# 2

# The z Transform

## 2–1 INTRODUCTION

A mathematical tool commonly used for the analysis and synthesis of discrete-time control systems is the $z$ transform. The role of the $z$ transform in discrete-time systems is similar to that of the Laplace transform in continuous-time systems.

In a linear discrete-time control system, a linear difference equation characterizes the dynamics of the system. To determine the system's response to a given input, such a difference equation must be solved. With the $z$ transform method, the solutions to linear difference equations become algebraic in nature. (Just as the Laplace transformation transforms linear time-invariant differential equations into algebraic equations in $s$, the $z$ transformation transforms linear time-invariant difference equations into algebraic equations in $z$.)

The main objective of this chapter is to present definitions of the $z$ transform, basic theorems associated with the $z$ transform, and methods for finding the inverse $z$ transform. Solving difference equations by the $z$ transform method is also discussed.

***Discrete-Time Signals.*** Discrete-time signals arise if the system involves a sampling operation of continuous-time signals. The sampled signal is $x(0), x(T)$, $x(2T), \ldots$, where $T$ is the sampling period. Such a sequence of values arising from the sampling operation is usually written as $x(kT)$. If the system involves an iterative process carried out by a digital computer, the signal involved is a number sequence $x(0), x(1), x(2), \ldots$ The sequence of numbers is usually written as $x(k)$, where the argument $k$ indicates the order in which the number occurs in the sequence, for example, $x(0), x(1), x(2), \ldots$ Although $x(k)$ is a number sequence, it can be considered as a sampled signal of $x(t)$ when the sampling period $T$ is 1 sec.

$x(t) = x(k)$ when $T = 1$ sec.

The $z$ transform applies to the continuous-time signal $x(t)$, sampled signal $x(kT)$, and the number sequence $x(k)$. In dealing with the $z$ transform, if no confusion occurs in the discussion, we occasionally use $x(kT)$ and $x(k)$ interchangeably. [That is, to simplify the presentation, we occasionally drop the explicit appearance of $T$ and write $x(kT)$ as $x(k)$.]

***Outline of the Chapter.*** Section 2–1 has presented introductory remarks. Section 2–2 presents the definition of the $z$ transform and associated subjects. Section 2–3 gives $z$ transforms of elementary functions. Important properties and theorems of the $z$ transform are presented in Section 2–4. Both analytical and computational methods for finding the inverse $z$ transform are discussed in Section 2–5. Section 2–6 presents the solution of difference equations by the $z$ transform method. Finally, Section 2–7 gives concluding comments.

## 2–2 THE z TRANSFORM

The $z$ transform method is an operational method that is very powerful when working with discrete-time systems. In what follows we shall define the $z$ transform of a time function or a number sequence.

In considering the $z$ transform of a time function $x(t)$, we consider only the sampled values of $x(t)$, that is, $x(0), x(T), x(2T), \ldots$, where $T$ is the sampling period.

The $z$ transform of a time function $x(t)$, where $t$ is nonnegative, or of a sequence of values $x(kT)$, where $k$ takes zero or positive integers and $T$ is the sampling period, is defined by the following equation:

$$X(z) = \mathcal{Z}[x(t)] = \mathcal{Z}[x(kT)] = \sum_{k=0}^{\infty} x(kT)z^{-k} \qquad (2\text{–}1)$$

For a sequence of numbers $x(k)$, the $z$ transform is defined by

$$X(z) = \mathcal{Z}[x(k)] = \sum_{k=0}^{\infty} x(k)z^{-k} \qquad (2\text{–}2)$$

one-sided z-transform

The $z$ transform defined by Equation (2–1) or (2–2) is referred to as the *one-sided z transform*.

The symbol $\mathcal{Z}$ denotes "the $z$ transform of." In the one-sided $z$ transform, we assume $x(t) = 0$ for $t < 0$ or $x(k) = 0$ for $k < 0$. Note that $z$ is a complex variable.

Note that, when dealing with a time sequence $x(kT)$ obtained by sampling a time signal $x(t)$, the $z$ transform $X(z)$ involves $T$ explicitly. However, for a number sequence $x(k)$, the $z$ transform $X(z)$ does not involve $T$ explicitly.

The $z$ transform of $x(t)$, where $-\infty < t < \infty$, or of $x(k)$, where $k$ takes integer values ($k = 0, \pm 1, \pm 2, \cdots$), is defined by

$$X(z) = \mathcal{Z}[x(t)] = \mathcal{Z}[x(kT)] = \sum_{k=-\infty}^{\infty} x(kT)z^{-k} \qquad (2\text{–}3)$$

or

$$X(z) = \mathcal{Z}[x(k)] = \sum_{k=-\infty}^{\infty} x(k)z^{-k} \qquad (2\text{–}4)$$

two-sided z transform

[illegible]

$$\boxed{X(z) = \sum_{i=0}^{\infty} x_i z^{-i}}$$

The $z$ transform defined by Equation (2–3) or (2–4) is referred to as the *two-sided z transform*. In the two-sided $z$ transform, the time function $x(t)$ is assumed to be nonzero for $t < 0$ and the sequence $x(k)$ is considered to have nonzero values for $k < 0$. Both the one-sided and two-sided $z$ transforms are series in powers of $z^{-1}$. (The latter involves both positive and negative powers of $z^{-1}$.) In this book, only the one-sided $z$ transform is considered in detail.

For most engineering applications the one-sided $z$ transform will have a convenient closed-form solution in its region of convergence. Note that whenever $X(z)$, an infinite series in $z^{-1}$, converges outside the circle $|z| = R$, where $R$ is called the *radius of absolute convergence*, in using the $z$ transform method for solving discrete-time problems it is not necessary each time to specify the values of $z$ over which $X(z)$ is convergent.

Notice that expansion of the right-hand side of Equation (2–1) gives

$$X(z) = x(0) + x(T)z^{-1} + x(2T)z^{-2} + \cdots + x(kT)z^{-k} + \cdots \qquad (2\text{–}5)$$

Equation (2–5) implies that the $z$ transform of any continuous-time function $x(t)$ may be written in the series form by inspection. The $z^{-k}$ in this series indicates the position in time at which the amplitude $x(kT)$ occurs. Conversely, if $X(z)$ is given in the series form as above, the inverse $z$ transform can be obtained by inspection as a sequence of the function $x(kT)$ that corresponds to the values of $x(t)$ at the respective instants of time.

If the $z$ transform is given as a ratio of two polynomials in $z$, then the inverse $z$ transform may be obtained by several different methods, such as the direct division method, the computational method, the partial-fraction-expansion method, and the inversion integral method (see Section 2–5 for details.)

## 2–3 z TRANSFORMS OF ELEMENTARY FUNCTIONS

In the following we shall present $z$ transforms of several elementary functions. It is noted that in one-sided $z$ transform theory, in sampling a discontinuous function $x(t)$, we assume that the function is continuous from the right; that is, if discontinuity occurs at $t = 0$, then we assume that $x(0)$ is equal to $x(0+)$ rather than to the average at the discontinuity, $[x(0-) + x(0+)]/2$.

***Unit-Step Function.*** Let us find the $z$ transform of the unit-step function

$$x(t) = \begin{cases} 1(t), & 0 \le t \\ 0, & t < 0 \end{cases}$$

As just noted, in sampling a unit-step function we assume that this function is continuous from the right; that is, $1(0) = 1$. Then, referring to Equation (2–1), we have

$$X(z) = \mathcal{Z}[1(t)] = \sum_{k=0}^{\infty} 1z^{-k} = \sum_{k=0}^{\infty} z^{-k}$$
$$= 1 + z^{-1} + z^{-2} + z^{-3} + \cdots$$

$S = 1 + r + r^2 + r^3 + \cdots$

$S = \dfrac{1}{1-r}$ $\Longrightarrow$ $X(z) = \dfrac{1}{1 - z^{-1}} = \dfrac{z}{z-1}$

$$= \frac{1}{1 - z^{-1}}$$

$$= \frac{z}{z - 1}$$

$X(z) = 1 + z^{-1} + z^{-2} + z^{-3} + \cdots$
for $|z| \leq 1$, $X(z) \to \infty$ (not converges)
for $|z| > 1$, $X(z) \to 0$ (converges)

Notice that the series converges if $|z| > 1$. In finding the $z$ transform, the variable $z$ acts as a dummy operator. It is not necessary to specify the region of $z$ over which $X(z)$ is convergent. It suffices to know that such a region exists. The $z$ transform $X(z)$ of a time function $x(t)$ obtained in this way is valid throughout the $z$ plane except at poles of $X(z)$.

It is noted that $1(k)$ as defined by

$$1(k) = \begin{cases} 1, & k = 0, 1, 2, \ldots \\ 0, & k < 0 \end{cases}$$

is commonly called a *unit-step sequence*.

***Unit-Ramp Function.*** Consider the unit-ramp function

$$x(t) = \begin{cases} t, & 0 \leq t \\ 0, & t < 0 \end{cases}$$

Notice that

$$x(kT) = kT, \qquad k = 0, 1, 2, \ldots$$

Figure 2–1 depicts the sampled unit-ramp signal. The magnitudes of the sampled values are proportional to the sampling period $T$. The $z$ transform of the unit-ramp function can be written as

$$X(z) = \mathcal{Z}[t] = \sum_{k=0}^{\infty} x(kT)z^{-k} = \sum_{k=0}^{\infty} kTz^{-k} = T\sum_{k=0}^{\infty} kz^{-k}$$

$$= T(z^{-1} + 2z^{-2} + 3z^{-3} + \cdots)$$

prove this!

$$= T\frac{z^{-1}}{(1 - z^{-1})^2}$$

$$= \frac{Tz}{(z - 1)^2}$$

From Kuo's Ex 3-4 Find the z-tr of the ramp func. $f(t) = t\,u(t)$ $\{f(kT) = kT$

$F(z) = \sum_{k=0}^{\infty} kT z^{-k} = Tz^{-1} + 2Tz^{-2} + 3Tz^{-3} + \cdots$ (1)

to express $F(z)$ in closed form, multiply both sides by $z^{-1}$,

$z^{-1}F(z) = Tz^{-2} + 2Tz^{-3} + \cdots$ (2)

(1)−(2) $\Rightarrow (1 - z^{-1})F(z) = Tz^{-1} + Tz^{-2} + Tz^{-3} + \cdots$ (3)

multiply both side of (3) by $z^{-1}$

$(z^{-1} - z^{-2})F(z) = Tz^{-2} + Tz^{-3} + Tz^{-4} + \cdots$ (4)

now do (3)−(4)

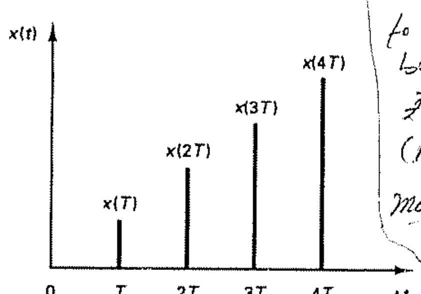


**Figure 2–1** Sampled unit-ramp signal.

$F(z)[1 - z^{-1} - z^{-1} + z^{-2}] = Tz^{-1}$

$(1 - 2z^{-1} + z^{-2})F(z) = Tz^{-1}$

$(1 - z^{-1})^2 F(z) = Tz^{-1}$

$\Rightarrow \boxed{F(z) = \frac{Tz^{-1}}{(1 - z^{-1})^2}} = \frac{Tz}{z^2(1 - 2z^{-1} + z^{-2})}$

$F(z) = \frac{Tz}{z^2 - 2z + 1}$

$F(z) = \frac{Tz}{(z - 1)^2}$

Note that it is a function of the sampling period $T$.

***Polynomial Function $a^k$.*** Let us obtain the $z$ transform of $x(k)$ as defined by

$$x(k) = \begin{cases} a^k, & k = 0, 1, 2, \ldots \\ 0, & k < 0 \end{cases}$$

where $a$ is a constant. Referring to the definition of the $z$ transform given by Equation (2–2), we obtain

$$\begin{aligned} X(z) = \mathcal{Z}[a^k] &= \sum_{k=0}^{\infty} x(k) z^{-k} = \sum_{k=0}^{\infty} a^k z^{-k} \\ &= 1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} + \cdots \\ &= \frac{1}{1 - az^{-1}} \\ &= \frac{z}{z - a} \end{aligned}$$

***Exponential Function.*** Let us find the $z$ transform of

$$x(t) = \begin{cases} e^{-at}, & 0 \le t \\ 0, & t < 0 \end{cases}$$

Since

$$x(kT) = e^{-akT}, \qquad k = 0, 1, 2, \ldots$$

we have

$$\begin{aligned} X(z) = \mathcal{Z}[e^{-at}] &= \sum_{k=0}^{\infty} x(kT) z^{-k} = \sum_{k=0}^{\infty} e^{-akT} z^{-k} \\ &= 1 + e^{-aT} z^{-1} + e^{-2aT} z^{-2} + e^{-3aT} z^{-3} + \cdots \\ &= \frac{1}{1 - e^{-aT} z^{-1}} \\ &= \frac{z}{z - e^{-aT}} \end{aligned}$$

***Sinusoidal Function.*** Consider the sinusoidal function

$$x(t) = \begin{cases} \sin \omega t, & 0 \le t \\ 0, & t < 0 \end{cases}$$

Noting that

$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

$$e^{-j\omega t} = \cos \omega t - j \sin \omega t$$

we have

$$\sin \omega t = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

$\cos \omega t = \frac{1}{2}(e^{j\omega t} - e^{-j\omega t})$

Since the $z$ transform of the exponential function is

$$\mathcal{Z}[e^{-at}] = \frac{1}{1 - e^{-aT}z^{-1}}$$

we have

$$\begin{aligned}
X(z) = \mathcal{Z}[\sin \omega t] &= \mathcal{Z}\left[\frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})\right] \\
&= \frac{1}{2j}\left(\frac{1}{1 - e^{j\omega T}z^{-1}} - \frac{1}{1 - e^{-j\omega T}z^{-1}}\right) \\
&= \frac{1}{2j}\frac{(e^{j\omega T} - e^{-j\omega T})z^{-1}}{1 - (e^{j\omega T} + e^{-j\omega T})z^{-1} + z^{-2}} \\
&= \frac{z^{-1}\sin \omega T}{1 - 2z^{-1}\cos \omega T + z^{-2}} \\
&= \frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1}
\end{aligned}$$

**Example 2–1**

Obtain the $z$ transform of the cosine function

$$x(t) = \begin{cases} \cos \omega t, & 0 \le t \\ 0, & t < 0 \end{cases}$$

If we proceed in a manner similar to the way we treated the $z$ transform of the sine function, we have

$$\begin{aligned}
X(z) = \mathcal{Z}[\cos \omega t] &= \tfrac{1}{2}\mathcal{Z}[e^{j\omega t} + e^{-j\omega t}] \\
&= \frac{1}{2}\left(\frac{1}{1 - e^{j\omega T}z^{-1}} + \frac{1}{1 - e^{-j\omega T}z^{-1}}\right) \\
&= \frac{1}{2}\frac{2 - (e^{-j\omega T} + e^{j\omega T})z^{-1}}{1 - (e^{j\omega T} + e^{-j\omega T})z^{-1} + z^{-2}} \\
&= \frac{1 - z^{-1}\cos \omega T}{1 - 2z^{-1}\cos \omega T + z^{-2}} \\
&= \frac{z^2 - z \cos \omega T}{z^2 - 2z \cos \omega T + 1}
\end{aligned}$$

HW

1 the z-transforms of

$x(t) = at$

$x(t) = \sin \omega t$

$x(t) = \cos \omega t$

**Example 2–2**

Obtain the $z$ transform of

$$X(s) = \frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{1}{s} - \frac{1}{s+1}$$

Whenever a function in $s$ is given, one approach for finding the corresponding $z$ transform is to convert $X(s)$ into $x(t)$ and then find the $z$ transform of $x(t)$. Another approach is to expand $X(s)$ into partial fractions and use a $z$ transform table to find the $z$ transforms of the expanded terms Still other approaches will be discussed in Section 3–3

The inverse Laplace transform of $X(s)$ is

Let them find it!

$$x(t) = 1 - e^{-t}, \qquad 0 \le t$$

Hence,

$$X(z) = \mathcal{Z}[1 - e^{-t}] = \frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-T}z^{-1}}$$

$$= \frac{(1 - e^{-T})z^{-1}}{(1 - z^{-1})(1 - e^{-T}z^{-1})}$$

$$= \frac{(1 - e^{-T})z}{(z - 1)(z - e^{-T})}$$

***Comments.*** Just as in working with the Laplace transformation, a table of $z$ transforms of commonly encountered functions is very useful for solving problems in the field of discrete-time systems. Table 2–1 is such a table.

