$$\int \frac{q(x)}{p(x)}\,dx$$, onde $P(x)$ polinômio de grau 2.

a) $$\int \frac{1}{(x-1)(x+3)}\,dx \;\star = \int \frac{1}{4}\cdot\frac{1}{x-1} - \frac{1}{4}\cdot\frac{1}{x+3}\,dx = \frac{1}{4}\int \frac{1}{x-1}\,dx^{*} - \frac{1}{4}\int \frac{1}{x+3}\,dx^{\odot}$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+3| + C,\quad C\in\mathbb{R}$$

duas raízes distintas

denominador igual

numerador igual

$$\star\quad \frac{1}{(x-1)(x+3)} = \frac{a}{x-1} + \frac{b}{x+3} = \frac{a(x+3)+b(x-1)}{(x-1)(x+3)}$$

$$\therefore\ 1 = ax + a3 + bx - b$$

$$0.x + 1 = 1 = (a+b)x + (a3-b)$$

polinômios iguais

$$\begin{cases} a+b=0 \\ 3a-b=1 \end{cases} \Leftrightarrow \begin{cases} a+b=0 \\ 4a=1 \end{cases} \Leftrightarrow \begin{cases} b=-a=-\frac{1}{4} \\ a=\frac{1}{4} \end{cases}$$

$$\Leftrightarrow \begin{cases} a=\frac{1}{4} \\ b=-\frac{1}{4} \end{cases}$$

$$* \quad \int \frac{1}{x-1}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|x-1| + C,\quad C\in\mathbb{R}$$

$u = x-1$

$du = dx$

$$\odot \quad \int \frac{1}{x+3}\,dx = \int \frac{1}{v}\,dv = \ln|v| + C = \ln|x+3| + C,\quad C\in\mathbb{R}$$

$v = x+3$

$dv = dx$

b) $\int \frac{1}{\boxed{x^2+4x+5}}dx \overset{*}{=} \int \frac{1}{\boxed{(x+2)^2+1}}\boxed{dx} = \int \frac{1}{u^2+1}\boxed{du} = \text{arctg}\, u + c$

Polinômio sem raízes reais

$u = x+2$ $\quad \boxed{du = dx}$

$= \text{arctg}(x+2) + c,$
$c \in \mathbb{R}$

* completamento de quadrado

$x^2+4x = x^2+2\cdot 2x+4-4 = (x+2)^2-4$

$\therefore \boxed{x^2+4x+5} = (x+2)^2+5-4 = \boxed{(x+2)^2+1}$

Queremos u^2+1

c) $\int \frac{x}{(x+2)(x+3)}dx = \int \frac{-2}{x+2} + \frac{3}{x+3}dx = -2\ln|x+2| + 3\ln|x+3| + c,$

$c \in \mathbb{R}$

duas raízes distintas

$$\frac{x}{(x+2)\cdot(x+3)} = \frac{a}{x+2} + \frac{b}{x+3} = \frac{a(x+3)+b(x+2)}{(x+2)\cdot(x+3)}$$

$\therefore x = ax+3a+bx+2b$

$\therefore x = (a+b)x + (3a+2b)$

$\therefore \begin{cases} a+b=1 \\ 3a+2b=0 \end{cases} \Leftrightarrow \begin{cases} a+b=1 \\ a=-2 \end{cases} \Leftrightarrow \begin{cases} a=-2 \\ b=1-a=3 \end{cases}$

d) $\displaystyle\int \frac{x}{x^2+2x+2}\,dx = \int \left[\frac{x+\frac{1}{2}}{x^2+2x+2} - \frac{1}{2}\cdot\frac{1}{x^2+2x+2}\right] dx$

$\displaystyle = \int \frac{x+\frac{1}{2}}{x^2+2x+2}\,dx^{☆} - \frac{1}{2}\int \frac{1}{x^2+2x+2}\,dx^{*}$

denominador não tem raízes

mudanças de variável distintos

$\boxed{u = x^2+2x+2}$

$du = (2x+1)\,dx$

$\therefore \boxed{\left(x+\frac{1}{2}\right)dx = \frac{1}{2}\,du}$

☆ $\displaystyle\int \frac{x+\frac{1}{2}}{x^2+2x+2}\,dx = \int \frac{1}{u}\,\frac{1}{2}\,du = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|x^2+2x+2| + c,\quad c\in\mathbb{R}$

* $\displaystyle\int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{(x+1)^2+1}\,dx = \int \frac{1}{w^2+1}\,dw = \operatorname{arctg}(w) + c$

