. Nevertheless there are still aircraft types for which a simple mechanical link between the cockpit controls and the control surfaces is appropriate. The remarks on controls-free stability in this Appendix apply to such aircraft, where a need to study controls-free stability may arise.

A1. GENERAL DEFINITIONS

A1.1 Controls Fixed and Controls Free

Stability is considered in two distinct conditions.

(a) With the controls set to trim in steady flight and then held fixed throughout the subsequent motion.

(b) With the controls set to trim with zero control hinge moments in steady flight and the controls left free throughout the subsequent motion.

An aircraft may be stable with controls fixed and unstable with controls free or vice versa. The control movements which the pilot has to make to maintain a changed state are related to the controls-fixed condition, while the control hinge moments required to be overcome influence the controls-free condition.

The difference between controls-fixed and controls-free stability depends on the degree of static mass-balance of the controls, on the design of their aerodynamic balance, and on the inertial properties of the control circuits, and is therefore adjustable within certain limits.

A1.2 Static Stability

For a simple initial presentation, static stability can be defined in a 'basic' case with engine off, when there are no effects, such as those due to slipstream, structural flexure, Reynolds number or compressibility, to cause variations in the pitching moment coefficient with speed, so that at the flight condition $\partial C_m/\partial V = 0$. In this case it can be said that, when the aircraft is flying in a straight glide and is displaced in the pitch plane, static stability is positive if the static moment (proportional to the angle of attack) tends to restore it to the initial condition, and negative if the moment increases the disturbance. To preserve the relationship between static stability and the dynamic motion in the general case, the definition must also include a change in speed when the aircraft is displaced from the equilibrium condition. The following more precise definition covers all cases.

Consider an aircraft for which pitch control is provided by an elevator, with trim tab, fitted to a tailplane. Suppose the aircraft is trimmed for straight flight at a given speed, angle of attack and engine condition and is then held at the same engine condition and elevator position but at a slightly higher speed and at the angle of attack appropriate to trimmed flight at that higher speed. The aircraft is then statically stable with controls fixed if the pitching moment tends to restore it to the original angle of attack. In flight the elevator position must be changed to maintain trimmed flight at the modified speed, the change being proportional to the restoring or disturbing moment. The aircraft is therefore stable with controls fixed if the elevator is moved downwards to retrim the aircraft at a slightly higher speed.

A similar argument applies with controls free. In this case the aircraft is statically stable if an upward change in elevator trim-tab angle is required for straight flight at a speed slightly higher than the trimmed speed.

A1.3 Stability in Dynamic Motion

The aircraft is said to be stable in dynamic motion, controls-fixed, if, when the controls are fixed, it returns to its initial trimmed state after a small temporary disturbance. Similarly, if the controls are left free, and the aircraft returns to the initial trimmed state, it is said to be stable, controls-free. The disturbance in speed and angle of attack may subside gradually, the amplitude being halved in $t_{1/2}$ seconds, or the speed and incidence may oscillate about the trimmed values. In both cases the damping is measured by $1/t_{1/2}$.

For an aircraft that is unstable in dynamic motion the speed and angle of attack may diverge from the trimmed values or may oscillate about them with ever-increasing amplitude. The rate of growth is then measured by the time $t_{2/1}$ to double the amplitude.

A number of different forms of damping criteria are discussed in Item No. 67038 (Reference 15) of the Dynamics Sub-series.

Immediately after a disturbance the motion is a combination of a number of modes but, after a short time, the heavily damped modes die out and the dynamic motion persists in the form of the least stable mode. If the aircraft is statically unstable, one of these modes will be divergent. Thus, a statically unstable aircraft is also unstable in dynamic motion, but a statically stable aircraft is not necessarily stable in dynamic motion.

A2. BASIC THEORY OF STATIC STABILITY

A2.1 Assumptions

(i) The force, pitching moment and hinge moments coefficients, C_L, C_D, C_m and C_B, are independent of variations in speed and air density. Slipstream effects and structural flexure are excluded and the aerodynamic coefficients are not affected through the motion by compressibility or Reynolds number effects.

(ii) The air density remains constant during the motion.

(iii) The lift coefficient, C_L, and the moment coefficients, C_m and C_B, are linear functions of angle of attack and control setting.

A2.2 Neutral Point

The neutral point is the position of the c.g. (centre of gravity) at which the static stability is neutral. For this c.g. position the same elevator angle will trim the aircraft at all speeds with controls fixed. Similarly, at the neutral point with controls free, if the elevator trim-tab angle is set to trim with controls free at one speed, there will be no control hinge moments for trim at any other speed.