impulse func. $\rightarrow \delta(t) \rightarrow \mathcal{L}[\delta(t)] = 1$, $\mathcal{Z}[\delta(t)] = 1$

**TABLE 2–1 TABLE OF z TRANSFORMS**

| | $X(s)$ | $x(t)$ | $x(kT)$ or $x(k)$ | $X(z)$ |
|---|---|---|---|---|
| 1. | — | — | Kronecker delta $\delta_0(k)$<br>1, $k = 0$<br>0, $k \neq 0$ | 1 |
| 2. | — | — | $\delta_0(n - k)$<br>1, $n = k$<br>0, $n \neq k$ | $z^{-k}$ $= 1 \cdot z^{-k}$ |
| 3. | $\frac{1}{s}$ | $1(t)$ step | $1(k)$ | $\frac{1}{1 - z^{-1}}$ |
| 4. | $\frac{1}{s + a}$ | $e^{-at}$ exponential | $e^{-akT}$ | $\frac{1}{1 - e^{-aT}z^{-1}}$ |
| 5. | $\frac{1}{s^2}$ | $t$ ramp | $kT$ | $\frac{Tz^{-1}}{(1 - z^{-1})^2}$ |
| 6. | $\frac{2}{s^3}$ | $t^2$ parabola | $(kT)^2$ | $\frac{T^2z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}$ $= \frac{T^2 z(z+1)}{(z-1)^3}$ |
| 7. | $\frac{6}{s^4}$ | $t^3$ | $(kT)^3$ | $\frac{T^3z^{-1}(1 + 4z^{-1} + z^{-2})}{(1 - z^{-1})^4}$ |
| 8. | $\frac{a}{s(s + a)}$ | $1 - e^{-at}$ | $1 - e^{-akT}$ | $\frac{(1 - e^{-aT})z^{-1}}{(1 - z^{-1})(1 - e^{-aT}z^{-1})}$ |
| 9. | $\frac{b - a}{(s + a)(s + b)}$ | $e^{-at} - e^{-bt}$ | $e^{-akT} - e^{-bkT}$ | $\frac{(e^{-aT} - e^{-bT})z^{-1}}{(1 - e^{-aT}z^{-1})(1 - e^{-bT}z^{-1})}$ |
| 10. | $\frac{1}{(s + a)^2}$ | $te^{-at}$ | $kTe^{-akT}$ | $\frac{Te^{-aT}z^{-1}}{(1 - e^{-aT}z^{-1})^2}$ |
| 11. | $\frac{s}{(s + a)^2}$ | $(1 - at)e^{-at}$ | $(1 - akT)e^{-akT}$ | $\frac{1 - (1 + aT)e^{-aT}z^{-1}}{(1 - e^{-aT}z^{-1})^2}$ |

see item #18

$$\frac{a}{s(s+a)} \rightarrow 1 - e^{-at} \rightarrow \frac{z(1 - e^{-aT})}{(z-1)(z - e^{-aT})} = \frac{(1 - e^{-aT})z^{-1}}{(1 - z^{-1})(1 - e^{-aT}z^{-1})}$$

(from [illegible]'s book)

**TABLE 2–1** (continued)

| | $X(s)$ | $x(t)$ | $x(kT)$ or $x(k)$ | $X(z)$ |
|---|---|---|---|---|
| 12. | $\frac{2}{(s+a)^3}$ | $t^2 e^{-at}$ | $(kT)^2 e^{-akT}$ | $\frac{T^2 e^{-aT}(1+e^{-aT}z^{-1})z^{-1}}{(1-e^{-aT}z^{-1})^3}$ |
| 13. | $\frac{a^2}{s^2(s+a)}$ | $at - 1 + e^{-at}$ | $akT - 1 + e^{-akT}$ | $\frac{[(aT-1+e^{-aT}) + (1-e^{-aT}-aTe^{-aT})z^{-1}]z^{-1}}{(1-z^{-1})^2(1-e^{-aT}z^{-1})}$ |
| 14 | $\frac{\omega}{s^2+\omega^2}$ | $\sin \omega t$ | $\sin \omega kT$ | $\frac{z^{-1}\sin \omega T}{1-2z^{-1}\cos \omega T + z^{-2}}$ |
| 15. | $\frac{s}{s^2+\omega^2}$ | $\cos \omega t$ | $\cos \omega kT$ | $\frac{1-z^{-1}\cos \omega T}{1-2z^{-1}\cos \omega T + z^{-2}}$ |
| 16. | $\frac{\omega}{(s+a)^2+\omega^2}$ | $e^{-at}\sin \omega t$ | $e^{-akT}\sin \omega kT$ | $\frac{e^{-aT}z^{-1}\sin \omega T}{1-2e^{-aT}z^{-1}\cos \omega T + e^{-2aT}z^{-2}}$ |
| 17. | $\frac{s+a}{(s+a)^2+\omega^2}$ | $e^{-at}\cos \omega t$ | $e^{-akT}\cos \omega kT$ | $\frac{1-e^{-aT}z^{-1}\cos \omega T}{1-2e^{-aT}z^{-1}\cos \omega T + e^{-2aT}z^{-2}}$ |
| 18. | | | $a^k$ | $\frac{1}{1-az^{-1}}$ |
| 19. | | | $a^{k-1}$<br>$k = 1, 2, 3, \ldots$ | $\frac{z^{-1}}{1-az^{-1}}$ |
| 20. | | | $ka^{k-1}$ | $\frac{z^{-1}}{(1-az^{-1})^2}$ |
| 21. | | | $k^2 a^{k-1}$ | $\frac{z^{-1}(1+az^{-1})}{(1-az^{-1})^3}$ |
| 22. | | | $k^3 a^{k-1}$ | $\frac{z^{-1}(1+4az^{-1}+a^2z^{-2})}{(1-az^{-1})^4}$ |
| 23. | | | $k^4 a^{k-1}$ | $\frac{z^{-1}(1+11az^{-1}+11a^2z^{-2}+a^3z^{-3})}{(1-az^{-1})^5}$ |
| 24 | | | $a^k \cos k\pi$ | $\frac{1}{1+az^{-1}}$ |
| 25 | | | $\frac{k(k-1)}{2!}$ | $\frac{z^{-2}}{(1-z^{-1})^3}$ |
| 26 | | $\frac{k(k-1)\cdots(k-m+2)}{(m-1)!}$ | | $\frac{z^{-m+1}}{(1-z^{-1})^m}$ |
| 27 | | | $\frac{k(k-1)}{2!}a^{k-2}$ | $\frac{z^{-2}}{(1-az^{-1})^3}$ |
| 23 | $\frac{k(k-1)\cdots(k-m+2)}{(m-1)!}a^{k-m+1}$ | | | $\frac{z^{-m+1}}{(1-az^{-1})^m}$ |

$x(t) = 0$, for $t < 0$

$x(kT) = x(k) = 0$, for $k < 0$

Unless otherwise noted, $k = 0, 1, 2, 3, \ldots$

## 2–4 IMPORTANT PROPERTIES AND THEOREMS OF THE z TRANSFORM

The use of the $z$ transform method in the analysis of discrete-time control systems may be facilitated if theorems of the $z$ transform are referred to. In this section we present important properties and useful theorems of the $z$ transform. We assume that the time function $x(t)$ is $z$-transformable and that $x(t)$ is zero for $t < 0$.

***Multiplication by a Constant.*** If $X(z)$ is the $z$ transform of $x(t)$, then

$$\mathcal{Z}[ax(t)] = a\,\mathcal{Z}[x(t)] = aX(z)$$

where $a$ is a constant.

To prove this, note that by definition

$$\mathcal{Z}[ax(t)] = \sum_{k=0}^{\infty} ax(kT)z^{-k} = a\sum_{k=0}^{\infty} x(kT)z^{-k} = aX(z)$$

***Linearity of the z Transform.*** The $z$ transform possesses an important property: linearity. This means that, if $f(k)$ and $g(k)$ are $z$-transformable and $\alpha$ and $\beta$ are scalars, then $x(k)$ formed by a linear combination

$$x(k) = \alpha f(k) + \beta g(k)$$

has the $z$ transform

$$X(z) = \alpha F(z) + \beta G(z)$$

where $F(z)$ and $G(z)$ are the $z$ transforms of $f(k)$ and $g(k)$, respectively.

The linearity property can be proved by referring to Equation (2–2) as follows:

$$\begin{aligned} X(z) &= \mathcal{Z}[x(k)] = \mathcal{Z}[\alpha f(k) + \beta g(k)] \\ &= \sum_{k=0}^{\infty}[\alpha f(k) + \beta g(k)]z^{-k} \\ &= \alpha\sum_{k=0}^{\infty} f(k)z^{-k} + \beta\sum_{k=0}^{\infty} g(k)z^{-k} \\ &= \alpha\mathcal{Z}[f(k)] + \beta\mathcal{Z}[g(k)] \\ &= \alpha F(z) + \beta G(z) \end{aligned}$$

(definition of z-tr

$X(z) = \mathcal{Z}[x(k)] = \sum_{k=0}^{\infty} x(k) z^{-k}$

***Multiplication by $a^k$.*** If $X(z)$ is the $z$ transform of $x(k)$, then the $z$ transform of $a^k x(k)$ can be given by $X(a^{-1}z)$:

$$\mathcal{Z}[a^k x(k)] = X(a^{-1}z) \tag{2–6}$$

This can be proved as follows:

$$\begin{aligned} \mathcal{Z}[a^k x(k)] &= \sum_{k=0}^{\infty} a^k x(k)z^{-k} = \sum_{k=0}^{\infty} x(k)(a^{-1}z)^{-k} \\ &= X(a^{-1}z) \end{aligned}$$

Real / Translation /

***Shifting Theorem.*** The shifting theorem presented here is also referred to as the real translation theorem. If $x(t) = 0$ for $t < 0$ and $x(t)$ has the $z$ transform $X(z)$, then time delay (right shift)

$$\mathcal{Z}[x(t-nT)] = z^{-n}X(z) \qquad \checkmark \tag{2-7}$$

and

time advance (left shift)

$$\mathcal{Z}[x(t+nT)] = z^{n}\left[X(z) - \sum_{k=0}^{n-1} x(kT)z^{-k}\right] \tag{2-8}$$

ask them to prove

where $n$ is zero or a positive integer.

To prove Equation (2–7), note that

$$\mathcal{Z}[x(t-nT)] = \sum_{k=0}^{\infty} x(kT-nT)z^{-k} \qquad \checkmark$$

$$= z^{-n}\sum_{k=0}^{\infty} x(kT-nT)z^{-(k-n)} \qquad \checkmark \tag{2-9}$$

By defining $m = k - n$, Equation (2–9) can be written as follows:

$$\mathcal{Z}[x(t-nT)] = z^{-n}\sum_{m=-n}^{\infty} x(mT)z^{-m} \qquad \checkmark$$

Since $x(mT) = 0$ for $m < 0$, we may change the lower limit of the summation from $m = -n$ to $m = 0$. Hence,

$$\mathcal{Z}[x(t-nT)] = z^{-n}\sum_{m=0}^{\infty} x(mT)z^{-m} = z^{-n}X(z) \qquad \checkmark \tag{2-10}$$

(Thus, multiplication of a $z$ transform by $z^{-n}$ has the effect of delaying the time function $x(t)$ by time $nT$. (That is, move the function to the right by time $nT$.)) $\Rightarrow$ $z^{-n}$ is called delay operator

To prove Equation (2–8), we note that (time advance)

$$\mathcal{Z}[x(t+nT)] = \sum_{k=0}^{\infty} x(kT+nT)z^{-k}$$

$$= z^{n}\sum_{k=0}^{\infty} x(kT+nT)z^{-(k+n)} \qquad \checkmark$$

$+T_1 \quad -T_1$

$$= z^{n}\left[\sum_{k=0}^{\infty} x(kT+nT)z^{-(k+n)} + \sum_{k=0}^{n-1} x(kT)z^{-k} - \sum_{k=0}^{n-1} x(kT)z^{-k}\right] \qquad \checkmark$$

$$= z^{n}\left[\sum_{k=0}^{\infty} x(kT)z^{-k} - \sum_{k=0}^{n-1} x(kT)z^{-k}\right]$$

açılarak eşitlik gösterilebilir

$$= z^{n}\left[X(z) - \sum_{k=0}^{n-1} x(kT)z^{-k}\right] \qquad \checkmark$$

For the number sequence $x(k)$, Equation (2–8) can be written as follows:

$$\mathcal{Z}[x(k+n)] = z^{n}\left[X(z) - \sum_{k=0}^{n-1} x(k)z^{-k}\right] \qquad \checkmark$$

From this last equation, we obtain

$$\mathcal{Z}[x(k+1)] = zX(z) - zx(0) \qquad \checkmark \tag{2-11}$$

$$\mathcal{Z}[x(k+2)] = z\mathcal{Z}[x(k+1)] - zx(1) = z^2X(z) - z^2x(0) - zx(1) \tag{2-12}$$

$$= z^2\left[X(z) - x(0) - x(1)z^{-1}\right]$$

$$= z^2X(z) - z^2x(0) - zx(1)$$

~~$z^2$ almalı~~

$$= z^n\left[x(nT)z^{-n} + x[(n+1)T]z^{-(n+1)} + x[(n+2)T]z^{-(n+2)} + \cdots + x(0) + x(T)z^{-1} + x(2T)z^{-2} + \cdots - \sum_{k=0}^{n-1} x(kT)z^{-k}\right]$$

$$= z^n\left[x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots + x[(n-1)T]z^{-(n-1)} + x(nT)z^{-n} + x[(n+1)T]z^{-(n+1)} + \cdots\right]$$

$$= z^n\left[X(z) - \sum_{k=0}^{n-1} x(kT)z^{-k}\right]$$



Similarly,

$$\mathcal{Z}[x(k+n)] = z^n X(z) - z^n x(0) - z^{n-1}x(1) - z^{n-2}x(2) - \cdots - zx(n-1) \qquad (2\text{–}13)$$

where $n$ is a positive integer.

Remember that multiplication of the $z$ transform $X(z)$ by $z$ has the effect of advancing the signal $x(kT)$ by one step (1 sampling period) and that multiplication of the $z$ transform $X(z)$ by $z^{-1}$ has the effect of delaying the signal $x(kT)$ by one step (1 sampling period).

**Example 2–3**

Find the $z$ transforms of unit-step functions that are delayed by 1 sampling period and 4 sampling periods, respectively, as shown in Figure 2–2(a) and (b).

Using the shifting theorem given by Equation (2–7), we have

$$\mathcal{Z}[1(t-T)] = z^{-1}\mathcal{Z}[1(t)] = z^{-1}\frac{1}{1-z^{-1}} = \frac{z^{-1}}{1-z^{-1}} = \frac{1}{z-1}$$

Also,

$$\mathcal{Z}[1(t-4T)] = z^{-4}\mathcal{Z}[1(t)] = z^{-4}\frac{1}{1-z^{-1}} = \frac{z^{-4}}{1-z^{-1}} = \frac{1}{z^4 - z^3} = \frac{1}{z^3(z-1)}$$

(Note that $z^{-1}$ represents a delay of 1 sampling period $T$, regardless of the value of $T$.)

**Example 2–4**

Obtain the $z$ transform of

$$f(a) = \begin{cases} a^{k-1}, & k = 1, 2, 3, \ldots \\ 0, & k \le 0 \end{cases}$$

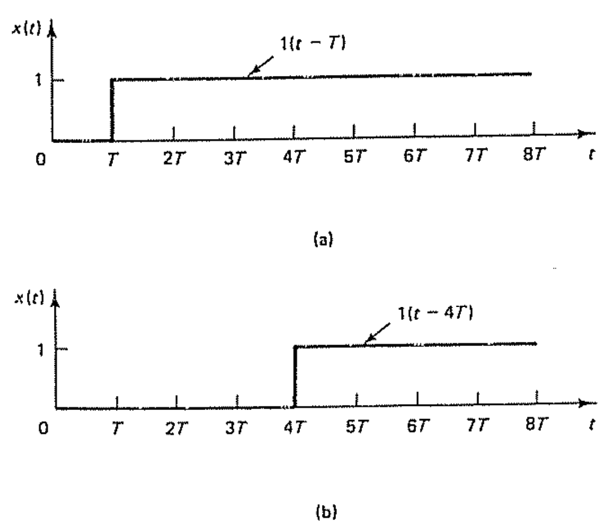


**Figure 2–2** (a) Unit-step function delayed by 1 sampling period; (b) unit-step function delayed by 4 sampling periods

$$f(a) = \begin{cases} a^{k-1}, & k = 1,2,3. \\ 0, & k \leq 0 \end{cases}$$



18 19 $x$ item

Referring to Equation (2–7), we have

$$\mathcal{Z}[x(k-1)] = z^{-1}X(z)$$

The $z$ transform of $a^k$ is

$$\mathcal{Z}[a^k] = \frac{1}{1 - az^{-1}}$$ (from Table 2-1 or p27)

and so

$$\mathcal{Z}[f(a)] = \mathcal{Z}[a^{k-1}] = z^{-1}\frac{1}{1 - az^{-1}} = \frac{z^{-1}}{1 - az^{-1}}$$

where $k = 1, 2, 3, \ldots$.