$= \operatorname{arctg}(x+1) + c,\quad c\in\mathbb{R}$

$w = x+1$

$\boxed{dw = dx}$

$x^2+2x = x^2+2x+1-1 = (x+1)^2 - 1$

$\therefore x^2+2x+2 = (x+1)^2+1$

$\displaystyle\therefore \int \frac{x}{x^2+2x+2}\,dx = \ln\sqrt{x^2+2x+2} - \frac{1}{2}\operatorname{arctg}(x+1) + c,\quad c\in\mathbb{R}$

e) $\displaystyle\int \frac{1}{x^2+x+1}\,dx = \int \frac{1}{\frac{3}{4}\left[\left(\boxed{\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}}\right)^2+1\right]}\,\boxed{dx} = \int \frac{1}{\frac{3}{4}\left[\boxed{u}^2+1\right]}\,\boxed{\frac{\sqrt{3}}{2}du}$

sem raízes reais

$\boxed{u = \frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}}$

$du = \frac{2}{\sqrt{3}}dx$

completamento de quadrado

$x^2+x = x^2+2\cdot\frac{1}{2}x+\frac{1}{4}-\frac{1}{4} = \left(x+\frac{1}{2}\right)^2-\frac{1}{4}$ $\therefore$ $\boxed{dx = \frac{\sqrt{3}}{2}du}$

$\therefore\ x^2+x+1 = \left(x+\frac{1}{2}\right)^2-\frac{1}{4}+1 = \left(x+\frac{1}{2}\right)^2+\frac{3}{4}$ não é 1.

$\therefore\ x^2+x+1 = \frac{3}{4}\left[\frac{4}{3}\cdot\left(x+\frac{1}{2}\right)^2+1\right]$

pra dentro

agora é 1

$= \frac{3}{4}\left[\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1\right] = \frac{3}{4}\left[\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)^2+1\right]$

$\therefore\ \displaystyle\int\frac{1}{x^2+x+1}\,dx = \int\frac{1}{\frac{3}{4}(u^2+1)}\frac{\sqrt{3}}{2}\,du = \int\frac{4}{3}\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{u^2+1}\,du$

$\displaystyle = \frac{2}{\sqrt{3}}\int\frac{1}{u^2+1}\,du = \frac{2}{\sqrt{3}}\operatorname{arctg}\boxed{u}+C,\ C\in\mathbb{R}$

$\displaystyle = \frac{2}{\sqrt{3}}\operatorname{arctg}\left(\boxed{\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}}\right)+C,\ C\in\mathbb{R}$

$$\int \frac{q(x)}{p(x)}\,dx \qquad \text{grau } q(x) > \text{grau } p(x)$$

$$q(x) = q_1(x) \cdot p(x) + r(x), \text{ onde}$$

$r(x) = 0$ ou
grau $r(x) <$ grau $p(x)$

$$\int \frac{q(x)}{p(x)}\,dx = \int \frac{q_1(x)\,p(x) + r(x)}{p(x)}\,dx$$

$$= \int \left(q_1(x) + \frac{r(x)}{p(x)} \right) dx$$

$$= \int q_1(x)\,dx + \int \frac{r(x)}{p(x)}\,dx$$

f) $\displaystyle\int \frac{x^4+1}{x^2+2x+2}\,dx$

divisão de polinômios:

$$\begin{array}{l|l} x^4 + 0x^3 + 0x^2 + 0x + 1 & x^2+2x+2 \\ \cline{2-2} -x^4 - 2x^3 - 2x^2 & x^2 - 2x + 2 \\ \hline \quad -2x^3 - 2x^2 + 0x & \\ \quad 2x^3 + 4x^2 + 4x & \\ \hline \qquad 2x^2 + 4x + 1 & \\ \qquad -2x^2 - 4x - 4 & \\ \hline \qquad\qquad -3 & \end{array}$$

$$= \int x^2 - 2x + 2\,dx + \int \frac{-3}{x^2+2x+2}\,dx = \frac{x^3}{3} - x^2 + 2x - 3\,\text{arctg}(x+1) + c, \quad c \in \mathbb{R}$$

$$\int -\frac{3}{x^2+2x+2}\,dx = -3\int \frac{1}{(x+1)^2+1}\,\boxed{dx} = -3\int \frac{1}{u^2+1}\,\boxed{du}$$

$$x^2+2x+2 = x^2+2x+1+1 = (x+1)^2+1$$

$$x+1 = u \qquad \boxed{dx = dx}$$

$$= -3\,\text{arctg}\,u + c = -3\,\text{arctg}(x+1) + c,$$

$c \in \mathbb{R}$