The distance of the neutral point behind some aircraft reference point is written as $h_n l$ with controls fixed, and as $h_{nf} l$ with controls free, where l is some characteristic length. If the distance of the c.g. from the reference point is denoted by $h_{cg} l$, the aircraft is statically stable with controls fixed when $h_{cg} < h_n$ and with controls free when $h_{cg} < h_{nf}$.

A2.3 Static or c.g. Margin

The degree of static stability is measured by the static margin which is equal in magnitude but opposite in sign to the rate of change of the incremental pitching moment coefficient with incremental C_L, where $C_L = W/\frac{1}{2}\rho V^2 S$.

In the basic theory the static margin is also equal to the c.g. margin, $h_n - h_{cg}$ or $h_{nf} - h_{cg}$, the distance of the c.g. ahead of the neutral point. Thus, with controls fixed, the static margin is

$$k_s = -\frac{\mathrm{d}C_m(\alpha)}{\mathrm{d}C_L(\alpha)} = -\frac{\partial C_m/\partial\alpha}{\partial C_L/\partial\alpha} = h_n - h_{cg} = k_{cg}, \tag{A2.1}$$

where C_m, about the c.g. and C_L are measured with the elevator and elevator trim-tab fixed. Similarly, with controls free, the static margin is

$$k_{sf} = -\left[\frac{\mathrm{d}C_m}{\mathrm{d}C_L}\right]_f = h_{nf} - h_{cg} = k_{cgf}, \tag{A2.2}$$

where C_m and C_L are measured with the elevator trim-tab fixed and the elevator free.

In flight tests (see Reference 1) the static margin with controls fixed is determined from the elevator position required to maintain steady flight at each speed. Thus,

$$k_s = -\frac{\partial C_m}{\partial\eta}\frac{\mathrm{d}\bar{\eta}}{\mathrm{d}C_L} \tag{A2.3}$$

where here, and in the following paragraph, the expressions with a bar refer to a trimmed state of flight at constant C_L, so that $\bar{\eta}$ is the elevator angle required for trim and $\partial C_m/\partial\eta$ is measured with C_L constant. Similarly, with elevator free

$$k_{sf} = -\frac{\partial C_m}{\partial\tau_\eta}\frac{\mathrm{d}\bar{\tau}_\eta}{\mathrm{d}C_L} \tag{A2.4}$$

where $\bar{\tau}_\eta$ is the elevator trim-tab angle for trim. Alternatively, τ_η may be kept constant and the hinge moment coefficient C_B determined from hinge moment measurements, in which case

$$k_{sf} = -\frac{\partial C_m}{\partial C_B}\frac{\mathrm{d}C_B}{\mathrm{d}C_L}. \tag{A2.5}$$

A3. MORE GENERAL THEORY OF STATIC STABILITY

A3.1 Assumptions

A more general theory (References 2 and 9) has been developed to include the effects of variations of the aerodynamic coefficients with speed. Of the assumptions of the basic theory (Section A2.1) only assumptions (ii) and (iii) are retained. It should be noted that because assumption (iii) remains, the effects of slipstream or wake are not completely covered because linear effects are assumed.

In the general theory the condition $C_L = W/\tfrac{1}{2}\rho V^2 S$ is replaced by $C_R = W/\tfrac{1}{2}\rho V^2 S$, where R is the resultant of the aerodynamic forces acting on the aircraft (lift, drag and thrust).

In general, the static margin is not equal to the distance of the c.g. ahead of the neutral point and a clear distinction between static and c.g. margins is essential. The theory stresses the importance of the static margin as the measure of static stability, the position of the neutral point being relatively insignificant.

A3.2 Static Margin

In place of Equation (A2.1) the controls-fixed static margin is equated to the differential $-\,\mathrm{d}C_m/\mathrm{d}C_R$, where C_m and C_R are functions of α and V, so that

$$k_s = -\frac{\mathrm{d}C_m(\alpha, V)}{\mathrm{d}C_R(\alpha, V)} = -\frac{(\partial C_m/\partial\alpha) + (\partial C_m/\partial V)(\mathrm{d}V/\mathrm{d}\alpha)}{(\partial C_R/\partial\alpha) + (\partial C_R/\partial V)(\mathrm{d}V/\mathrm{d}\alpha)}\,. \tag{A3.1}$$

The differential $\mathrm{d}V/\mathrm{d}\alpha$ is obtained from the imposed condition relating V and C_R in steady flight in a uniform atmosphere,

$$C_R V^2 = \frac{W}{\tfrac{1}{2}\rho S} \text{ (= constant for constant } \rho\text{)}, \tag{A3.2}$$

and can be written

$$\frac{\mathrm{d}V}{\mathrm{d}\alpha} = \frac{V(\partial C_R/\partial\alpha)}{2C_R + V(\partial C_R/\partial V)}\,. \tag{A3.3}$$

Equations (A3.1) and (A3.3) are used in References 2 and 9 to show that the constant term, E_1 of the characteristic stability equation for longitudinal motion, see Section 5.3, has the same sign and is proportional to k_s.