**Example 2–5**

Consider the function $y(k)$, which is a sum of functions $x(h)$, where $h = 0, 1, 2, \ldots, k$, such that

$$y(k) = \sum_{h=0}^{k} x(h), \qquad k = 0, 1, 2, \ldots$$

where $y(k) = 0$ for $k < 0$. Obtain the $z$ transform of $y(k)$.

First note that

$$y(k) = x(0) + x(1) + \cdots + x(k-1) + x(k)$$
$$y(k-1) = x(0) + x(1) + \cdots + x(k-1)$$

Hence,

$$y(k) - y(k-1) = x(k), \qquad k = 0, 1, 2, \ldots$$

Therefore,

$$\mathcal{Z}[y(k) - y(k-1)] = \mathcal{Z}[x(k)]$$

or

$$Y(z) - z^{-1}Y(z) = X(z)$$

which yields

$$Y(z) = \frac{1}{1 - z^{-1}}X(z)$$

where $X(z) = \mathcal{Z}[x(k)]$.

***Complex Translation Theorem.*** If $x(t)$ has the $z$ transform $X(z)$, then the $z$ transform of $e^{-at}x(t)$ can be given by $X(ze^{aT})$. This is known as the *complex translation theorem*.

To prove this theorem, note that

$$\mathcal{Z}[e^{-at}x(t)] = \sum_{k=0}^{\infty} x(kT)e^{-akT}z^{-k} = \sum_{k=0}^{\infty} x(kT)(ze^{aT})^{-k} = X(ze^{aT}) \qquad (2\text{–}14)$$

Thus, we see that replacing $z$ in $X(z)$ by $ze^{aT}$ gives the $z$ transform of $e^{-at}x(t)$.

**Example 2–6**

Given the $z$ transforms of $\sin \omega t$ and $\cos \omega t$, obtain the $z$ transforms of $e^{-at}\sin \omega t$ and $e^{-at}\cos \omega t$, respectively, by using the complex translation theorem.

Tr. theorem:

$$[e^{\mp at} x(t)] = [X(s \pm a)]_{z=e^{Ts}} = X(ze^{\pm aT})$$

Proof

$$\mathcal{Z}[e^{-aT}x(t)] = \sum_{k=0}^{\infty} x(kT)e^{-akT}z^{-k}, \quad \text{let } (z_1)^{-k} = (ze^{aT})^{-k} \Rightarrow z_1^{-k} = e^{-akT}z^{-k}$$

$$\Rightarrow = \sum_{k=0}^{\infty} x(kT)z_1^{-k} = X(z_1) = X(ze^{aT})$$

Noting that

$$\mathcal{Z}[\sin \omega t] = \frac{z^{-1} \sin \omega T}{1 - 2z^{-1} \cos \omega T + z^{-2}}$$

using the comp. fr. theo.

we substitute $ze^{aT}$ for $z$ to obtain the $z$ transform of $e^{-at} \sin \omega t$ as follows:

$$\mathcal{Z}[e^{-at} \sin \omega t] = \frac{e^{-aT} z^{-1} \sin \omega T}{1 - 2e^{-aT} z^{-1} \cos \omega T + e^{-2aT} z^{-2}}$$

Similarly, for the cosine function, we have

$$\mathcal{Z}[\cos \omega t] = \frac{1 - z^{-1} \cos \omega T}{1 - 2z^{-1} \cos \omega T + z^{-2}}$$

By substituting $ze^{aT}$ for $z$ in the $z$ transform of $\cos \omega t$, we obtain

$$\mathcal{Z}[e^{-at} \cos \omega t] = \frac{1 - e^{-aT} z^{-1} \cos \omega T}{1 - 2e^{-aT} z^{-1} \cos \omega T + e^{-2aT} z^{-2}}$$

**Example 2–7**

Obtain the $z$ transform of $te^{-at}$.

Notice that

$$\mathcal{Z}[t] = \frac{Tz^{-1}}{(1 - z^{-1})^2} = X(z)$$

Thus,

$$\mathcal{Z}[te^{-at}] = X(ze^{aT}) = \frac{Te^{-aT} z^{-1}}{(1 - e^{-aT} z^{-1})^2}$$

init. val thm n.s

$f(0^+) = \lim_{s \to \infty} sF(s)$

final value theorm s

$f(\infty) = \lim_{s \to 0} sF(s)$

***Initial Value Theorem.*** If $x(t)$ has the $z$ transform $X(z)$ and if $\lim_{z \to \infty} X(z)$ exists, then the initial value $x(0)$ of $x(t)$ or $x(k)$ is given by

$$x(0) = \lim_{z \to \infty} X(z) \tag{2–15}$$

To prove this theorem, note that

$$X(z) = \sum_{k=0}^{\infty} x(k) z^{-k} = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

Letting $z \to \infty$ in this last equation, we obtain Equation (2–15). The behavior of the signal in the neighborhood of $t = 0$ or $k = 0$ can thus be determined by the behavior of $X(z)$ at $z = \infty$.

The initial value theorem is convenient for checking $z$ transform calculations for possible errors. Since $x(0)$ is usually known, a check of the initial value by $\lim_{z \to \infty} X(z)$ can easily spot errors in $X(z)$, if any exist.

$x(0) = \lim_{z \to \infty} X(z)$

**Example 2–8**

Determine the initial value $x(0)$ if the $z$ transform of $x(t)$ is given by

$$X(z) = \frac{(1 - e^{-T})z^{-1}}{(1 - z^{-1})(1 - e^{-T} z^{-1})}$$

By using the initial value theorem, we find

$$x(0) = \lim_{z \to \infty} \frac{(1 - e^{-T})z^{-1}}{(1 - z^{-1})(1 - e^{-T} z^{-1})} = 0$$

Referring to Example 2–2, notice that this $X(z)$ was the $z$ transform of

$$x(t) = 1 - e^{-t}$$

and thus $x(0) = 0$, which agrees with the result obtained earlier.

*Final Value Theorem.* Suppose that $x(k)$, where $x(k) = 0$ for $k < 0$, has the $z$ transform $X(z)$ and that all the poles of $X(z)$ lie inside the unit circle, with the possible exception of a simple pole at $z = 1$. [This is the condition for the stability of $X(z)$, or the condition for $x(k)$ $(k = 0, 1, 2, \ldots)$ to remain finite.] Then the final value of $x(k)$, that is, the value of $x(k)$ as $k$ approaches infinity, can be given by

$$\lim_{k\to\infty} x(k) = \lim_{z\to 1} [(1 - z^{-1})X(z)] \tag{2–16}$$

To prove the final value theorem, note that

$$\mathcal{Z}[x(k)] = X(z) = \sum_{k=0}^{\infty} x(k)z^{-k} \qquad (1)$$

$$\mathcal{Z}[x(k-1)] = z^{-1}X(z) = \sum_{k=0}^{\infty} x(k-1)z^{-k} \qquad (2)$$

Hence, taking (1) − (2),

$$\sum_{k=0}^{\infty} x(k)z^{-k} - \sum_{k=0}^{\infty} x(k-1)z^{-k} = X(z) - z^{-1}X(z)$$

Taking the limit as $z$ approaches unity, we have

$$\lim_{z\to 1}\left[\sum_{k=0}^{\infty} x(k)z^{-k} - \sum_{k=0}^{\infty} x(k-1)z^{-k}\right] = \lim_{z\to 1}[(1 - z^{-1})X(z)]$$

Because of the assumed stability condition and the condition that $x(k) = 0$ for $k < 0$, the left-hand side of this last equation becomes

$$\sum_{k=0}^{\infty}[x(k) - x(k-1)] = [x(0) - x(-1)] + [x(1) - x(0)]$$

$$z^{-k} = 1 \qquad + [x(2) - x(1)] + \cdots = x(\infty) = \lim_{k\to\infty} x(k)$$

Hence,

$$\lim_{k\to\infty} x(k) = \lim_{z\to 1}[(1 - z^{-1})X(z)]$$

which is Equation (2–16). The final value theorem is very useful in determining the behavior of $x(k)$ as $k \to \infty$ from its $z$ transform $X(z)$. without the need for taking inv. z-tr.

**Example 2–9**

Determine the final value $x(\infty)$ of

$$X(z) = \frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-aT}z^{-1}}, \qquad a > 0$$

by using the final value theorem.

By applying the final value theorem to the given $X(z)$, we obtain

$$\begin{aligned} x(\infty) &= \lim_{z \to 1} [(1 - z^{-1})X(z)] \\ &= \lim_{z \to 1} \left[ (1 - z^{-1}) \left( \frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-aT} z^{-1}} \right) \right] \\ &= \lim_{z \to 1} \left( 1 - \frac{1 - z^{-1}}{1 - e^{-aT} z^{-1}} \right) = 1 \end{aligned}$$

It is noted that the given $X(z)$ is actually the $z$ transform of

$$x(t) = 1 - e^{-at}$$

By substituting $t = \infty$ in this equation, we have

$$x(\infty) = \lim_{t \to \infty} (1 - e^{-at}) = 1$$

As a matter of course, the two results agree.

***Summary.*** In this section we have presented important properties and theorems of the $z$ transform that will prove to be useful in solving many $z$ transform problems. For the purpose of convenient reference, these important properties and theorems are summarized in Table 2–2. (Many of the theorems presented in this table are discussed in this section. Those not discussed here but included in the table are derived or proved in Appendix B.)

## *2–5 THE INVERSE z TRANSFORM*

The $z$ transformation serves the same role for discrete-time control systems that the Laplace transformation serves for continuous-time control systems. For the $z$ transform to be useful, we must be familiar with methods for finding the inverse $z$ transform.

The notation for the inverse $z$ transform is $\mathcal{Z}^{-1}$. The inverse $z$ transform of $X(z)$ yields the corresponding time sequence $x(k)$.

It should be noted that only the time sequence at the sampling instants is obtained from the inverse $z$ transform. Thus, the inverse $z$ transform of $X(z)$ yields a unique $x(k)$, but does not yield a unique $x(t)$. This means that the inverse $z$ transform yields a time sequence that specifies the values of $x(t)$ only at discrete instants of time, $t = 0, T, 2T, \ldots$, and says nothing about the values of $x(t)$ at all other times. That is, many different time functions $x(t)$ can have the same $x(kT)$. See Figure 2–3.

When $X(z)$, the $z$ transform of $x(kT)$ or $x(k)$, is given, the operation that determines the corresponding $x(kT)$ or $x(k)$ is called the *inverse z transformation*. An obvious method for finding the inverse $z$ transform is to refer to a $z$ transform table. However, unless we refer to an extensive $z$ transform table, we may not be able to find the inverse $z$ transform of a complicated function of $z$. (If we use a less extensive table of $z$ transforms, it is necessary to express a complex $z$ transform as a sum of simpler $z$ transforms. Refer to the partial-fraction-expansion method presented in this section.)

Other than referring to $z$ transform tables, four methods for obtaining the inverse $z$ transform are commonly available:

TABLE 2–2 IMPORTANT PROPERTIES AND THEOREMS OF THE $z$ TRANSFORM

| | $x(t)$ or $x(k)$ | $\mathcal{Z}[x(t)]$ or $\mathcal{Z}[x(k)]$ |
|---|---|---|
| 1. | $ax(t)$ | $aX(z)$ |
| 2. | $ax_1(t) + bx_2(t)$ | $aX_1(z) + bX_2(z)$ |
| 3. | $x(t + T)$ or $x(k + 1)$ | $zX(z) - zx(0)$ |
| 4. | $x(t + 2T)$ | $z^2X(z) - z^2x(0) - zx(T)$ |
| 5. | $x(k + 2)$ | $z^2X(z) - z^2x(0) - zx(1)$ |
| 6. | $x(t + kT)$ | $z^kX(z) - z^kx(0) - z^{k-1}x(T) - \cdots - zx(kT - T)$ |
| 7. | $x(t - kT)$ | $z^{-k}X(z)$ |
| 8. | $x(n + k)$ | $z^kX(z) - z^kx(0) - z^{k-1}x(1) - \cdots - zx(k - 1)$ |
| 9. | $x(n - k)$ | $z^{-k}X(z)$ |
| 10. | $tx(t)$ | $-Tz\frac{d}{dz}X(z)$ |
| 11. | $kx(k)$ | $-z\frac{d}{dz}X(z)$ |
| 12. | $e^{-at}x(t)$ | $X(ze^{aT})$ |
| 13. | $e^{-ak}x(k)$ | $X(ze^{a})$ |
| 14. | $a^kx(k)$ | $X\left(\frac{z}{a}\right)$ |
| 15. | $ka^kx(k)$ | $-z\frac{d}{dz}X\left(\frac{z}{a}\right)$ |
| 16. | $x(0)$ | $\lim_{z\to\infty} X(z)$ if the limit exists |
| 17. | $x(\infty)$ | $\lim_{z\to 1} [(1 - z^{-1})X(z)]$ if $(1 - z^{-1})X(z)$ is analytic on and outside the unit circle |
| 18. | $\nabla x(k) = x(k) - x(k - 1)$ | $(1 - z^{-1})X(z)$ |
| 19. | $\Delta x(k) = x(k + 1) - x(k)$ | $(z - 1)X(z) - zx(0)$ |
| 20. | $\sum_{k=0}^{n} x(k)$ | $\frac{1}{1 - z^{-1}}X(z)$ |
| 21. | $\frac{\partial}{\partial a}x(t, a)$ | $\frac{\partial}{\partial a}X(z, a)$ |
| 22. | $k^mx(k)$ | $\left(-z\frac{d}{dz}\right)^m X(z)$ |
| 23. | $\sum_{k=0}^{n} x(kT)y(nT - kT)$ | $X(z)Y(z)$ |
| 24. | $\sum_{k=0}^{\infty} x(k)$ | $X(1)$ |

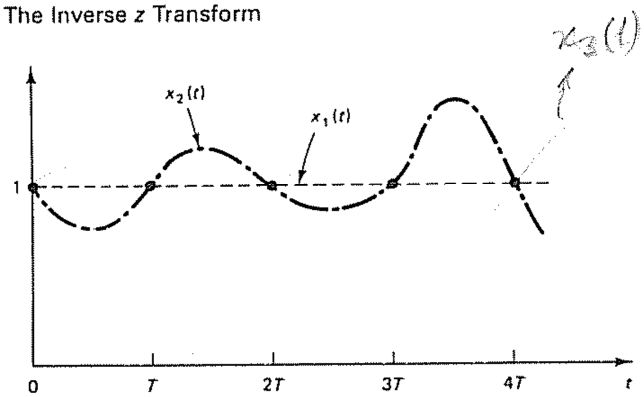


**Figure 2–3** Two different continuous-time functions, $x_1(t)$ and $x_2(t)$, that have the same values at $t = 0, T, 2T, \ldots$.

1. Direct division method
2. Computational method
3. Partial-fraction-expansion method
4. Inversion integral method

In obtaining the inverse $z$ transform, we assume, as usual, that the time sequence $x(kT)$ or $x(k)$ is zero for $k < 0$.

Before we present the four methods, however, a few comments on poles and zeros of the pulse transfer function are in order.

***Poles and Zeros in the z Plane.*** In engineering applications of the $z$ transform method, $X(z)$ may have the form

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^n + a_1 z^{n-1} + \cdots + a_n} \qquad (m \leq n) \tag{2–17}$$

or

$$X(z) = \frac{b_0(z - z_1)(z - z_2)\cdots(z - z_m)}{(z - p_1)(z - p_2)\cdots(z - p_n)}$$

where the $p_i$'s $(i = 1, 2, \ldots, n)$ are the poles of $X(z)$ and the $z_j$'s $(j = 1, 2, \ldots, m)$ the zeros of $X(z)$.

The locations of the poles and zeros of $X(z)$ determine the characteristics of $x(k)$, the sequence of values or numbers. As in the case of the $s$ plane analysis of linear continuous-time control systems, we often use a graphical display in the $z$ plane of the locations of the poles and zeros of $X(z)$.