In place of Equation (A2.3) there is the general result

$$k_s = -\frac{\partial C_m}{\partial\eta}\frac{\mathrm{d}\bar{\eta}}{\mathrm{d}C_R}\,. \tag{A3.4}$$

The static margin at a given C_R is measured at $C_m = 0$, *i.e.* with the elevator in the correct position for trim at that C_R. In the estimation of $\mathrm{d}\bar{\eta}/\mathrm{d}C_R$ from flight tests this condition is automatically satisfied. The slope $\partial C_m/\partial\eta$ is measured at the trimmed condition.

Similar conditions apply to the measurement of the static margin with controls free,

$$k_{sf} = -\left[\frac{\mathrm{d}C_m}{\mathrm{d}C_R}\right]_f = -\frac{\partial C_m}{\partial \tau_\eta}\frac{\mathrm{d}\bar{\tau}_\eta}{\mathrm{d}C_R}$$

$$= -\frac{\partial C_m}{\partial C_B}\frac{\mathrm{d}C_B}{\mathrm{d}C_R}\,. \tag{A3.5}$$

The slope $\mathrm{d}C_B/\mathrm{d}C_R$ must be measured at the elevator trim-tab setting for which the hinge moment coefficient $C_B = 0$ at the chosen C_R.

It is shown in Section 5.3.1 that for many aircraft in controls-fixed motion a necessary but not sufficient requirement for stability is $k_s > 0$. In controls-free motion there is a similar requirement, $k_{sf} > 0$. Both results follow because the static margin is proportional to, and has the same sign as, the constant term in the characteristic equation that contains the roots of the equations of motion.

A3.3 Neutral Point and c.g. Margin

The neutral point at h_n or h_{nf} is the c.g. position for which the static margin is zero.

The c.g. margin is given by:

$$k_{cg} = h_n - h_{cg} \text{ with controls free} \tag{A3.6}$$

or

$$k_{cgf} = h_{nf} - h_{cg} \text{ with controls fixed.}$$

At any particular speed, k_s (or k_{sf} is proportional to k_{cg} (or k_{cgf}) but the factor of proportionality varies with speed and may become infinite or negative. Thus it is possible for the c.g. margin to be negative while the static margin is positive. In such cases the static margin and not the c.g. margin defines the degree of static stability.

A4. THEORY OF MANOEUVRABILITY

A completely general criterion for response to elevator control, or manoeuvrability in the longitudinal plane, can be defined in terms of the ratio of the control movement or hinge moment in a pull-out from a dive to the centripetal acceleration built up in the pull-out.

This ratio varies with the pilot's technique and the stage reached in the pull-out. Accordingly for the purpose of establishing a definite criterion of manoeuvrability the following approximations and assumptions are made.

(a) Steady flight in an arc of a circle in the vertical plane at constant speed and constant centripetal acceleration ng can be maintained. Effects related to variations in speed are therefore absent.

(b) Assumptions (ii) and (iii) of the basic theory (Section A2.1) are retained.

(c) Changes in the gravity component due to change in flight path during the manoeuvre can be neglected in comparison with ng.

The difference, $\Delta\bar{\eta}$ (or ΔB), between the elevator angle (or hinge moment) to maintain the centripetal acceleration ng, (*i.e.* $nW = L - W$) and the angle (or moment) to trim in straight flight at the same speed is proportional to ng, and hence the following definitions apply:

(1) elevator angle per $g = \Delta\bar{\eta}/n$ (A4.1)

(2) elevator angle moment per $g = \Delta B/n$.

The manoeuvre points, controls-fixed and controls-free, are the c.g. positions at which $\Delta\bar{\eta} = 0$ and $\Delta B = 0$, respectively, and are located at distances h_m and h_{mf} behind the aircraft reference point.

The manoeuvre margins are equal to the distances of the c.g. ahead of the manoeuvre points. They are written as

$$k_m = h_m - h_{cg} \text{ with controls fixed} \qquad (A4.2)$$

or

$$k_{mf} = h_{mf} - h_{cgf} \text{ with controls free ,}$$

and $\Delta\bar{\eta}/n$ and $\Delta B/n$ are proportional to $k_m C_L$ and $k_{mf} C_L$ respectively. In the general theory the factor of proportionality is a function of speed.