Note that in control engineering and signal processing $X(z)$ is frequently expressed as a ratio of polynomials in $z^{-1}$, as follows:

multiply (2-17) by $\frac{z^{-n}}{z^{-n}}$ $\Rightarrow$

$$X(z) = \frac{b_0 z^{-(n-m)} + b_1 z^{-(n-m+1)} + \cdots + b_m z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}} \tag{2–18}$$

where $z^{-1}$ is interpreted as the unit delay operator. In this chapter, where the basic properties and theorems of the $z$ transform method are presented, $X(z)$ may be expressed in terms of powers of $z$, as given by Equation (2–17), or in terms of powers of $z^{-1}$, as given by Equation (2–18), depending on the circumstances.

In finding the poles and zeros of $X(z)$, it is convenient to express $X(z)$ as a ratio of polynomials in $z$. For example,

$$X(z) = \frac{z^2 + 0.5z}{z^2 + 3z + 2} = \frac{z(z + 0.5)}{(z + 1)(z + 2)}$$

Clearly, $X(z)$ has poles at $z = -1$ and $z = -2$ and zeros at $z = 0$ and $z = -0.5$. If $X(z)$ is written as a ratio of polynomials in $z^{-1}$, however, the preceding $X(z)$ can be written as

$$X(z) = \frac{1 + 0.5z^{-1}}{1 + 3z^{-1} + 2z^{-2}} = \frac{1 + 0.5z^{-1}}{(1 + z^{-1})(1 + 2z^{-1})}$$

Although poles at $z = -1$ and $z = -2$ and a zero at $z = -0.5$ are clearly seen from the expression, a zero at $z = 0$ is not explicitly shown, and so the beginner may fail to see the existence of a zero at $z = 0$. Therefore, in dealing with the poles and zeros of $X(z)$, it is preferable to express $X(z)$ as a ratio of polynomials in $z$, rather than polynomials in $z^{-1}$. In addition, in obtaining the inverse $z$ transform by use of the inversion integral method, it is desirable to express $X(z)$ as a ratio of polynomials in $z$, rather than polynomials in $z^{-1}$, to avoid any possible errors in determining the number of poles at the origin of function $X(z)z^{k-1}$.

A. ***Direct Division Method.*** In the direct division method we obtain the inverse $z$ transform by expanding $X(z)$ into an infinite power series in $z^{-1}$. This method is useful when it is difficult to obtain the closed-form expression for the inverse $z$ transform or it is desired to find only the first several terms of $x(k)$.

The direct division method stems from the fact that if $X(z)$ is expanded into a power series in $z^{-1}$, that is, if

$$\begin{aligned} X(z) &= \sum_{k=0}^{\infty} x(kT)z^{-k} \\ &= x(0) + x(T)z^{-1} + x(2T)z^{-2} + \cdots + x(kT)z^{-k} + \cdots \end{aligned}$$

or

$$\begin{aligned} X(z) &= \sum_{k=0}^{\infty} x(k)z^{-k} \\ &= x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots + x(k)z^{-k} + \cdots \end{aligned}$$

then $x(kT)$ or $x(k)$ is the coefficient of the $z^{-k}$ term. Hence, the values of $x(kT)$ or $x(k)$ for $k = 0, 1, 2, \ldots$ can be determined by inspection.

If $X(z)$ is given in the form of a rational function, the expansion into an infinite power series in increasing powers of $z^{-1}$ can be accomplished by simply dividing the numerator by the denominator, where both the numerator and denominator of $X(z)$ are written in increasing powers of $z^{-1}$. If the resulting series is convergent, the coefficients of the $z^{-k}$ term in the series are the values $x(kT)$ of the time sequence or the values of $x(k)$ of the number sequence.

Although the present method gives the values of $x(0), x(T), x(2T), \ldots$ or the values of $x(0), x(1), x(2), \ldots$ in a sequential manner, it is usually difficult to obtain an expression for the general term from a set of values of $x(kT)$ or $x(k)$.

**Example 2–10**

Find $x(k)$ for $k = 0, 1, 2, 3, 4$ when $X(z)$ is given by

$$X(z) = \frac{10z + 5}{(z - 1)(z - 0.2)} \qquad = \frac{10z + 5}{z^2 - 1.2z + 0.2}$$

First, rewrite $X(z)$ as a ratio of polynomials in $z^{-1}$, as follows:

$$X(z) = \frac{10z^{-1} + 5z^{-2}}{1 - 1.2z^{-1} + 0.2z^{-2}}$$

Dividing the numerator by the denominator, we have

$$\begin{array}{r l}
 & 10z^{-1} + 17z^{-2} + 18.4z^{-3} + 18.68z^{-4} + \cdots \\
1 - 1.2z^{-1} + 0.2z^{-2} \,\big) & 10z^{-1} + 5z^{-2} \\
 & \underline{10z^{-1} - 12z^{-2} + 2z^{-3}} \\
 & \quad 17z^{-2} - 2z^{-3} \\
 & \quad \underline{17z^{-2} - 20.4z^{-3} + 3.4z^{-4}} \\
 & \qquad 18.4z^{-3} - 3.4z^{-4} \\
 & \qquad \underline{18.4z^{-3} - 22.08z^{-4} + 3.68z^{-5}} \\
 & \qquad\quad 18.68z^{-4} - 3.68^{-5} \\
 & \qquad\quad \underline{18.68z^{-4} - 22.416z^{-5} + 3.736z^{-6}}
\end{array}$$

$10z^{-1} + 5z^{-2} \;\big|\; 1 - 1.2z^{-1} + 0.2z^{-2}$

$\mp 10z^{-1} \mp 12z^{-2} \pm 2z^{-3} \;\big|\; 10z^{-1} + 17z^{-2} + 18.4z^{-3} \ldots$

$17z^{-2} - 2z^{-3}$

$\mp 17z^{-2} \mp 20.4z^{-3} \pm 3.4z^{-4}$

$18.4z^{-3} - 3.4z^{-4}$

Thus,

$$X(z) = 10z^{-1} + 17z^{-2} + 18.4z^{-3} + 18.68z^{-4} + \cdots$$

By comparing this infinite series expansion of $X(z)$ with $X(z) = \sum_{k=0}^{\infty} x(k)z^{-k}$, we obtain

$$x(0) = 0$$
$$x(1) = 10$$
$$x(2) = 17$$
$$x(3) = 18.4$$
$$x(4) = 18.68$$

As seen from this example, the direct division method may be carried out by hand calculations if only the first several terms of the sequence are desired. In general, the method does not yield a closed-form expression for $x(k)$, except in special cases.

**Example 2–11**

Find $x(k)$ when $X(z)$ is given by

$$X(z) = \frac{1}{z + 1} = \frac{z^{-1}}{1 + z^{-1}}$$

By dividing the numerator by the denominator, we obtain

$$X(z) = \frac{z^{-1}}{1 + z^{-1}} = z^{-1} - z^{-2} + z^{-3} - z^{-4} + \cdots$$

By comparing this infinite series expansion of $X(z)$ with $X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$, we obtain

There is no a closed-form expression for $x(k)$

$$x(0) = 0$$
$$x(1) = 1$$
$$x(2) = -1$$
$$x(3) = 1$$
$$x(4) = -1$$
$$\vdots$$

This is an alternating signal of 1 and $-1$, which starts from $k = 1$. Figure 2–4 shows a plot of this signal.

**Example 2–12**

Obtain the inverse $z$ transform of

$$X(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

The transform $X(z)$ is already in the form of a power series in $z^{-1}$. Since $X(z)$ has a finite number of terms, it corresponds to a signal of finite length. By inspection, we find

$$x(0) = 1$$
$$x(1) = 2$$
$$x(2) = 3$$
$$x(3) = 4$$

All other $x(k)$ values are zero.

2. *Computational Method.* In what follows, we present two computational approaches to obtain the inverse $z$ transform.

1. MATLAB approach
2. Difference equation approach

Consider a system $G(z)$ defined by

$$G(z) = \frac{0.4673z^{-1} - 0.3393z^{-2}}{1 - 1.5327z^{-1} + 0.6607z^{-2}} \tag{2–19}$$

In finding the inverse $z$ transform, we utilize the Kronecker delta function $\delta_0(kT)$, where

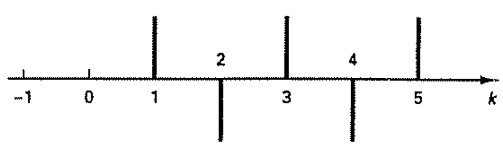


**Figure 2–4** Alternating signal of 1 and $-1$ starting from $k = 1$.

$$\delta_0(kT) = 1, \qquad \text{for } k = 0$$
$$= 0, \qquad \text{for } k \neq 0$$

Assume that $x(k)$, the input to the system $G(z)$, is the Kronecker delta input, or

$$x(k) = 1, \qquad \text{for } k = 0$$
$$= 0, \qquad \text{for } k \neq 0$$

The $z$ transform of the Kronecker delta input is

$$X(z) = 1$$

Using the Kronecker delta input, Equation (2-19) can be rewritten as

$$G(z) = \frac{Y(z)}{X(z)} = \frac{0.4673z^{-1} - 0.3393z^{-2}}{1 - 1.5327z^{-1} + 0.6607z^{-2}}$$
$$= \frac{0.4673z - 0.3393}{z^2 - 1.5327z + 0.6607} \qquad (2\text{-}20)$$

*MATLAB Approach.* MATLAB can be used for finding the inverse $z$ transform. Referring to Equation (2-20), the input $X(z)$ is the $z$ transform of the Kronecker delta input. In MATLAB the Kronecker delta input is given by

```
x = [1   zeros(1,N)]
```

where N corresponds to the end of the discrete-time duration of the process considered.

Since the $z$ transform of the Kronecker delta input $X(z)$ is equal to unity, the response of the system to this input is

$$Y(z) = G(z) = \frac{0.4673z^{-1} - 0.3393z^{-2}}{1 - 1.5327z^{-1} + 0.6607z^{-2}} = \frac{0.4673z - 0.3393}{z^2 - 1.5327z + 0.6607}$$

Hence the inverse $z$ transform of $G(z)$ is given by $y(0), y(1), y(2), \ldots$. Let us obtain $y(k)$ up to $k = 40$.

To obtain the inverse $z$ transform of $G(z)$ with MATLAB, we proceed as follows: Enter the numerator and denominator as follows:

```
num = [0    0.4673   -0.3393]
den = [1   -1.5327    0.6607]
```

Enter the Kronecker delta input.

$G(z) = \frac{Y(z)}{X(z)}$

```
x = [1   zeros(1,40)]
```

Then enter the command

$Y(z) = G(z)\underbrace{X(z)}_{1}$

```
y = filter(num,den,x)
```

to obtain the response $y(k)$ from $k = 0$ to $k = 40$.

$X(z) \rightarrow \boxed{G(z)} \rightarrow Y(z)$

Summarizing, the MATLAB program to obtain the inverse $z$ transform or the response to the Kronecker delta input is as shown in MATLAB Program 2-1.

MATLAB Program 2–1

```
% -------- Finding inverse z transform --------

% ***** Finding the inverse z transform of G(z) is the same as
% finding the response of the system Y(z)/X(z) = G(z) to the
% Kronecker delta input *****

% ***** Enter the numerator and denominator of G(z) *****

num = [0  0.4673  -0.3393];
den = [1  -1.5327  0.6607];

% ***** Enter the Kronecker delta input x and filter command
% y = filter(num,den,x) *****

x = [1  zeros(1,40)];
y = filter(num,den,x)
```

If this program is executed, the screen will show the output $y(k)$ from $k = 0$ to 40 as follows:

```
y =
  Columns 1 through 7
        0    0.4673   0.3769   0.2690   0.1632   0.0725   0.0032
  Columns 8 through 14
  -0.0429  -0.0679  -0.0758  -0.0712  -0.0591  -0.0436  -0.0277
  Columns 15 through 21
  -0.0137  -0.0027   0.0050   0.0094   0.0111   0.0108   0.0092
  Columns 22 through 28
   0.0070   0.0046   0.0025   0.0007  -0.0005  -0.0013  -0.0016
  Columns 29 through 35
  -0.0016  -0.0014  -0.0011  -0.0008  -0.0004  -0.0002   0.0000
  Columns 36 through 41
   0.0002   0.0002   0.0002   0.0002   0.0002   0.0001
```

(Note that MATLAB computations begin from column 1 and end at column 41, rather than from column 0 to column 40.) These values give the inverse $z$ transform of $G(z)$. That is,

$$y(0) = 0$$

$$y(1) = 0.4673$$

$$y(2) = 0.3769$$

$$y(3) = 0.2690$$

$$\vdots$$

$$y(40) = 0.0001$$

To plot the values of the inverse $z$ transform of $G(z)$, follow the procedure given in the following.

***Plotting Response to the Kronecker Delta Input.*** Consider the system given by Equation (2–20). A possible MATLAB program to obtain the response of this system to the Kronecker delta input is shown in MATLAB Program 2–2. The corresponding plot is shown in Figure 2–5.

MATLAB Program 2–2

```
% -------- Response to Kronecker delta input --------

num = [0  0.4673  -0.3393];
den = [1  -1.5327  0.6607];
x = [1  zeros(1,40)];
v = [0  40  -1  1];
axis(v);
k = 0:40;
y = filter(num,den,x);
plot(k,y,'o')
grid
title('Response to Kronecker Delta Input')
xlabel('k')
ylabel('y(k)')
```

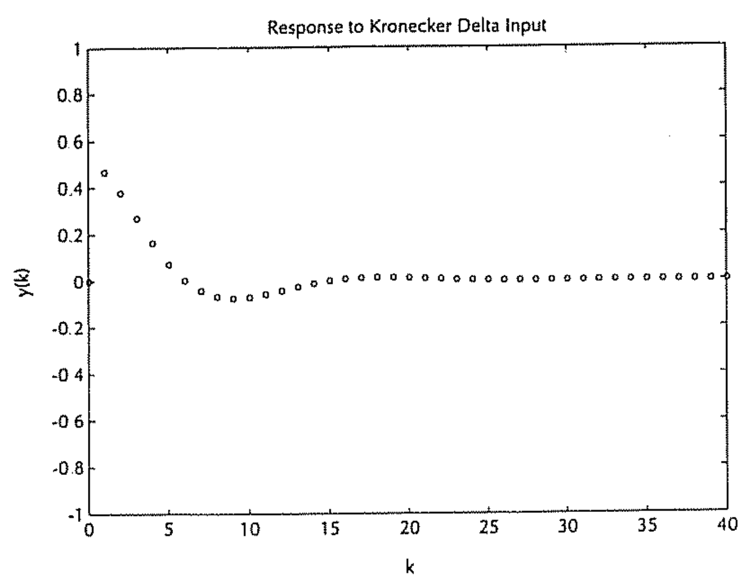


**Figure 2–5** Response of the system defined by Equation (2–20) to the Kronecker delta input.

If we wish to connect consecutive points (open circles, o) by straight lines, we need to modify the plot command from plot(k,y,o) to plot(k,y,'o',k,y,'–').

*Difference Equation Approach.* Noting that Equation (2–20) can be written as

$$(z^2 - 1.5327z + 0.6607)Y(z) = (0.4673z - 0.3393)X(z)$$

we can convert this equation into the difference equation as follows:

$$y(k+2) - 1.5327y(k+1) + 0.6607y(k) = 0.4673x(k+1) - 0.3393x(k) \qquad (2\text{–}21)$$

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where $x(0) = 1$ and $x(k) = 0$ for $k \neq 0$, and $y(k) = 0$ for $k < 0$. [$x(k)$ is the Kronecker delta input.]

$=1$ ; $)=0,\ k\neq 0$ ; $=0\ k<0$

The initial data $y(0)$ and $y(1)$ can be determined as follows: By substituting $k = -2$ into Equation (2–21), we find

$$y(0) - 1.5327\underbrace{y(-1)}_{0} + 0.6607\underbrace{y(-2)}_{0} = 0.4673\underbrace{x(-1)}_{0} - 0.3393\underbrace{x(-2)}_{0}$$

from which we get

$$y(0) = 0$$

Next, by substituting $k = -1$ into Equation (2–21), we obtain

$$y(1) - 1.5327\underbrace{y(0)}_{0} + 0.6607\underbrace{y(-1)}_{0} = 0.4673\underbrace{x(0)}_{1} - 0.3393\underbrace{x(-1)}_{0}$$

from which we get

$$y(1) = 0.4673$$

Finding the inverse $z$ transform of $Y(z)$ now becomes a matter of solving the following difference equation for $y(k)$:

$$y(k+2) - 1.5327y(k+1) + 0.6607y(k) = 0.4673x(k+1) - 0.3393x(k) \qquad (2\text{–}22) = 2\text{-}21$$

with the initial data $y(0) = 0$, $y(1) = 0.4673$, $x(0) = 1$, and $x(k) = 0$ for $k \neq 0$.

HW → Equation (2–22) can be solved easily by hand, or by use of BASIC, FORTRAN, or other.

3. ***Partial-Fraction-Expansion Method.*** The partial-fraction expansion method presented here, which is parallel to the partial-fraction-expansion method used in Laplace transformation, is widely used in routine problems involving $z$ transforms. The method requires that all terms in the partial fraction expansion be easily recognizable in the table of $z$ transform pairs.

To find the inverse $z$ transform, if $X(z)$ has one or more zeros at the origin ($z = 0$), then $X(z)/z$ or $X(z)$ is expanded into a sum of simple first- or second-order terms by partial fraction expansion, and a $z$ transform table is used to find the corresponding time function of each expanded term. It is noted that the only reason that we expand $X(z)/z$ into partial fractions is that each expanded term has a form that may easily be found from commonly available $z$ transform tables.

**Example 2–13**

Before we discuss the partial-fraction-expansion method, we shall review the shifting theorem. Consider the following $X(z)$:

$$X(z) = \frac{z^{-1}}{1 - az^{-1}}$$

By writing $zX(z)$ as $Y(z)$, we obtain

$$zX(z) = Y(z) = \frac{1}{1 - az^{-1}}$$

Referring to Table 2–1, the inverse $z$ transform of $Y(z)$ can be obtained as follows:

item 12

$$\mathcal{Z}^{-1}[Y(z)] = y(k) = a^k$$

Hence, the inverse $z$ transform of $X(z) = z^{-1}Y(z)$ is given by

$y(k) = a^k$

$$\mathcal{Z}^{-1}[X(z)] = x(k) = y(k - 1)$$

$y(k-1) = a^{k-1}$

Since $y(k)$ is assumed to be zero for all $k < 0$, we have

$$x(k) = \begin{cases} y(k - 1) = a^{k-1}, & k = 1, 2, 3, \ldots \\ 0, & k \le 0 \end{cases}$$

Consider $X(z)$ as given by

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_{m-1} z + b_m}{z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n}, \qquad m \le n$$

To expand $X(z)$ into partial fractions, we first factor the denominator polynomial of $X(z)$ and find the poles of $X(z)$:

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_{m-1} z + b_m}{(z - p_1)(z - p_2) \cdots (z - p_n)}$$

We then expand $X(z)/z$ into partial fractions so that each term is easily recognizable in a table of $z$ transforms. If the shifting theorem is utilized in taking inverse $z$ transforms, however, $X(z)$, instead of $X(z)/z$, may be expanded into partial fractions. The inverse $z$ transform of $X(z)$ is obtained as the sum of the inverse $z$ transforms of the partial fractions.

A commonly used procedure for the case where all the poles are of simple order and there is at least one zero at the origin (that is, $b_m = 0$) is to divide both sides of $X(z)$ by $z$ and then expand $X(z)/z$ into partial fractions. Once $X(z)/z$ is expanded, it will be of the form

$$\frac{X(z)}{z} = \frac{a_1}{z - p_1} + \frac{a_2}{z - p_2} + \cdots + \frac{a_n}{z - p_n}$$

The coefficient $a_i$ can be determined by multiplying both sides of this last equation by $z - p_i$ and setting $z = p_i$. This will result in zero for all the terms on the right-hand side except the $a_i$ term, in which the multiplicative factor $z - p_i$ has been canceled by the denominator. Hence, we have

$$a_i = \left[ (z - p_i) \frac{X(z)}{z} \right]_{z = p_i}$$

Note that such determination of $a_i$ is valid only for simple poles.

If $X(z)/z$ involves a multiple pole, for example, a double pole at $z = p_1$ and no other poles, then $X(z)/z$ will have the form

$$\frac{X(z)}{z} = \frac{c_1}{(z - p_1)^2} + \frac{c_2}{z - p_1}$$

The coefficients $c_1$ and $c_2$ are determined from

$$c_1 = \left[(z - p_1)^2 \frac{X(z)}{z}\right]_{z=p_1}$$

$$c_2 = \left\{\frac{d}{dz}\left[(z - p_1)^2 \frac{X(z)}{z}\right]\right\}_{z=p_1}$$

It is noted that if $X(z)/z$ involves a triple pole at $z = p_1$, then the partial fractions must include a term $(z + p_1)/(z - p_1)^3$. (See Problem A–2–8.)

**Example 2–14**

Given the $z$ transform

$$X(z) = \frac{(1 - e^{-aT})z}{(z - 1)(z - e^{-aT})} \Rightarrow \frac{X(z)}{z} = \frac{1 - e^{-aT}}{(z-1)(z - e^{-aT})} = \frac{a_1}{z-1} + \frac{a_2}{z - e^{-aT}}$$

$a_1 = 1$

$a_2 = -1$

where $a$ is a constant and $T$ is the sampling period, determine the inverse $z$ transform $x(kT)$ by use of the partial-fraction-expansion method.

The partial fraction expansion of $X(z)/z$ is found to be

$$\frac{X(z)}{z} = \frac{1}{z - 1} - \frac{1}{z - e^{-aT}}$$

Thus,

$$X(z) = \frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-aT}z^{-1}}$$

From Table 2–1 we find

$$\mathcal{Z}^{-1}\left[\frac{1}{1 - z^{-1}}\right] = 1$$

$$\mathcal{Z}^{-1}\left[\frac{1}{1 - e^{-aT}z^{-1}}\right] = e^{-akT}$$

Hence, the inverse $z$ transform of $X(z)$ is

$$x(kT) = 1 - e^{-akT}, \qquad k = 0, 1, 2, \ldots$$

**Example 2–15**

Let us obtain the inverse $z$ transform of

HW

$$X(z) = \frac{z^2 + z + 2}{(z - 1)(z^2 - z + 1)}$$

by use of the partial-fraction-expansion method.

We may expand $X(z)$ into partial fractions as follows:

$$X(z) = \frac{4}{z - 1} + \frac{-3z + 2}{z^2 - z + 1} = \frac{4z^{-1}}{1 - z^{-1}} + \frac{-3z^{-1} + 2z^{-2}}{1 - z^{-1} + z^{-2}}$$

Noting that the two poles involved in the quadratic term of this last equation are complex conjugates, we rewrite $X(z)$ as follows:

$$X(z) = \frac{4z^{-1}}{1 - z^{-1}} - 3\left(\frac{z^{-1} - 0.5z^{-2}}{1 - z^{-1} + z^{-2}}\right) + \frac{0.5z^{-2}}{1 - z^{-1} + z^{-2}}$$

$$= 4z^{-1}\frac{1}{1 - z^{-1}} - 3z^{-1}\frac{1 - 0.5z^{-1}}{1 - z^{-1} + z^{-2}} + z^{-1}\frac{0.5z^{-1}}{1 - z^{-1} + z^{-2}}$$

Since

$$\mathcal{Z}[e^{-akT}\cos\omega kT] = \frac{1 - e^{-aT}z^{-1}\cos\omega T}{1 - 2e^{-aT}z^{-1}\cos\omega T + e^{-2aT}z^{-2}}$$

$$\mathcal{Z}[e^{-akT}\sin\omega kT] = \frac{e^{-aT}z^{-1}\sin\omega T}{1 - 2e^{-aT}z^{-1}\cos\omega T + e^{-2aT}z^{-2}}$$

by identifying $e^{-2aT} = 1$ and $\cos\omega T = \frac{1}{2}$ in this case, we have $\omega T = \pi/3$ and $\sin\omega T = \sqrt{3}/2$. Hence, we obtain

$e^{-aT} = e^{-2aT} = 1$

$$\mathcal{Z}^{-1}\left[\frac{1 - 0.5z^{-1}}{1 - z^{-1} + z^{-2}}\right] = 1^k \cos\frac{k\pi}{3}$$

$e^{-akT} = 1^k$

and

$$\mathcal{Z}^{-1}\left[\frac{0.5z^{-1}}{1 - z^{-1} + z^{-2}}\right] = \mathcal{Z}^{-1}\left[\frac{1}{\sqrt{3}}\frac{(\sqrt{3}/2)z^{-1}}{1 - z^{-1} + z^{-2}}\right] = \frac{1}{\sqrt{3}}1^k \sin\frac{k\pi}{3}$$

Thus, we have

$$x(k) = 4(1^{k-1}) - 3(1^{k-1})\cos\frac{(k-1)\pi}{3} + \frac{1}{\sqrt{3}}(1^{k-1})\sin\frac{(k-1)\pi}{3}$$

$u(k-1)$

Rewriting, we have

$$x(k) = \begin{cases} 4 - 3\cos\dfrac{(k-1)\pi}{3} + \dfrac{1}{\sqrt{3}}\sin\dfrac{(k-1)\pi}{3}, & k = 1, 2, 3, \ldots \\ 0, & k \le 0 \end{cases}$$

The first several values of $x(k)$ are given by

$$x(0) = 0$$
$$x(1) = 1$$
$$x(2) = 3$$
$$x(3) = 6$$
$$x(4) = 7$$
$$x(5) = 5$$
$$\vdots$$

Note that the inverse $z$ transform of $X(z)$ can also be obtained as follows:

$$X(z) = 4z^{-1}\frac{1}{1 - z^{-1}} - 3\left(\frac{z^{-1}}{1 - z^{-1} + z^{-2}}\right) + 2z^{-1}\frac{z^{-1}}{1 - z^{-1} + z^{-2}}$$

Since

$$\mathcal{Z}^{-1}\left[\frac{z^{-1}}{1 - z^{-1}}\right] = \begin{cases} 1, & k = 1, 2, 3, \ldots \\ 0, & k \le 0 \end{cases}$$

and

$$\mathcal{Z}^{-1}\left[\frac{z^{-1}}{1 - z^{-1} + z^{-2}}\right] = \frac{2}{\sqrt{3}}(1^k)\sin\frac{k\pi}{3}$$

we have

$$x(k) = \begin{cases} 4 - 2\sqrt{3}\sin\dfrac{k\pi}{3} + \dfrac{4}{\sqrt{3}}\sin\dfrac{(k-1)\pi}{3}, & k = 1, 2, 3, \ldots \\ 0, & k \le 0 \end{cases}$$

Although this solution may look different from the one obtained earlier, both solutions are correct and yield the same values for $x(k)$.

4. *Inversion Integral Method.* This is a useful technique for obtaining the inverse $z$ transform. The inversion integral for the $z$ transform $X(z)$ is given by

$$\mathcal{Z}^{-1}[X(z)] = x(kT) = x(k) = \frac{1}{2\pi j}\oint_C X(z)z^{k-1}\,dz \qquad (2\text{–}23)$$

where $C$ is a circle with its center at the origin of the $z$ plane such that all poles of $X(z)z^{k-1}$ are inside it. [For the derivation of Equation (2–23), see Appendix B.] → B-3 p586

The equation for giving the inverse $z$ transform in terms of residues can be derived by using theory of complex variables. It can be obtained as follows:

$$\begin{aligned} x(kT) = x(k) &= K_1 + K_2 + \cdots + K_m \\ &= \sum_{i=1}^{m}[\text{residue of } X(z)z^{k-1} \text{ at pole } z = z_i \text{ of } X(z)z^{k-1}] \end{aligned} \qquad (2\text{–}24)$$

where $K_1, K_2, \ldots, K_m$ denote the residues of $X(z)z^{k-1}$ at poles $z_1, z_2, \ldots, z_m$, respectively. (For the derivation of this equation, see Appendix B.) In evaluating residues, note that if the denominator of $X(z)z^{k-1}$ contains a simple pole $z = z_i$ then the corresponding residue $K$ is given by

$$K = \lim_{z \to z_i}[(z - z_i)X(z)z^{k-1}] \qquad (2\text{–}25)$$

If $X(z)z^{k-1}$ contains a multiple pole $z_j$ of order $q$, then the residue $K$ is given by

$$K = \frac{1}{(q-1)!}\lim_{z \to z_j}\frac{d^{q-1}}{dz^{q-1}}[(z - z_j)^q X(z)z^{k-1}] \qquad (2\text{–}26)$$

Note that the values of $k$ in Equations (2–24), (2–25), and (2–26) are nonnegative integer values.

If $X(z)$ has a zero of order $r$ at the origin, then $X(z)z^{k-1}$ in Equation (2–24) will involve a zero of order $r + k - 1$ at the origin. If $r \ge 1$, then $r + k - 1 \ge 0$ for $k \ge 0$, and there is no pole at $z = 0$ in $X(z)z^{k-1}$. However, if $r \le 0$, then there will be a pole at $z = 0$ for one or more nonnegative values of $k$. In such a case, separate inversion of Equation (2–24) is necessary for each such value of $k$. (See Problem A–2–9.)

It should be noted that the inversion integral method, when evaluated by residues, is a very simple technique for obtaining the inverse $z$ transform, provided that $X(z)z^{k-1}$ has no poles at the origin, $z = 0$. If, however, $X(z)z^{k-1}$ has a simple pole or a multiple pole at $z = 0$, then calculations may become cumbersome and the partial-fraction-expansion method may prove to be simpler to apply. On the other hand, in certain problems the partial-fraction-expansion approach may become laborious. Then, the inversion integral method proves to be very convenient.

**Example 2-16**

Obtain $x(kT)$ by using the inversion integral method when $X(z)$ is given by

$$X(z) = \frac{z(1 - e^{-aT})}{(z - 1)(z - e^{-aT})}$$

Note that

$$X(z)z^{k-1} = \frac{(1 - e^{-aT})z^k}{(z - 1)(z - e^{-aT})}$$

For $k = 0, 1, 2, \ldots, X(z)z^{k-1}$ has two simple poles, $z = z_1 = 1$ and $z = z_2 = e^{-aT}$. Hence, from Equation (2-24), we have

$$\begin{aligned} x(k) &= \sum_{i=1}^{2}\left[\text{residue of } \frac{(1 - e^{-aT})z^k}{(z - 1)(z - e^{-aT})} \text{ at pole } z = z_i\right] \\ &= K_1 + K_2 \end{aligned}$$

where

$$\begin{aligned} K_1 &= [\text{residue at simple pole } z = 1] \\ &= \lim_{z \to 1}\left[(z - 1)\frac{(1 - e^{-aT})z^k}{(z - 1)(z - e^{-aT})}\right] = 1 \\ K_2 &= [\text{residue at simple pole } z = e^{-aT}] \\ &= \lim_{z \to e^{-aT}}\left[(z - e^{-aT})\frac{(1 - e^{-aT})z^k}{(z - 1)(z - e^{-aT})}\right] = -e^{-akT} \end{aligned}$$

Hence,

$$x(kT) = K_1 + K_2 = 1 - e^{-akT}, \qquad k = 0, 1, 2, \ldots$$

**Example 2-17**

Obtain the inverse z transform of

$$X(z) = \frac{z^2}{(z - 1)^2(z - e^{-aT})}$$

by using the inversion integral method.

Notice that

$$X(z)z^{k-1} = \frac{z^{k+1}}{(z - 1)^2(z - e^{-aT})}$$

For $k = 0, 1, 2, \ldots, X(z)z^{k-1}$ has a simple pole at $z = z_1 = e^{-aT}$ and a double pole at $z = z_2 = 1$. Hence, from Equation (2-24), we obtain

$$\begin{aligned} x(k) &= \sum_{i=1}^{2}\left[\text{residue of } \frac{z^{k+1}}{(z - 1)^2(z - e^{-aT})} \text{ at pole } z = z_i\right] \\ &= K_1 + K_2 \end{aligned}$$

where

$$
\begin{aligned}
K_1 &= [\text{residue at simple pole } z = e^{-aT}] \\
&= \lim_{z \to e^{-aT}} \left[ (z - e^{-aT}) \frac{z^{k+1}}{(z-1)^2(z - e^{-aT})} \right] = \frac{e^{-a(k+1)T}}{(1 - e^{-aT})^2} \\
K_2 &= [\text{residue at double pole } z = 1] \\
&= \frac{1}{(2-1)!} \lim_{z \to 1} \frac{d}{dz} \left[ (z-1)^2 \frac{z^{k+1}}{(z-1)^2(z - e^{-aT})} \right] \\
&= \lim_{z \to 1} \frac{d}{dz} \left( \frac{z^{k+1}}{z - e^{-aT}} \right) \\
&= \lim_{z \to 1} \frac{(k+1)z^k(z - e^{-aT}) - z^{k+1}}{(z - e^{-aT})^2} \\
&= \frac{k}{1 - e^{-aT}} - \frac{e^{-aT}}{(1 - e^{-aT})^2}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
x(kT) = K_1 + K_2 &= \frac{e^{-aT} e^{-akT}}{(1 - e^{-aT})^2} + \frac{k}{1 - e^{-aT}} - \frac{e^{-aT}}{(1 - e^{-aT})^2} \\
&= \frac{kT}{T(1 - e^{-aT})} - \frac{e^{-aT}(1 - e^{-akT})}{(1 - e^{-aT})^2}, \qquad k = 0, 1, 2, \ldots
\end{aligned}
$$

## 2–6 z TRANSFORM METHOD FOR SOLVING DIFFERENCE EQUATIONS

Difference equations can be solved easily by use of a digital computer, provided the numerical values of all coefficients and parameters are given. However, closed-form expressions for $x(k)$ cannot be obtained from the computer solution, except for very special cases. The usefulness of the $z$ transform method is that it enables us to obtain the closed-form expression for $x(k)$.