A5. RELATIONSHIP BETWEEN STATIC AND MANOEUVRE MARGINS AND STABILITY OF DYNAMIC MOTION

A5.1 Basic Theory

Within the simplifying assumptions of the basic theory of Section A2.1 there is a close relationship between the static and manoeuvre margins, given by

$$k_m = k_s - \frac{M_q}{\mu} = h_n - h_{cg} - \frac{M_q}{\mu}, \qquad (A5.1)$$

where it has been assumed that Z_q is small in relation to μ. Moreover, if the pitch damping is entirely due to the tailplane

$$k_{mf} = k_{sf} - F\frac{M_q}{\mu} = h_{nf} - h_{cg} - F\frac{M_q}{\mu}, \qquad (A5.2)$$

where F is the ratio of the controls-free to controls-fixed tailplane lift-curve slopes.

The terms $-M_q/\mu$ and $-FM_q/\mu$ are due to the angular velocity in the pull-out. The pitch damping derivative M_q is aero-normalised for datum conditions ρ_e and V_e, corresponding to steady values of ρ and V, and $l_0 = l$. The relative density parameter $\mu = m_e/\tfrac{1}{2}\rho_e S l_0$ where the datum mass is $m_e = W/g$. At low speeds M_q/μ is small and the static and manoeuvre margins are nearly equal.

Equations (A5.1) and (A5.2) express the fact that the manoeuvre point in the basic case lies aft of the c.g. position for neutral static stability by an amount such that in pitching motion with the c.g. at the manoeuvre point the couple due to the static instability is just balanced by the damping couple.

Flight with the c.g. in the region of the manoeuvre point is characterised by a rapid change of normal acceleration following small, and possibly inadvertent, movements of the elevator.

A5.2 General Theory

It must be noted that in general both static and manoeuvre margins are functions of speed and vary independently of each other. There is no close relationship between them and Equations (A5.1) and (A5.2) no longer hold, see Reference 9.

A5.3 Stability of Dynamic Motion

For most aircraft the motion following a disturbance in the plane of symmetry consists of two oscillatory modes, as shown in Section 5.3.1. Necessary but not sufficient requirements for the stability of those modes are that the static and manoeuvre margins be positive.

When the static and manoeuvre margins are both large and positive, the motion with controls fixed consists of

(a) a slowly damped oscillation, commonly known as the phugoid, with a period of the order of 30 seconds.

(b) a well damped pitching oscillation with a period typically of from 2 to 8 seconds, generally referred to as the short period oscillation.

With controls free there is an additional heavily damped rapid oscillation of the elevator, and the short period pitching oscillation may under certain conditions become unstable. Reference 2 gives the following summary of the types of instability that will occur when the margins become small or negative.

Type of Instability

(a)	k_m	positive,	k_s	negative.	Slow divergence
(b)	k_m	negative,	k_s	negative.	Rapid divergence
(c)	k_m	negative,	k_s	positive.	Very unstable oscillation or rapid divergence
(d)	k_m	positive,	k_s	large and positive.	Unstable phugoid oscillation when k_s is large and k_m is small

The behaviour with controls free is similar.

APPENDIX B DESCRIPTION OF LATERAL MODES OF MOTION

B1. INTRODUCTION

It has been shown in Section 5.3.2 that the disturbed lateral motion of an aircraft that has been slightly disturbed from straight and symmetrical flight, with controls fixed, is determined by the four non-trivial roots of a characteristic lateral stability quintic. For an aircraft with wings of high aspect ratio and modest sweep there are normally three modes, a rolling subsidence, a spiral motion and a lateral oscillation known as Dutch roll. In what follows an attempt is made to explain in physical terms how each of these modes develops, its nature, and how the three freedoms of sideslip, roll rate and yaw rate are coupled in the three modes. It should be stressed that the argument presents the development of the motion in a sequential manner whereas in reality it occurs in a simultaneous fashion.

B2. ROLLING SUBSIDENCE

Assume that the aircraft has been given a slight rolling velocity p in the positive sense, the starboard wing dropping. This rolling velocity decays very rapidly, aperiodically in unstalled flight (rapid subsidence), due to the large damping moment from the wing (negative L_p). This rolling moment derivative due to rolling is large in comparison with the other rolling derivatives for aircraft with wings of high aspect ratio and modest sweep; in such a case a good approximation to the initial behaviour of an aircraft in roll is obtained by assuming it to be free to roll only.

Due to the rolling velocity there is a yawing moment (N_p) and since this is normally negative, the aircraft will tend to yaw to port, that is, the starboard wing will move forward. Due to the angle of bank consequent on the rolling moment, the aircraft starts sideslipping to starboard, and the sideslip in turn will induce a yawing moment. For an aircraft with positive weathercock stability, this yawing moment will tend to turn the aircraft into the direction towards which it is sideslipping, that is, to starboard. It is seen that the yawing moments due to rolling and sideslipping oppose each other, the direction of yawing depending on which of these has the predominant effect.