Consider the linear time-invariant discrete-time system characterized by the following linear difference equation:

$$
\begin{aligned}
&x(k) + a_1 x(k-1) + \cdots + a_n x(k-n) \\
&\qquad = b_0 u(k) + b_1 u(k-1) + \cdots + b_n u(k-n) \qquad (2\text{–}27)
\end{aligned}
$$

where $u(k)$ and $x(k)$ are the system's input and output, respectively, at the $k$th iteration. In describing such a difference equation in the $z$ plane, we take the $z$ transform of each term in the equation.

Let us define

$$\mathcal{Z}[x(k)] = X(z)$$

Then $x(k+1), x(k+2), x(k+3), \ldots$ and $x(k-1), x(k-2), x(k-3), \ldots$ can be expressed in terms of $X(z)$ and the initial conditions. Their exact $z$ transforms were derived in Section 2–4 and are summarized in Table 2–3 for convenient reference.

Next we present two example problems for solving difference equations by the $z$ transform method.

$$\mathcal{Z}[x(k-n)] = z^{-n}X(z)$$
$$\mathcal{Z}[x(k+n)] = z^{n}\{X(z) - \sum_{k=0}^{n-1} x(k)z^{-k}\}$$



TABLE 2-3 $z$ TRANSFORMS OF $x(k+m)$ AND $x(k-m)$

| Discrete function | $z$ Transform |
|---|---|
| $x(k+4)$ | $z^4X(z) - z^4x(0) - z^3x(1) - z^2x(2) - zx(3)$ |
| $x(k+3)$ | $z^3X(z) - z^3x(0) - z^2x(1) - zx(2)$ |
| $x(k+2)$ | $z^2X(z) - z^2x(0) - zx(1)$ |
| $x(k+1)$ | $zX(z) - zx(0)$ |
| $x(k)$ | $X(z)$ |
| $x(k-1)$ | $z^{-1}X(z)$ |
| $x(k-2)$ | $z^{-2}X(z)$ |
| $x(k-3)$ | $z^{-3}X(z)$ |
| $x(k-4)$ | $z^{-4}X(z)$ |

**Example 2-18**

Solve the following difference equation by use of the $z$ transform method:

$$x(k+2) + 3x(k+1) + 2x(k) = 0, \qquad x(0) = 0, \qquad x(1) = 1$$

First note that the $z$ transforms of $x(k+2)$, $x(k+1)$, and $x(k)$ are given, respectively, by

$$\mathcal{Z}[x(k+2)] = z^2X(z) - z^2x(0) - zx(1)$$
$$\mathcal{Z}[x(k+1)] = zX(z) - zx(0)$$
$$\mathcal{Z}[x(k)] = X(z)$$

Taking the $z$ transforms of both sides of the given difference equation, we obtain

$$z^2X(z) - z^2x(0) - zx(1) + 3zX(z) - 3zx(0) + 2X(z) = 0$$

Substituting the initial data and simplifying gives

$$X(z) = \frac{z}{z^2 + 3z + 2} = \frac{z}{(z+1)(z+2)} = \frac{z}{z+1} - \frac{z}{z+2}$$
$$= \frac{1}{1+z^{-1}} - \frac{1}{1+2z^{-1}}$$

Noting that

$$\mathcal{Z}^{-1}\left[\frac{1}{1+z^{-1}}\right] = (-1)^k, \qquad \mathcal{Z}^{-1}\left[\frac{1}{1+2z^{-1}}\right] = (-2)^k$$

(see Table 2-1 Item 18) p. 29-30

we have

$$x(k) = (-1)^k - (-2)^k, \qquad k = 0, 1, 2, \ldots$$

**Example 2-19**

Obtain the solution of the following difference equation in terms of $x(0)$ and $x(1)$:

$$x(k+2) + (a+b)x(k+1) + abx(k) = 0$$

where $a$ and $b$ are constants and $k = 0, 1, 2, \ldots$

The $z$ transform of this difference equation can be given by

$$[z^2X(z) - z^2x(0) - zx(1)] + (a + b)[zX(z) - zx(0)] + abX(z) = 0$$

or

$$[z^2 + (a + b)z + ab]X(z) = [z^2 + (a + b)z]x(0) + zx(1)$$

Solving this last equation for $X(z)$ gives

$$X(z) = \frac{[z^2 + (a + b)z]x(0) + zx(1)}{z^2 + (a + b)z + ab}$$

Notice that constants $a$ and $b$ are the negatives of the two roots of the characteristic equation. We shall now consider separately two cases: **(a)** $a \neq b$ and **(b)** $a = b$.

(a) For the case where $a \neq b$, expanding $X(z)/z$ into partial fractions, we obtain

$$\frac{X(z)}{z} = \frac{bx(0) + x(1)}{b - a}\frac{1}{z + a} + \frac{ax(0) + x(1)}{a - b}\frac{1}{z + b}, \qquad a \neq b$$

from which we get

$$X(z) = \frac{bx(0) + x(1)}{b - a}\frac{1}{1 + az^{-1}} + \frac{ax(0) + x(1)}{a - b}\frac{1}{1 + bz^{-1}}$$

The inverse $z$ transform of $X(z)$ gives

$$x(k) = \frac{bx(0) + x(1)}{b - a}(-a)^k + \frac{ax(0) + x(1)}{a - b}(-b)^k, \qquad a \neq b$$

where $k = 0, 1, 2, \ldots$.

(b) For the case where $a = b$, the $z$ transform $X(z)$ becomes

$$\begin{aligned} X(z) &= \frac{(z^2 + 2az)x(0) + zx(1)}{z^2 + 2az + a^2} \\ &= \frac{zx(0)}{z + a} + \frac{z[ax(0) + x(1)]}{(z + a)^2} \\ &= \frac{x(0)}{1 + az^{-1}} + \frac{[ax(0) + x(1)]z^{-1}}{(1 + az^{-1})^2} \end{aligned}$$

The inverse $z$ transform of $X(z)$ gives

$$x(k) = x(0)(-a)^k + [ax(0) + x(1)]k(-a)^{k-1}, \qquad a = b$$

where $k = 0, 1, 2, \ldots$.

## 2-7 CONCLUDING COMMENTS

In this chapter the basic theory of the $z$ transform method has been presented. The $z$ transform serves the same purpose for linear time-invariant discrete-time systems as the Laplace transform provides for linear time-invariant continuous-time systems.

The computer method of analyzing data in discrete time results in difference equations. With the $z$ transform method, linear time-invariant difference equations can be transformed into algebraic equations. This facilitates the transient response analysis of the digital control system. Also, the $z$ transform method allows us to use conventional analysis and design techniques available to analog (continuous-time) control systems, such as the root-locus technique. Frequency-response analysis and design can be carried out by converting the $z$ plane into the $w$ plane. Also, the $z$-transformed characteristic equation allows us to apply a simple stability test, such as the Jury stability criterion. These subjects will be discussed in detail in Chapters 3 and 4.

## EXAMPLE PROBLEMS AND SOLUTIONS

**Problem A-2-1**

Obtain the $z$ transform of $\mathbf{G}^k$, where $\mathbf{G}$ is an $n \times n$ constant matrix.

**Solution** By definition, the $z$ transform of $\mathbf{G}^k$ is

$$\begin{aligned}\mathcal{Z}[\mathbf{G}^k] &= \sum_{k=0}^{\infty} \mathbf{G}^k z^{-k} \\ &= \mathbf{I} + \mathbf{G}z^{-1} + \mathbf{G}^2 z^{-2} + \mathbf{G}^3 z^{-3} + \cdots \\ &= (\mathbf{I} - \mathbf{G}z^{-1})^{-1} \\ &= (z\mathbf{I} - \mathbf{G})^{-1} z\end{aligned}$$

Note that $\mathbf{G}^k$ can be obtained by taking the inverse $z$ transform of $(\mathbf{I} - \mathbf{G}z^{-1})^{-1}$ or $(z\mathbf{I} - \mathbf{G})^{-1}z$. That is,

$$\mathbf{G}^k = \mathcal{Z}^{-1}[(\mathbf{I} - \mathbf{G}z^{-1})^{-1}] = \mathcal{Z}^{-1}[(z\mathbf{I} - \mathbf{G})^{-1} z]$$

**Problem A-2-2**

Obtain the $z$ transform of $k^2$.

**Solution** By definition, the $z$ transform of $k^2$ is

$$\begin{aligned}\mathcal{Z}[k^2] &= \sum_{k=0}^{\infty} k^2 z^{-k} = z^{-1} + 4z^{-2} + 9z^{-3} + 16z^{-4} + \cdots \\ &= z^{-1}(1 + z^{-1})(1 + 3z^{-1} + 6z^{-2} + 10z^{-3} + 15z^{-4} + \cdots) \\ &= \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}\end{aligned}$$

Here we have used the closed-form expression $(1 - z^{-1})^{-3}$ for the infinite series involved in the problem. (See Appendix B.)

**Problem A-2-3**

Obtain the $z$ transform of $ka^{k-1}$ by two methods.

**Solution**

*Method 1.* By definition, the $z$ transform of $ka^{k-1}$ is given by

$$\begin{aligned}\mathcal{Z}[ka^{k-1}] &= \sum_{k=0}^{\infty} ka^{k-1} z^{-k} \\ &= z^{-1} + 2az^{-2} + 3a^2 z^{-3} + 4a^3 z^{-4} + \cdots \\ &= z^{-1}(1 + 2az^{-1} + 3a^2 z^{-2} + 4a^3 z^{-3} + \cdots) \\ &= \frac{z^{-1}}{(1 - az^{-1})^2}\end{aligned}$$

*Method 2.* The summation expression for the $z$ transform of $ka^{k-1}$ can also be written as follows:

$$\mathcal{Z}[ka^{k-1}] = \sum_{k=0}^{\infty} ka^{k-1} z^{-k} = a^{-1} \sum_{k=0}^{\infty} ka^k z^{-k} = \frac{1}{a} \sum_{k=0}^{\infty} k\left(\frac{z}{a}\right)^{-k}$$
$$= \frac{1}{a} \frac{(z/a)^{-1}}{[1 - (z/a)^{-1}]^2} = \frac{z^{-1}}{(1 - az^{-1})^2}$$

**Problem A–2–4**

Show that

$$\mathcal{Z}\left[\sum_{h=0}^{k} x(h)\right] = \frac{1}{1 - z^{-1}} X(z)$$

$$\mathcal{Z}\left[\sum_{h=0}^{k-1} x(h)\right] = \frac{z^{-1}}{1 - z^{-1}} X(z)$$

and

$$\sum_{k=0}^{\infty} x(k) = \lim_{z \to 1} X(z) \tag{2–28}$$

Also show that

$$\mathcal{Z}\left[\sum_{h=i}^{k} x(h)\right] = \frac{1}{1 - z^{-1}}\left[X(z) - \sum_{h=0}^{i-1} x(h) z^{-h}\right] \tag{2–29}$$

where $1 \le i \le k - 1$.

**Solution** Define

$$y(k) = \sum_{h=0}^{k} x(h), \qquad k = 0, 1, 2, \ldots$$

so that

$$y(0) = x(0)$$
$$y(1) = x(0) + x(1)$$
$$y(2) = x(0) + x(1) + x(2)$$
$$\vdots$$
$$y(k) = x(0) + x(1) + x(2) + \cdots + x(k)$$

Then, clearly

$$y(k) - y(k-1) = x(k) \quad \checkmark$$

By writing the $z$ transforms of $x(k)$ and $y(k)$ as $X(z)$ and $Y(z)$, respectively, and by taking the $z$ transform of this last equation, we have

$$Y(z) - z^{-1} Y(z) = X(z)$$

Hence,

$$Y(z) = \frac{1}{1 - z^{-1}} X(z)$$

or

$$\mathcal{Z}\left[\sum_{h=0}^{k} x(h)\right] = \mathcal{Z}[y(k)] = Y(z) = \frac{1}{1 - z^{-1}} X(z)$$

and

$$\mathcal{Z}\left[\sum_{h=0}^{k-1} x(h)\right] = \mathcal{Z}[y(k-1)] = z^{-1} Y(z) = \frac{z^{-1}}{1 - z^{-1}} X(z)$$

By using the final value theorem, we find

$$\lim_{k\to\infty} y(k) = \lim_{k\to\infty}\left[\sum_{h=0}^{k} x(h)\right] = \lim_{z\to 1}\left[(1 - z^{-1})\frac{1}{1 - z^{-1}} X(z)\right]$$

or

$$\sum_{h=0}^{\infty} x(h) = \sum_{k=0}^{\infty} x(k) = \lim_{z\to 1} X(z)$$

Next, to prove Equation (2–29), first define

$$\tilde{y}(k) = \sum_{h=i}^{k} x(h) = x(i) + x(i+1) + \cdots + x(k)$$

where $1 \le i \le k - 1$. Define also

$$\tilde{X}(z) = x(i)z^{-i} + x(i+1)z^{-(i+1)} + \cdots + x(k)z^{-k} + \cdots$$

Then, noting that

$$X(z) = \mathcal{Z}[x(k)] = \sum_{k=0}^{\infty} x(k)z^{-k} = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

we obtain

$$\tilde{X}(z) = X(z) - \sum_{h=0}^{i-1} x(h)z^{-h}$$

Since

$$\tilde{y}(k) - \tilde{y}(k-1) = x(k), \qquad k = i, i+1, i+2, \ldots$$

the $z$ transform of this last equation becomes

$$\tilde{Y}(z) - z^{-1}\tilde{Y}(z) = \tilde{X}(z)$$

[Note that the $z$ transform of $x(k)$, which begins with $k = i$, is $\tilde{X}(z)$, not $X(z)$.] Thus,

$$\mathcal{Z}\left[\sum_{h=i}^{k} x(h)\right] = \tilde{Y}(z) = \frac{1}{1 - z^{-1}}\tilde{X}(z) = \frac{1}{1 - z^{-1}}\left[X(z) - \sum_{h=0}^{i-1} x(h)z^{-h}\right]$$

**Problem A–2–5**

Obtain the $z$ transform of the curve $x(t)$ shown in Figure 2–6. Assume that the sampling period $T$ is 1 sec.

**Solution** From Figure 2–6 we obtain

$$x(0) = 0$$

$$x(1) = 0.25$$

$$x(2) = 0.50$$

$$x(3) = 0.75$$

$$x(k) = 1, \qquad k = 4, 5, 6, \ldots$$

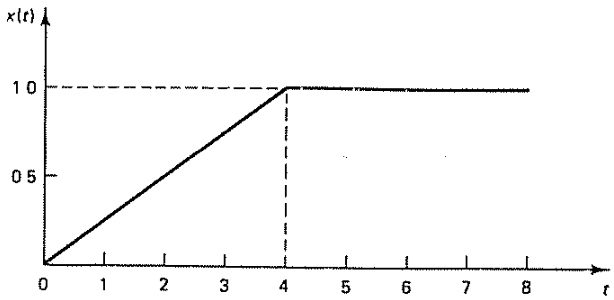


Figure 2–6 Curve $x(t)$.

Then the $z$ transform of $x(k)$ can be given by

$$\begin{aligned}X(z) &= \sum_{k=0}^{\infty} x(k)z^{-k}\\ &= 0.25z^{-1} + 0.50z^{-2} + 0.75z^{-3} + z^{-4} + z^{-5} + z^{-6} + \cdots\\ &= 0.25(z^{-1} + 2z^{-2} + 3z^{-3}) + z^{-4}\frac{1}{1 - z^{-1}}\\ &= \frac{z^{-1} + z^{-2} + z^{-3} + z^{-4}}{4(1 - z^{-1})}\\ &= \frac{1}{4}\frac{z^{-1}(1 + z^{-1} + z^{-2} + z^{-3})(1 - z^{-1})}{(1 - z^{-1})^2}\\ &= \frac{1}{4}\frac{z^{-1}(1 - z^{-4})}{(1 - z^{-1})^2}\end{aligned}$$

Notice that the curve $x(t)$ can be written as

$$x(t) = \tfrac{1}{4}t - \tfrac{1}{4}(t - 4)1(t - 4)$$

where $1(t - 4)$ is the unit-step function occurring at $t = 4$. Since the sampling period $T = 1$ sec, the $z$ transform of $x(t)$ can also be obtained as follows:

$$\begin{aligned}X(z) &= \mathcal{Z}[x(t)] = \mathcal{Z}[\tfrac{1}{4}t] - \mathcal{Z}[\tfrac{1}{4}(t - 4)1(t - 4)]\\ &= \frac{1}{4}\frac{z^{-1}}{(1 - z^{-1})^2} - \frac{1}{4}\frac{z^{-4}z^{-1}}{(1 - z^{-1})^2}\\ &= \frac{1}{4}\frac{z^{-1}(1 - z^{-4})}{(1 - z^{-1})^2}\end{aligned}$$

**Problem A–2–6**

Consider $X(z)$, where

$$X(z) = \frac{2z^3 + z}{(z - 2)^2(z - 1)}$$

Obtain the inverse $z$ transform of $X(z)$.