Both the sideslipping and the yawing velocities will normally cause a rolling moment. That due to sideslip normally acts in the sense tending to reduce the angle of bank, *i.e.* L_v is negative, thus being additional to the damping moment from the wing. The direction of the yawing velocity may vary and the resulting rolling moment due to yawing may therefore either tend to increase or decrease the original rolling velocity.

B3. SPIRAL MOTION

Consider the aircraft to experience a slight positive yawing velocity, nose to starboard. The yawing moment due to yawing of a normal aircraft is always damping (N_r negative), and by itself would cause the yawing velocity to decay; but due to the increase in relative velocity over the port wing and decrease over the starboard wing, there is a rolling moment associated with yawing (L_r) which causes the aircraft to roll in a positive sense, so that the starboard wing drops and a sideslip to starboard follows. The effect of these latter motions must now be considered in more detail.

Let it be assumed first that the aircraft has no rolling moment due to sideslip $(L_v = 0)$. If the aircraft has positive weathercock stability the sideslip velocity creates a yawing moment tending to turn the aircraft into the direction to which it is sideslipping (N_v positive), that is, to starboard. This, as shown already, induces a positive rolling moment which will cause the aircraft to roll further to starboard, thereby increasing in turn the sideslip velocity. In the absence of a restoring moment the aircraft will tend to fall into a tightening turn; this type of instability is known as ‘spiral’ instability.

The moment required to stop the sideslip is one which will eliminate the angle of bank. The source of rolling moment that can accomplish this in the present case is that due to sideslip, that is, a negative L_v is required. The effect of this negative L_v must overbalance the positive rolling moment due to the yawing velocity (L_r) which the aircraft has acquired.

The same requirement applies regardless of whether the motion is, in the first instance, due to a yawing or rolling disturbance or more directly due to an angle of bank.

B4. LATERAL OSCILLATION (DUTCH ROLL)

Assume again an initial yawing moment to starboard, leading to a clockwise rolling moment and consequently to a sideslip to starboard. As already indicated it is assumed that the rolling moment due to sideslip outweighs that due to yawing, so an anticlockwise roll will develop, and because of inertia the angle of bank will change sign, and this in turn will ultimately reverse the sign of the sideslip angle. The same sequence of events will now occur in the opposite direction, and an oscillation will result.

For the class of aircraft to which the above description specifically applies, the ratio of the rolling motion to the yawing motion depends on the values of particular aerodynamic derivatives and the inertia characteristics. The ratio is always small to modest for the type of aircraft referred to in the opening paragraph of this Appendix.

To determine the amplitude ratios and the phase relationships of the motions involved in the Dutch roll it is necessary to insert the solution for the Dutch roll in the equations of motion. It is convenient to use the complex form of the solution for this purpose (see Reference 7). Such calculations show that, in addition to the result already referred to for the roll to yaw ratio, the ratio of the sideslip angle β to the yaw angle ψ is only slightly removed from unity and the two angles are nearly 180º out of phase. This implies that during the Dutch roll motion the aircraft's centre of gravity deviates but little from the rectilinear.

The foregoing argument applies whatever the initial displacements of the aircraft may have been, since the rolling, yawing and sideslipping motion are all coupled.

B5. VARIATION OF DUTCH ROLL CHARACTERISTICS

As indicated in the preceding Section the character of the Dutch roll motion can vary widely from aircraft to aircraft. The changes in the amplitude ratios, phase differences, damping and frequency are the result of the wide range of aircraft geometry and speed range (and hence aerodynamic derivatives) on the one hand, and mass and mass distribution on the other.

Decrease of wing aspect ratio, increase of sweep and greater concentration of the mass within the comparatively longer fuselage result in a marked increase in the roll to yaw ratio. In general, therefore, this ratio may vary from small, yielding a mainly yawing oscillation with a slight amount of rolling motion, to large, yielding an oscillation in which the roll to yaw ratio is so great that the Dutch roll approaches a pure rolling oscillation (see References 7, 10 and 11).

As already mentioned the damping and frequency of the Dutch roll are affected to a similar extent. These changes place different emphasis on the various aerodynamic derivatives.

In the same way some changes will occur in the other two lateral modes, the roll subsidence and the spiral motion, but to a lesser extent, although under some extreme flight conditions these two aperiodic modes may combine to give a long period lateral oscillation.

APPENDIX C CONVERSION FACTORS

C1. TABLES

TABLE C1.1 CONVERSION FACTORS FOR AERODYNAMIC DERIVATIVES

Dimensional form of derivative (1)	*Divisor for forming aero-normalised derivative* (2)	*Aero-normalised derivative* (1)/(2)	*Factor for obtaining dynamic-normalised concise derivative* (3)	*Dynamic-normalised concise derivative* (3) × (1)/(2)
$\mathring{L}_p$ $\mathring{M}_q$ $\mathring{N}_p$ $\mathring{L}_r$ $\mathring{N}_r$	$\frac{1}{2}\rho_e V_e S l_0^2$	L_p M_q N_p L_r N_r	$-1/i_x$ $-1/i_y$ $-1/i_z$ $-1/i_x$ $-1/i_z$	l_p m_q n_p l_r n_r
$\mathring{M}_u$ $\mathring{L}_v$ $\mathring{N}_v$ $\mathring{M}_w$	$\frac{1}{2}\rho_e V_e S l_0$	M_u L_v N_v M_w	$-\mu/i_y$ $-\mu/i_x$ $-\mu/i_z$ $-\mu/i_y$	m_u l_v n_v m_w
$\mathring{M}_{\dot{w}}$	$\frac{1}{2}\rho_e S l_0^2$	$M_{\dot{w}}$	$-1/i_y$	$m_{\dot{w}}$
$\mathring{L}_\zeta$ $\mathring{M}_\eta$ $\mathring{N}_\zeta$ $\mathring{L}_\xi$ $\mathring{N}_\xi$	$\frac{1}{2}\rho_e V_e^2 S l_0$	L_ζ M_η N_ζ L_ξ N_ξ	$-\mu/i_x$ $-\mu/i_y$ $-\mu/i_z$ $-\mu/i_x$ $-\mu/i_z$	l_ζ m_η n_ζ l_ξ n_ξ
$\mathring{Y}_p$ $\mathring{X}_q$ $\mathring{Z}_q$ $\mathring{Y}_r$	$\frac{1}{2}\rho_e V_e S l_0$	Y_p X_q Z_q Y_r	$-1/\mu$ $-1/\mu$ $-1/\mu$ $-1/\mu$	y_p x_q z_q y_r
$\mathring{X}_u$ $\mathring{Z}_u$ $\mathring{Y}_v$ $\mathring{X}_w$ $\mathring{Z}_w$	$\frac{1}{2}\rho_e V_e S$	X_u Z_u Y_v X_w Z_w	-1 -1 -1 -1 -1	x_u z_u y_v x_w z_w
$\mathring{X}_{\dot{w}}$ $\mathring{Z}_{\dot{w}}$	$\frac{1}{2}\rho_e S l_0$	$X_{\dot{w}}$ $Z_{\dot{w}}$	$-1/\mu$ $-1/\mu$	$x_{\dot{w}}$ $z_{\dot{w}}$
$\mathring{X}_\eta$ $\mathring{Z}_\eta$ $\mathring{Y}_\zeta$ $\mathring{Y}_\xi$	$\frac{1}{2}\rho_e V_e^2 S$	X_η Z_η Y_ζ Y_ξ	-1 -1 -1 -1	x_η z_η y_ζ y_ξ
$\mu = m_e / \frac{1}{2}\rho_e S l_0$ $i_x, i_y, i_z = I_x / m_e l_0^2, I_y / m_e l_0^2, I_z / m_e l_0^2$				

TABLE C1.2 COMPARISON OF AMERICAN AND AERO-NORMALISED DERIVATIVES

Aero-normalised derivative (1)	*American equivalent* (2)	*Factor to obtain* (2) *from* (1)	*Aero-normalised derivative* (1)	*American equivalent* (2)	*Factor to obtain* (2) *from* (1)
X_u	C_{xu}	1	Y_v	$C_{Y\beta}$	1
X_w	$C_{x\alpha}$	1	Y_p	C_{Yp}	$2l_0/b$
X_q	C_{xq}	$2l_0/\bar{\bar{c}}$	Y_r	C_{Yr}	$2l_0/b$
$X_{\dot{w}}$	$C_{x\dot{\alpha}}$	$2l_0/\bar{\bar{c}}$	Y_ξ	$C_{Y\xi}$	1
X_η	$C_{x\eta}$	1	Y_ζ	$C_{Y\zeta}$	1
Z_u	C_{zu}	1	N_v	$C_{n\beta}$	l_0/b
Z_w	$C_{z\alpha}$	1	N_p	C_{np}	$2(l_0/b)^2$
Z_q	C_{zq}	$2l_0/\bar{\bar{c}}$	N_r	C_{nr}	$2(l_0/b)^2$
$Z_{\dot{w}}$	$C_{z\dot{\alpha}}$	$2l_0/\bar{\bar{c}}$	N_ξ	$C_{n\xi}$	l_0/b
Z_η	$C_{z\eta}$	1	N_ζ	$C_{n\zeta}$	l_0/b
M_u	C_{mu}	$l_0/\bar{\bar{c}}$	L_v	$C_{l\beta}$	l_0/b
M_w	$C_{m\alpha}$	$l_0/\bar{\bar{c}}$	L_p	C_{lp}	$2(l_0/b)^2$
M_q	C_{mq}	$2(l_0/\bar{\bar{c}})^2$	L_r	C_{lr}	$2(l_0/b)^2$
$M_{\dot{w}}$	$C_{m\dot{\alpha}}$	$2(l_0/\bar{\bar{c}})^2$	L_ξ	$C_{l\xi}$	l_0/b
M_η	$C_{m\eta}$	$l_0/\bar{\bar{c}}$	L_ζ	$C_{l\zeta}$	l_0/b