**Solution** We shall expand $X(z)/z$ into partial fractions as follows:

$$\frac{X(z)}{z} = \frac{2z^2 + 1}{(z - 2)^2(z - 1)} = \frac{9}{(z - 2)^2} - \frac{1}{z - 2} + \frac{3}{z - 1}$$

Then

$$X(z) = \frac{9z^{-1}}{(1 - 2z^{-1})^2} - \frac{1}{1 - 2z^{-1}} + \frac{3}{1 - z^{-1}}$$

The inverse $z$ transforms of the individual terms give

$$\mathcal{Z}^{-1}\left[\frac{z^{-1}}{(1 - 2z^{-1})^2}\right] = k(2^{k-1}), \qquad k = 0, 1, 2, \ldots$$

$$\mathcal{Z}^{-1}\left[\frac{1}{1 - 2z^{-1}}\right] = 2^k, \qquad k = 0, 1, 2, \ldots$$

$$\mathcal{Z}^{-1}\left[\frac{1}{1 - z^{-1}}\right] = 1$$

and therefore

$$x(k) = 9k(2^{k-1}) - 2^k + 3, \qquad k = 0, 1, 2, \ldots$$

**Problem A-2-7**

Obtain the inverse $z$ transform of

$$X(z) = \frac{z + 2}{(z - 2)z^2}$$

**Solution** Expanding $X(z)$ into partial fractions, we obtain

$$X(z) = \frac{1}{z - 2} - \frac{1}{z^2} - \frac{1}{z} = \frac{z^{-1}}{1 - 2z^{-1}} - z^{-2} - z^{-1}$$

[Note that in this example $X(z)$ involves a double pole at $z = 0$. Hence the partial fraction expansion must include the terms $1/(z^2)$ and $1/z$.] By referring to Table 2-1, we find the inverse $z$ transform of each term of this last equation. That is,

$$\mathcal{Z}^{-1}\left[\frac{z^{-1}}{1 - 2z^{-1}}\right] = \begin{cases} 2^{k-1}, & k = 1, 2, 3, \ldots \\ 0, & k \le 0 \end{cases}$$

$$\mathcal{Z}^{-1}[z^{-2}] = \begin{cases} 1, & k = 2 \\ 0, & k \ne 2 \end{cases}$$

$$\mathcal{Z}^{-1}[z^{-1}] = \begin{cases} 1, & k = 1 \\ 0, & k \ne 1 \end{cases}$$

Hence, the inverse $z$ transform of $X(z)$ can be given by

$$x(k) = \begin{cases} 0 - 0 - 0 = 0, & k = 0 \\ 1 - 0 - 1 = 0, & k = 1 \\ 2 - 1 - 0 = 1, & k = 2 \\ 2^{k-1} - 0 - 0 = 2^{k-1}, & k = 3, 4, 5, \ldots \end{cases}$$

Rewriting, we have

$$x(k) = \begin{cases} 0, & k = 0, 1 \\ 1, & k = 2 \\ 2^{k-1}, & k = 3, 4, 5, \ldots \end{cases}$$

To verify this result, the direct division method may be applied to this problem. Noting that

$$X(z) = \frac{z+2}{(z-2)z^2} = \frac{z^{-2} + 2z^{-3}}{1 - 2z^{-1}}$$
$$= z^{-2} + 4z^{-3} + 8z^{-4} + 16z^{-5} + 32z^{-6} + \cdots$$
$$= z^{-2} + (2^{3-1})z^{-3} + (2^{4-1})z^{-4} + (2^{5-1})z^{-5} + (2^{6-1})z^{-6} + \cdots$$

we find

$$x(k) = \begin{cases} 0, & k = 0, 1 \\ 1, & k = 2 \\ 2^{k-1}, & k = 3, 4, 5, \ldots \end{cases}$$

**Problem A–2–8**

Obtain the inverse $z$ transform of

$$X(z) = \frac{z^{-2}}{(1 - z^{-1})^3}$$

**Solution** The inverse $z$ transform of $z^{-2}/(1 - z^{-1})^3$ is not available from most $z$ transform tables. It is possible, however, to write the given $X(z)$ as a sum of $z$ transforms that are commonly available in $z$ transform tables. Since the denominator of $X(z)$ is $(1 - z^{-1})^3$ and the $z$ transform of $k^2$ is $z^{-1}(1 + z^{-1})/(1 - z^{-1})^3$, let us rewrite $X(z)$ as

$$X(z) = \frac{z^{-2}}{(1 - z^{-1})^3} = \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3} - \frac{z^{-1}}{(1 - z^{-1})^3}$$
$$= \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3} - \frac{z^{-1} - z^{-2} + z^{-2}}{(1 - z^{-1})^3}$$

or

$$\frac{z^{-2}}{(1 - z^{-1})^3} = \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3} - \frac{z^{-1}}{(1 - z^{-1})^2} - \frac{z^{-2}}{(1 - z^{-1})^3}$$

from which we obtain the following partial fraction expansion:

$$\frac{z^{-2}}{(1 - z^{-1})^3} = \frac{1}{2}\left[\frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3} - \frac{z^{-1}}{(1 - z^{-1})^2}\right]$$

The $z$ transforms of the two terms on the right-hand side of this last equation can be found from Table 2–1. Thus,

$$x(k) = \mathcal{Z}^{-1}\left[\frac{z^{-2}}{(1 - z^{-1})^3}\right] = \frac{1}{2}(k^2 - k) = \frac{1}{2}k(k - 1), \qquad k = 0, 1, 2, \ldots$$

It is noted that if the given $X(z)$ is expanded into other partial fractions then the inverse $z$ transform may not be obtained.

As an alternative approach, the inverse $z$ transform of $X(z)$ may be obtained by use of the inversion integral method. First, note that

$$X(z)z^{k-1} = \frac{z^k}{(z - 1)^3}$$

Hence, for $k = 0, 1, 2, \ldots, X(z)z^{k-1}$ has a triple pole at $z = 1$. Referring to Equation (2–24), we have

$$x(k) = \left[\text{residue of } \frac{z^k}{(z - 1)^3} \text{ at triple pole } z = 1\right]$$

$$= \frac{1}{(3-1)!} \lim_{z \to 1} \frac{d^2}{dz^2}\left[(z-1)^3 \frac{z^k}{(z-1)^3}\right]$$

$$= \frac{1}{2!} \lim_{z \to 1} \frac{d^2}{dz^2}(z^k)$$

$$= \frac{1}{2} \lim_{z \to 1} \frac{d}{dz}(kz^{k-1})$$

$$= \frac{1}{2} \lim_{z \to 1} [k(k-1)z^{k-2}]$$

$$= \frac{1}{2}k(k-1), \qquad k = 0, 1, 2, \ldots$$

**Problem A–2–9**

Using the inversion integral method, obtain the inverse $z$ transform of

$$X(z) = \frac{10}{(z-1)(z-2)}$$

**Solution** Note that

$$X(z)z^{k-1} = \frac{10z^{k-1}}{(z-1)(z-2)}$$

For $k = 0$, notice that $X(z)z^{k-1}$ becomes

$$X(z)z^{k-1} = \frac{10}{(z-1)(z-2)z}, \qquad k = 0$$

Hence, for $k = 0$, $X(z)z^{k-1}$ has three simple poles, $z = z_1 = 1$, $z = z_2 = 2$, and $z = z_3 = 0$. For $k = 1, 2, 3, \ldots$, however, $X(z)z^{k-1}$ has only two simple poles, $z = z_1 = 1$ and $z = z_2 = 2$. Therefore, we must consider $x(0)$ and $x(k)$ (where $k = 1, 2, 3, \ldots$) separately.

*For $k = 0$.* For this case, referring to Equation (2–24), we have

$$x(0) = \sum_{i=1}^{3}\left[\text{residue of } \frac{10}{(z-1)(z-2)z} \text{ at pole } z = z_i\right]$$
$$= K_1 + K_2 + K_3$$

where

$$K_1 = [\text{residue at simple pole } z = 1]$$
$$= \lim_{z \to 1}\left[(z-1)\frac{10}{(z-1)(z-2)z}\right] = -10$$

$$K_2 = [\text{residue at simple pole } z = 2]$$
$$= \lim_{z \to 2}\left[(z-2)\frac{10}{(z-1)(z-2)z}\right] = 5$$

$$K_3 = [\text{residue at simple pole } z = 0]$$
$$= \lim_{z \to 0}\left[z\frac{10}{(z-1)(z-2)z}\right] = 5$$

Hence,

$$x(0) = K_1 + K_2 + K_3 = -10 + 5 + 5 = 0$$

*For $k = 1, 2, 3, \ldots$.* For this case, Equation (2–24) becomes

$$x(k) = \sum_{i=1}^{2}\left[\text{residue of } \frac{10z^{k-1}}{(z-1)(z-2)} \text{ at pole } z = z_i\right]$$
$$= K_1 + K_2$$

where

$$K_1 = [\text{residue at simple pole } z = 1]$$
$$= \lim_{z\to 1}\left[(z-1)\frac{10z^{k-1}}{(z-1)(z-2)}\right] = -10$$
$$K_2 = [\text{residue at simple pole } z = 2]$$
$$= \lim_{z\to 2}\left[(z-2)\frac{10z^{k-1}}{(z-1)(z-2)}\right] = 10(2^{k-1})$$

Thus,

$$x(k) = K_1 + K_2 = -10 + 10(2^{k-1}) = 10(2^{k-1} - 1), \qquad k = 1, 2, 3, \ldots$$

Hence, the inverse $z$ transform of the given $X(z)$ can be written

$$x(k) = \begin{cases} 0, & k = 0 \\ 10(2^{k-1} - 1), & k = 1, 2, 3, \ldots \end{cases}$$

An alternative way to write $x(k)$ for $k \geq 0$ is

$$x(k) = 5\delta_0(k) + 10(2^{k-1} - 1), \qquad k = 0, 1, 2, \ldots$$

where $\delta_0(k)$ is the Kronecker delta function and is given by

$$\delta_0(k) = \begin{cases} 1, & \text{for } k = 0 \\ 0, & \text{for } k \neq 0 \end{cases}$$

**Problem A–2–10**

Obtain the inverse $z$ transform of

$$X(z) = \frac{z(z+2)}{(z-1)^2} \tag{2–30}$$

by use of the four methods presented in Section 2–5.

**Solution**

*Method 1: Direct division method.* We first rewrite $X(z)$ as a ratio of two polynomials in $z^{-1}$:

$$X(z) = \frac{1 + 2z^{-1}}{(1 - z^{-1})^2} = \frac{1 + 2z^{-1}}{1 - 2z^{-1} + z^{-2}}$$

Dividing the numerator by the denominator, we get

$$X(z) = 1 + 4z^{-1} + 7z^{-2} + 10z^{-3} + \cdots$$

Hence,

$$x(0) = 1$$

$$x(1) = 4$$
$$x(2) = 7$$
$$x(3) = 10$$
$$\vdots$$

*Method 2: Computational method (MATLAB approach).* $X(z)$ can be written as

$$X(z) = \frac{z^2 + 2z}{z^2 - 2z + 1}$$

Hence, the inverse $z$ transform of $X(z)$ can be obtained with MATLAB as follows: Define

```
num = [1   2   0]
den = [1  -2   1]
```

If the values of $x(k)$ for $k = 0, 1, 2, \ldots, 30$ are desired, then enter the Kronecker delta input as follows:

```
u = [1   zeros(1,30)]
```

Then enter the command

```
x = filter(num,den,u)
```

See MATLAB Program 2-3. [The screen will show the output $x(k)$ from $k = 0$ to $k = 30$.] (MATLAB computations begin from column 1 and end at column 31, rather

MATLAB Program 2-3

```
num = [1  2  0];
den = [1  -2  1];
u = [1  zeros(1,30)];
x = filter(num,den,u)

x =

 Columns 1 through 12

      1    4    7   10   13   16   19   22   25   28   31   34

 Columns 13 through 24

     37   40   43   46   49   52   55   58   61   64   67   70

 Columns 25 through 31

     73   76   79   82   85   88   91
```

than from column 0 to column 30.) The values $x(k)$ give the inverse $z$ transform of $X(z)$. That is,

$$\begin{aligned} x(0) &= 1 \\ x(1) &= 4 \\ x(2) &= 7 \\ &\vdots \\ x(30) &= 91 \end{aligned}$$

*Method 3: Partial-fraction-expansion method.* We expand $X(z)$ into the following partial fractions:

$$X(z) = \frac{z(z+2)}{(z-1)^2} = 1 + \frac{3z}{(z-1)^2} + \frac{1}{z-1} = 1 + \frac{3z^{-1}}{(1-z^{-1})^2} + \frac{z^{-1}}{1-z^{-1}}$$

Then, noting that

$$\mathcal{Z}^{-1}[1] = \begin{cases} 1, & k = 0 \\ 0, & k = 1, 2, 3, \ldots \end{cases}$$

$$\mathcal{Z}^{-1}\left[\frac{z^{-1}}{(1-z^{-1})^2}\right] = k, \qquad k = 0, 1, 2, \ldots$$

$$\mathcal{Z}^{-1}\left[\frac{z^{-1}}{1-z^{-1}}\right] = \begin{cases} 1, & k = 1, 2, 3, \ldots \\ 0, & k \leq 0 \end{cases}$$

we obtain

$$\begin{aligned} x(0) &= 1 \\ x(k) &= 3k + 1, \qquad k = 1, 2, 3, \ldots \end{aligned}$$

which can be combined into one equation as follows:

$$x(k) = 3k + 1, \qquad k = 0, 1, 2, \ldots$$

Note that if we expand $X(z)$ into the following partial fractions

$$X(z) = 1 + \frac{4}{z-1} + \frac{3}{(z-1)^2} = 1 + \frac{4z^{-1}}{1-z^{-1}} + \frac{3z^{-2}}{(1-z^{-1})^2}$$

then the inverse $z$ transform of $X(z)$ becomes

$$\begin{aligned} x(0) &= 1 \\ x(k) &= 4 + 3(k-1) = 3k + 1, \qquad k = 1, 2, 3, \ldots \end{aligned}$$

or

$$x(k) = 3k + 1, \qquad k = 0, 1, 2, \ldots$$

which is the same as the result obtained by expanding $X(z)$ into the other partial fractions. [Remember that $X(z)$ can be expanded into different partial fractions, but the final result for the inverse $z$ transform is the same.]