NOTES:

(i) In some systems the differentiation with respect to u is carried out without the assumption that the coefficients are based on a constant value of V. In such cases (for small α and ignoring Mach variations)

$$C_{xu} = X_u - 2C_x,$$

$$C_{zu} = Z_u - 2C_z$$

and $$C_{mu} = M_u - 2C_m.$$

(ii) Elevator, aileron and rudder angles may be denoted by δ_e, δ_a and δ_r instead of η, ξ and ζ, respectively. Also, sometimes δ_a represents the sum of port and starboard aileron angles and is then equal to 2ξ.

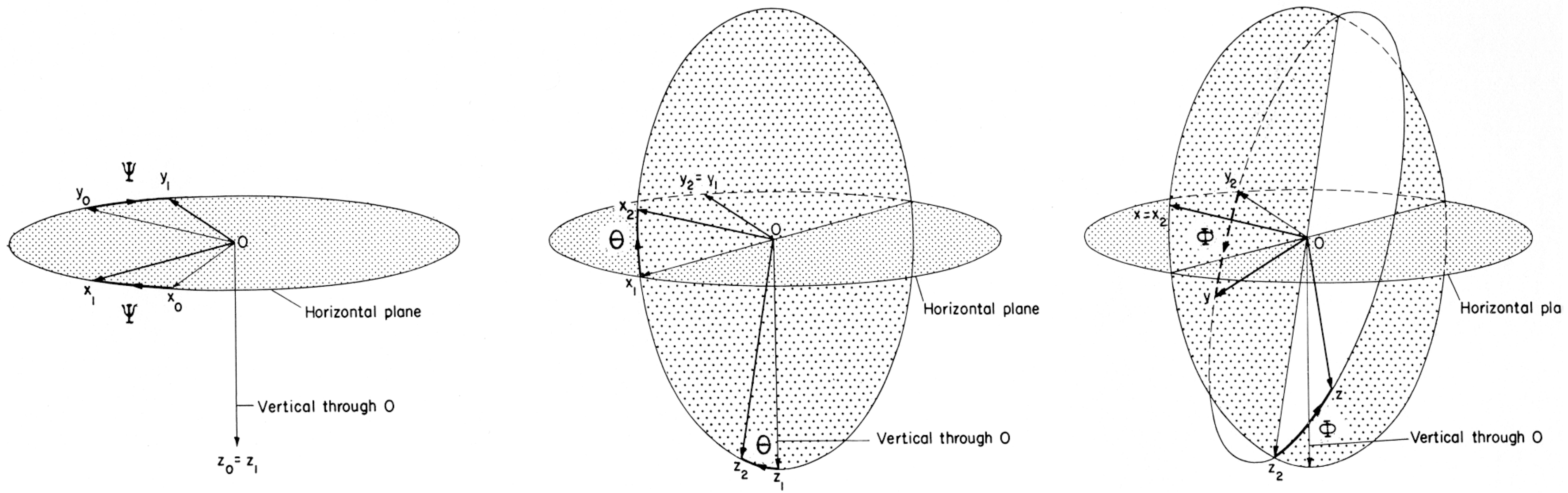
(iii) In an aerodynamic body-axis system, alternatives to C_{zu}, $C_{z\alpha}$ *etc*. are $-C_{Lu}$, $-C_{L\alpha}$ *etc*.

TABLE C1.3
CONVERSION FACTORS FOR DYNAMIC-NORMALISED CONCISE DERIVATIVES

Dimensional form of concise derivative (1)	*Multiplying factor to obtain dynamic-normalised concise derivative* (2)	*Dynamic-normalised concise derivative* (1) × (2)
$\mathring{l}_p$ $\mathring{l}_r$ $\mathring{m}_q$ $\mathring{n}_p$ $\mathring{n}_r$	τ	l_p l_r m_q n_p n_r
$\mathring{l}_v$ $\mathring{m}_u$ $\mathring{m}_w$ $\mathring{n}_v$	$\tau^2 V_e$	l_v m_u m_w n_v
$\mathring{m}_{\dot{w}}$	τV_e	$m_{\dot{w}}$
$\mathring{l}_\zeta$ $\mathring{l}_\xi$ $\mathring{m}_\eta$ $\mathring{n}_\zeta$ $\mathring{n}_\xi$	τ^2	l_ζ l_ξ m_η n_ζ n_ξ
$\mathring{x}_q$ $\mathring{y}_p$ $\mathring{y}_r$ $\mathring{z}_q$	$1/V_e$	x_q y_p y_r z_q
$\mathring{x}_u$ $\mathring{x}_w$ $\mathring{y}_v$ $\mathring{z}_u$ $\mathring{z}_w$	τ	x_u x_w y_v z_u z_w
$\mathring{x}_{\dot{w}}$ $\mathring{z}_{\dot{w}}$	1	$x_{\dot{w}}$ $z_{\dot{w}}$
$\mathring{x}_\eta$ $\mathring{z}_\eta$ $\mathring{y}_\zeta$ $\mathring{y}_\xi$	τ/V_e	x_η z_η y_ζ y_ξ
$\tau = m_e / \tfrac{1}{2}\rho_e V_e S$		

TABLE C1.4
CONVERSION FACTORS FOR OBTAINING DYNAMIC-NORMALISED QUANTITIES

Quantity in dimensional units (1)	*Multiplying factor to obtain dynamic-normalised quantity* (2)	*Dynamic-normalised quantity* (1) × (2)
$\mathring{g}$	τ/V_e	g
$\mathring{p}$, $\mathring{q}$, $\mathring{r}$	τ	p , q , r
$\mathring{u}$, $\mathring{v}$, $\mathring{w}$, $\mathring{V}$	$1/V_e$	u , v , w , V
$\tau = m_e/\frac{1}{2}\rho_e V_e S$		



(i) FIRST ROTATION:
THROUGH Ψ ABOUT $0z_0\,(\equiv 0z_1)$

(ii) SECOND ROTATION:
THROUGH Θ ABOUT $0y_1\,(\equiv 0y_2)$

(iii) THIRD ROTATION:
THROUGH Φ ABOUT $0x_2\,(\equiv 0x)$

$0x_0y_0z_0$ Initial position of axis system, coincident with Earth-axes system
$0x_1y_1z_1$ Position of axis system after first rotation $(0z_1 \equiv 0z_0)$
$0x_2y_2z_2$ Position of axis system after second rotation $(0y_2 \equiv 0y_1)$
$0xyz$ Final position of axis system $(0x \equiv 0x_2)$, coincident with body-axis system

FIGURE 1 SEQUENCE OF ROTATIONS DEFINING ATTITUDE ANGLES Ψ, Θ, Φ

THE PREPARATION OF THIS DATA ITEM

The work on this particular Item which supersedes Item Nos Aero A.00.00.02, Aero A.00.00.03, Aero A.00.00.04 and 67039 was monitored and guided by the Aerodynamics Committee which first met in 1942 and now has the following membership:

Chairman
Mr H.C. Garner – Independent

Vice-Chairman
Mr P.K. Jones – British Aerospace plc, Aircraft Group, Manchester

Members
Mr E.A. Boyd – Cranfield Institute of Technology
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Dr J.W. Flower – Bristol University
Mr A. Hipp – British Aerospace plc, Dynamics Group, Stevenage
Mr R. Jordan – Aircraft Research Association
Mr J. Kloos* – Saab-Scania, Linköping, Sweden
Mr J.R.C. Pedersen – Independent
Mr R. Sanderson – Messerschmitt-Bölkow-Blohm GmbH, Hamburg, Germany
Mr A.E. Sewell* – McDonnell Douglas, Long Beach, Calif., USA
Mr F.W. Stanhope – Rolls-Royce Ltd, Derby
Mr H. Vogel – British Aerospace plc, Aircraft Group, Weybridge.

Particular assistance has been provided during the preparation of this Item by the Dynamics Committee which has the following membership:

Chairman
Dr H.H.B.M. Thomas – Independent

Members
Prof.G.T.S.Done – The City University, London
Prof. G.H.Hancock – Queen Mary College, London University
Mr M.R. Heath – British Aerospace Dynamics Group, Stevenage
Dr M. Tobak* – NASA, Ames Research Centre, Calif., USA
Dr M.A. Woodhead – University of Salford.

* Corresponding Member

The member of staff who undertook the technical work involved in the initial assessment of the available information and the construction and subsequent development of the Item was

Mr R.W. Gilbey – Senior Engineer.