*Method 4: Inversion integral method.* First, note that

$$X(z)z^{k-1} = \frac{(z+2)z^k}{(z-1)^2}$$

For $k = 0, 1, 2, \ldots, X(z)z^{k-1}$ has a double pole at $z = 1$. Hence, referring to Equation (2-24), we have

$$x(k) = \left[\text{residue of } \frac{(z+2)z^k}{(z-1)^2} \text{ at double pole } z = 1\right]$$

Thus,

$$x(k) = \frac{1}{(2-1)!}\lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2\frac{(z+2)z^k}{(z-1)^2}\right]$$

$$= \lim_{z\to 1}\frac{d}{dz}[(z+2)z^k]$$

$$= 3k + 1, \qquad k = 0, 1, 2, \ldots$$

**Problem A-2-11**

Solve the following difference equation:

$$2x(k) - 2x(k-1) + x(k-2) = u(k)$$

where $x(k) = 0$ for $k < 0$ and

$$u(k) = \begin{cases} 1, & k = 0, 1, 2, \ldots \\ 0, & k < 0 \end{cases}$$

**Solution** By taking the $z$ transform of the given difference equation,

$$2X(z) - 2z^{-1}X(z) + z^{-2}X(z) = \frac{1}{1 - z^{-1}}$$

Solving this last equation for $X(z)$, we obtain

$$X(z) = \frac{1}{1 - z^{-1}}\,\frac{1}{2 - 2z^{-1} + z^{-2}} = \frac{z^3}{(z-1)(2z^2 - 2z + 1)}$$

Expanding $X(z)$ into partial fractions, we get

$$X(z) = \frac{z}{z-1} + \frac{-z^2 + z}{2z^2 - 2z + 1} = \frac{1}{1 - z^{-1}} + \frac{-1 + z^{-1}}{2 - 2z^{-1} + z^{-2}}$$

Notice that the two poles involved in the quadratic term in this last equation are complex conjugates. Hence, we rewrite $X(z)$ as follows:

$$X(z) = \frac{1}{1 - z^{-1}} - \frac{1}{2}\,\frac{1 - 0.5z^{-1}}{1 - z^{-1} + 0.5z^{-2}} + \frac{1}{2}\,\frac{0.5z^{-1}}{1 - z^{-1} + 0.5z^{-2}}$$

By referring to the formulas for the $z$ transforms of damped cosine and damped sine functions, we identify $e^{-2aT} = 0.5$ and $\cos\omega T = 1/\sqrt{2}$ for this problem. Hence, we get $\omega T = \pi/4$, $\sin\omega T = 1/\sqrt{2}$, and $e^{-aT} = 1/\sqrt{2}$. Then the inverse $z$ transform of $X(z)$ can be written as

$$x(k) = 1 - \tfrac{1}{2}e^{-akT}\cos\omega kT + \tfrac{1}{2}e^{-akT}\sin\omega kT$$

$$= 1 - \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^k\cos\frac{k\pi}{4} + \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^k\sin\frac{k\pi}{4}, \qquad k = 0, 1, 2, \ldots$$

from which we obtain

$$x(0) = 0.5$$

$$x(1) = 1$$

$$x(2) = 1.25$$
$$x(3) = 1.25$$
$$x(4) = 1.125$$
$$\vdots$$

**Problem A–2–12**

Consider the difference equation

$$x(k+2) - 1.3679x(k+1) + 0.3679x(k) = 0.3679u(k+1) + 0.2642u(k)$$

where $x(k)$ is the output and $x(k) = 0$ for $k \leq 0$ and where $u(k)$ is the input and is given by

$$u(k) = 0, \qquad k < 0$$
$$u(0) = 1$$
$$u(1) = 0.2142$$
$$u(2) = -0.2142$$
$$u(k) = 0, \qquad k = 3, 4, 5, \ldots$$

Determine the output $x(k)$.

**Solution** Taking the $z$ transform of the given difference equation, we obtain

$$[z^2X(z) - z^2x(0) - zx(1)] - 1.3679[zX(z) - zx(0)] + 0.3679X(z)$$
$$= 0.3679[zU(z) - zu(0)] + 0.2642U(z) \qquad (2\text{–}31)$$

By substituting $k = -1$ into the given difference equation, we find

$$x(1) - 1.3679x(0) + 0.3679x(-1) = 0.3679u(0) + 0.2642u(-1)$$

Since $x(0) = x(-1) = 0$ and since $u(-1) = 0$ and $u(0) = 1$, we obtain

$$x(1) = 0.3679u(0) = 0.3679$$

By substituting the initial data

$$x(0) = 0, \qquad x(1) = 0.3679, \qquad u(0) = 1$$

into Equation (2–31), we get

$$z^2X(z) - 0.3679z - 1.3679zX(z) + 0.3679X(z)$$
$$= 0.3679zU(z) - 0.3679z + 0.2642U(z)$$

Solving for $X(z)$, we find

$$X(z) = \frac{0.3679z + 0.2642}{z^2 - 1.3679z + 0.3679}U(z)$$

The $z$ transform of the input $u(k)$ is

$$U(z) = \mathcal{Z}[u(k)] = 1 + 0.2142z^{-1} - 0.2142z^{-2}$$

Hence,

$$X(z) = \frac{0.3679z + 0.2642}{z^2 - 1.3679z + 0.3679}(1 + 0.2142z^{-1} - 0.2142z^{-2})$$

$$= \frac{0.3679z^{-1} + 0.3430z^{-2} - 0.02221z^{-3} - 0.05659z^{-4}}{1 - 1.3679z^{-1} + 0.3679z^{-2}}$$

$$= 0.3679z^{-1} + 0.8463z^{-2} + z^{-3} + z^{-4} + z^{-5} + \cdots$$

Thus, the inverse $z$ transform of $X(z)$ gives

$$x(0) = 0$$

$$x(1) = 0.3679$$

$$x(2) = 0.8463$$

$$x(k) = 1, \qquad k = 3, 4, 5, \ldots$$

**Problem A–2–13**

Consider the difference equation

$$x(k + 2) = x(k + 1) + x(k)$$

where $x(0) = 0$ and $x(1) = 1$. Note that $x(2) = 1, x(3) = 2, x(4) = 3, \ldots$. The series $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ is known as the Fibonacci series. Obtain the general solution $x(k)$ in a closed form. Show that the limiting value of $x(k + 1)/x(k)$ as $k$ approaches infinity is $(1 + \sqrt{5})/2$, or approximately 1.6180.

**Solution** By taking the $z$ transform of this difference equation, we obtain

$$z^2X(z) - z^2x(0) - zx(1) = zX(z) - zx(0) + X(z)$$

Solving for $X(z)$ gives

$$X(z) = \frac{z^2x(0) + zx(1) - zx(0)}{z^2 - z - 1}$$

By substituting the initial data $x(0) = 0$ and $x(1) = 1$ into this last equation, we have

$$X(z) = \frac{z}{z^2 - z - 1}$$

$$= \frac{1}{\sqrt{5}}\left(\frac{z}{z - \dfrac{1 + \sqrt{5}}{2}} - \frac{z}{z - \dfrac{1 - \sqrt{5}}{2}}\right)$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1}{1 - \dfrac{1 + \sqrt{5}}{2}z^{-1}} - \frac{1}{1 - \dfrac{1 - \sqrt{5}}{2}z^{-1}}\right)$$

The inverse $z$ transform of $X(z)$ is

$$x(k) = \frac{1}{\sqrt{5}}\left[\left(\frac{1 + \sqrt{5}}{2}\right)^k - \left(\frac{1 - \sqrt{5}}{2}\right)^k\right], \qquad k = 0, 1, 2, \ldots$$

Note that although this last equation involves $\sqrt{5}$ the square roots in the right-hand side of this last equation cancel out, and the values of $x(k)$ for $k = 0, 1, 2, \ldots$ turn out to be positive integers.

The limiting value of $x(k+1)/x(k)$ as $k$ approaches infinity is obtained as follows:

$$\lim_{k\to\infty}\frac{x(k+1)}{x(k)} = \lim_{k\to\infty}\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}}$$

Since $|(1-\sqrt{5})/2| < 1$,

$$\lim_{k\to\infty}\left(\frac{1-\sqrt{5}}{2}\right)^{k} \to 0$$

Hence,

$$\lim_{k\to\infty}\frac{x(k+1)}{x(k)} = \lim_{k\to\infty}\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}}{\left(\frac{1+\sqrt{5}}{2}\right)^{k}} = \frac{1+\sqrt{5}}{2} = 1.6180$$

**Problem A–2–14**

Referring to Problem A–2–13, write a MATLAB program to generate the Fibonacci series. Carry out the Fibonacci series to $k = 30$.

**Solution** The $z$ transform of the difference equation

$$x(k+2) = x(k+1) + x(k)$$

is given by

$$z^2X(z) - z^2x(0) - zx(1) = zX(z) - zx(0) + X(z)$$

Solving this equation for $X(z)$ and substituting the initial data $x(0) = 0$ and $x(1) = 1$, we get

$$X(z) = \frac{z}{z^2 - z - 1}$$

The inverse $z$ transform of $X(z)$ will give the Fibonacci series.

To get the inverse $z$ transform of $X(z)$, obtain the response of this system to the Kronecker delta input. MATLAB Program 2–4 will yield the Fibonacci series.

MATLAB Program 2–4

```
% ---------- Fibonacci series ----------

% ***** The Fibonacci series can be generated as the
% response of X(z) to the Kronecker delta input, where
% X(z) = z/(z^2 - z - 1) *****

num = [0  1  0];
den = [1  -1  -1];
u = [1  zeros(1,30)];
x = filter(num,den,u)
```

The filtered output $y$ shown next gives the Fibonacci series.

```
x =

  Columns 1 through 6

         0          1          1          2          3          5

  Columns 7 through 12

         8         13         21         34         55         89

  Columns 13 through 18

       144        233        377        610        987       1597

  Columns 19 through 24

      2584       4181       6765      10946      17711      28657

  Columns 25 through 30

     46368      75025     121393     196418     317811     514229

  Column 31

    832040
```

Note that column 1 corresponds to $k = 0$ and column 31 corresponds to $k = 30$. The Fibonacci series is given by

$$x(0) = 0$$
$$x(1) = 1$$
$$x(2) = 1$$
$$x(3) = 2$$
$$x(4) = 3$$
$$x(5) = 5$$
$$\vdots$$
$$x(29) = 514{,}229$$
$$x(30) = 832{,}040$$

**Problem A–2–15**

Consider the difference equation

$$x(k + 2) + \alpha x(k + 1) + \beta x(k) = 0 \tag{2–32}$$

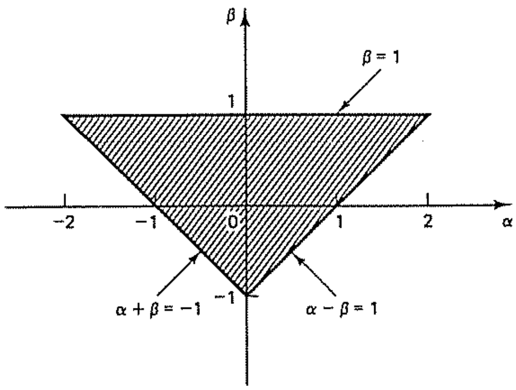


**Figure 2–7** Region in the $\alpha\beta$ plane in which the solution series of Equation (2–32), subjected to initial conditions, is finite.

Find the conditions on $\alpha$ and $\beta$ for which the solution series $x(k)$ for $k = 0, 1, 2, \ldots$, subjected to initial conditions, is finite.

**Solution** Let us define

$$\alpha = a + b, \qquad \beta = ab$$

Then, referring to Example 2–19, the solution $x(k)$ for $k = 0, 1, 2, \ldots$ can be given by

$$x(k) = \begin{cases} \dfrac{bx(0) + x(1)}{b - a}(-a)^k + \dfrac{ax(0) + x(1)}{a - b}(-b)^k, & a \neq b \\ x(0)(-a)^k + [ax(0) + x(1)]k(-a)^{k-1}, & a = b \end{cases}$$

The solution series $x(k)$ for $k = 0, 1, 2, \ldots$, subjected to initial conditions $x(0)$ and $x(1)$, is finite if the absolute values of $a$ and $b$ are less than unity. Thus, on the $\alpha\beta$ plane, three critical points can be located:

$$\alpha = 2, \qquad \beta = 1$$
$$\alpha = -2, \qquad \beta = 1$$
$$\alpha = 0, \qquad \beta = -1$$

The interior of the region bounded by lines connecting these points satisfies the condition $|a| < 1, |b| < 1$. The boundary lines can be given by $\beta = 1$, $\alpha - \beta = 1$, and $\alpha + \beta = -1$. See Figure 2–7. If point $(\alpha, \beta)$ lies inside the shaded triangular region, then the solution series $x(k)$ for $k = 0, 1, 2, \ldots$, subjected to initial conditions $x(0)$ and $x(1)$, is finite.

## PROBLEMS

**Problem B–2–1**

Obtain the $z$ transform of

$$x(t) = \frac{1}{a}(1 - e^{-at})$$

where $a$ is a constant.

**Problem B–2–2**

Obtain the $z$ transform of $k^3$.

**Problem B–2–3**

Obtain the $z$ transform of $t^2 e^{-at}$.

**Problem B–2–4**

Obtain the $z$ transform of the following $x(k)$:

$$x(k) = 9k(2^{k-1}) - 2^k + 3, \qquad k = 0, 1, 2, \ldots$$

Assume that $x(k) = 0$ for $k < 0$.

**Problem B–2–5**

Find the $z$ transform of

$$x(k) = \sum_{h=0}^{k} a^h$$

where $a$ is a constant.

**Problem B–2–6**

Show that

$$\mathcal{Z}[k(k-1)a^{k-2}] = \frac{(2!)z}{(z-a)^3}$$

$$\mathcal{Z}[k(k-1)\cdots(k-h+1)a^{k-h}] = \frac{(h!)z}{(z-a)^{h+1}}$$

**Problem B–2–7**

Obtain the $z$ transform of the curve $x(t)$ shown in Figure 2–8.

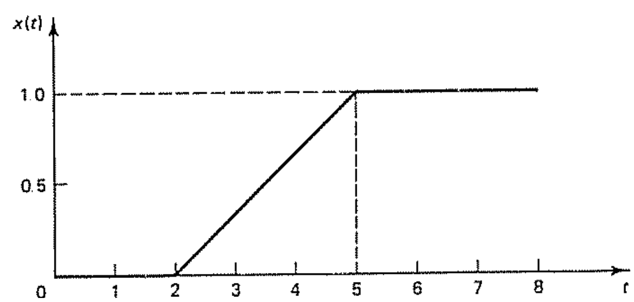


**Figure 2–8** Curve $x(t)$.

**Problem B–2–8**

Obtain the inverse $z$ transform of

$$X(z) = \frac{1 + 2z + 3z^2 + 4z^3 + 5z^4}{z^4}$$

**Problem B–2–9**

Find the inverse $z$ transform of

$$X(z) = \frac{z^{-1}(0.5 - z^{-1})}{(1 - 0.5z^{-1})(1 - 0.8z^{-1})^2}$$

Use (1) the partial-fraction-expansion method and (2) the MATLAB method. Write a MATLAB program for finding $x(k)$, the inverse $z$ transform of $X(z)$.

**Problem B–2–10**

Given the $z$ transform

$$X(z) = \frac{z^{-1}}{(1 - z^{-1})(1 + 1.3z^{-1} + 0.4z^{-2})}$$

determine the initial and final values of $x(k)$. Also find $x(k)$, the inverse $z$ transform of $X(z)$, in a closed form.

**Problem B–2–11**

Obtain the inverse $z$ transform of

$$X(z) = \frac{1 + z^{-1} - z^{-2}}{1 - z^{-1}}$$

Use (1) the inversion integral method and (2) the MATLAB method.

**Problem B–2–12**

Obtain the inverse $z$ transform of

$$X(z) = \frac{z^{-3}}{(1 - z^{-1})(1 - 0.2z^{-1})}$$

in a closed form.

**Problem B–2–13**

By using the inversion integral method, obtain the inverse $z$ transform of

$$X(z) = \frac{1 + 6z^{-2} + z^{-3}}{(1 - z^{-1})(1 - 0.2z^{-1})}$$

**Problem B–2–14**

Find the inverse $z$ transform of

$$X(z) = \frac{z^{-1}(1 - z^{-2})}{(1 + z^{-2})^2}$$

Use (1) the direct division method and (2) the MATLAB method.

**Problem B–2–15**

Obtain the inverse $z$ transform of

$$X(z) = \frac{0.368z^2 + 0.478z + 0.154}{(z - 1)z^2}$$

by use of the inversion integral method

**Problem B–2–16**

Find the solution of the following difference equation:

$$x(k + 2) - 1.3x(k + 1) + 0.4x(k) = u(k)$$

where $x(0) = x(1) = 0$ and $x(k) = 0$ for $k < 0$. For the input function $u(k)$, consider the following two cases:

$$u(k) = \begin{cases} 1, & k = 0, 1, 2, \ldots \\ 0, & k < 0 \end{cases}$$

and

$$u(0) = 1$$

$$u(k) = 0, \qquad k \neq 0$$

Solve this problem both analytically and computationally with MATLAB.

**Problem B–2–17**

Solve the following difference equation:

$$x(k + 2) - x(k + 1) + 0.25x(k) = u(k + 2)$$

where $x(0) = 1$ and $x(1) = 2$. The input function $u(k)$ is given by

$$u(k) = 1, \qquad k = 0, 1, 2, \ldots$$

Solve this problem both analytically and computationally with MATLAB.

**Problem B–2–18**

Consider the difference equation:

$$x(k + 2) - 1.3679x(k + 1) + 0.3679x(k) = 0.3679u(k + 1) + 0.2642u(k)$$

where $x(k) = 0$ for $k \leq 0$. The input $u(k)$ is given by

$$u(k) = 0, \qquad k < 0$$

$$u(0) = 1.5820$$

$$u(1) = -0.5820$$

$$u(k) = 0, \qquad k = 2, 3, 4, \ldots$$

Determine the output $x(k)$. Solve this problem both analytically and computationally with MATLAB.