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A fuzzy hybrid integrated framework for portfolio optimization in private banking

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ABSTRACT

Decision-making processes in private banking must comply with standards for risk management and transparency enforced by banking regulations. Therefore, investors must be supported throughout a risk-informed decision process. This paper contributes to the literature by presenting a hybrid integrated framework that considers personal features of the investor and additional characteristics imposed by regulations, for which linguistic evaluations are used with regard to risk exposure. The proposed approach for personal investment portfolios considers legal aspects and investor's preferences as an input to the novel fuzzy multiple-attribute decision making approach for sorting problems proposed in this paper, called FTOPSIS-Class. Then, the next step of the proposed framework uses the sorting results for a fuzzy multi-objective optimization model that considers the risk and return associated with the investor's profile over three objectives. The contributions of this paper are illustrated and validated by using a numerical application in line with a new trend for modern portfolio theory which enables a real world investor's characteristics to be considered throughout the decision-making process.

1. Introduction

When planning personal finance, an individual should consider the suitability of a range of banking products, or private equity investment, and insurance or employer-sponsored retirement plans, social security benefits, and income tax management, during his or her life based on dynamic objectives. Currently, banks provide a wide range of investment options, including funds, shares, and exchange-traded funds (ETFs) that can be accessed by a personal investor even with a budget as small as US$ 500. Given the events that characterize financial markets nowadays, which allows investors to expand their investment alternatives under different market contexts, a consensus has been created that financial decisions require flexible and customized frameworks that take several factors, criteria, and requirements into consideration (Zopounidis & Doumpos, 2013).

Ever since Markowitz set out the Mean-Variance Theory (Markowitz, 1952), portfolio optimization has been growing in importance which has led to the literature on portfolio selection becoming extensive. This was commemorated by Markowitz (2014) and Kolm, Tütüncü, and Fabozzi (2014) in a 60-year special issue of the European Journal of Operational Research which discusses trends in portfolio selection as well as how the most common branches of the original mean-variance optimization problem have developed since the fifty-year retrospective on portfolio selection presented by Rubinstein (2002). Currently, research on modern portfolio theory is being concentrated on two main streams (Anagnostopoulos & Mamanis, 2010): (i) incorporating alternative risk measures; and (ii) incorporating real features in the mathematical formulations, e.g., additional characteristics that the investor wishes to consider or is obliged to comply with, due to legislation. This research work focuses on the latter issue in portfolio modeling of private banking.

Personal finance is highly regulated in all countries by equivalents of Securities and Exchange Commissions. Such institutions regularly set out instructions which state that: (i) the product, service or operation should be convergent with the investor' objectives; (ii) the financial situation of the investor should be compatible with the product service and operation; and (iii) the client should possess enough knowledge to understand the risks relating to the product, service or operation. In summary, every financial institution must define and inform a profile in which each investor can be categorized. However, there is not a standard process or

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questionnaire for conducting such analysis. Some financial institutions have proposed some tools to undertake this task. Simultaneously, banks should classify their funds into categories based on market and credit risks. Unless otherwise clearly stated by the investor, banks are obliged to match the clients' profile with the total risk classification of funds.

Financial decision-making in private banking is also often affected by many other issues, such as social, economic, political, and psychological matters. More specifically, agents frequently have information that is characterized by vague linguistic descriptions such as high risk, low returns, etc. This kind of information has an important influence on financial markets, and is characterized by vagueness and ambiguity, which can be linked to fuzzy values in many cases. As a consequence, the literature on applying fuzzy sets in finance is growing (Chourmouziadis & Chatzoglou, 2016; Liu & Zhang, 2015; Perez & Gomez, 2016).

Further, portfolio selection in personal banking is, in general, carried out by a bank manager, whose experience in the finance market is sometimes limited, who chooses funds based on the investor's profile. In some cases, which are quite common in Brazil, banks do not always act in the best interests of the client, due to their acting in their own interest. This involves either selling funds which have high administration fees, or recommending those that are in step with the commercial strategies of the bank. One of the objectives of this study is also to offer an environment that improves the relationship between banks and investors, thereby simultaneously increasing the confidence that the two agents have in each other. In this environment, the bank should stimulate the client to participate in the process, and by so doing, this enhances the regard that the client will have for the advisor's professional practice and reinforces the ethical responsibility that the advisor will feel when exercising his/her duties.

Several models have been developed for portfolio optimization. Most of them have proposed mathematical formulations which take into consideration the classic risk-return trade-off, and restrict the analysis to two criteria (Prigent, 2007). Lately, some studies extended the standard Markowitz mean-variance approach by introducing cardinality constraints (Chang, Meade, Beasley, & Sharaiha, 2000), and by using different risk measures (Chang, Yang, & Chang, 2009). Currently, multi-objective optimization (MOO) has become the predominant way of formulating and solving the problem (Steuer, Qi, & Hirschberger, 2007). Steuer, Qi, and Hirschberger (2005) and Zopounidis and Doumpos (2013) presented interesting surveys on applying MOO approaches in the portfolio selection problem. In particular, multi-objective evolutionary algorithms (MOEA) (Anagnostopoulos & Mamanis, 2010; Metaxiotis & Liagkouras, 2012; Ong, Huang, & Tzeng, 2005) have become of increasing interest due to their ability to handle non-convex and non-linear criteria, and to consider real features. The recognition that several items of information are vague in portfolio selection has led to fuzzy sets being integrated into the modeling approaches that have been developed. Bermúdez, Segura, and Vercher (2012) combined MOEA and fuzzy sets to deal with cardinality-constrained portfolio selection. Perez and Gomez (2016) presented a general nonlinear binary MOO model with some fuzzy parameters with a view to representing information not fully known by investors. Calvo, Ivorra, and Liern (2016) used fuzzy sets to represent a non-financial criterion in a risk-return bi-objective problem.

However, the application of modeling-based approaches on private banking contexts is still scant. Gonzalez-Carrasco, Colomo-Palacios, Lopez-Cuadrado, Garcı, and Ruiz-Mezcua (2012) proposed a private banking recommendation system to determine the investor profile and the corresponding, most suitable investment alternatives based on Artificial Intelligence techniques. The system was mainly developed to recommend a set of products based on social attributes and psychological aspects of the investor, and in the characteristics of investment products, but it is not able to select a portfolio, considering several real features, such as the number and (in)compatibility of assets. Ali, Akçay, Sayman, Yılmaz, and Özçelik (2016) introduced an optimization-based approach to cross-selling investment products to private banking, considering simultaneously customers' and banks' interests. A non-linear integer programming model was developed aimed at maximizing customers' return and bank's profitability by introducing new products in the existing portfolio of the customer. The authors claimed that a win-win situation can be created. However, they do not consider the regulation issues concerning private banking, and only focus on customer's return, neglecting long-term objectives. Although these two studies have interesting and useful ideas on portfolio selection, to the best of our knowledge there is no procedure that simultaneously comprises the following relevant issues involved in personal finance: (i) the ability to handle a client's declared preferences concerning his/her perceptions of risk, and to explicitly consider the client's knowledge and long-term objectives in the modeling framework; and (ii) the ability of considering regulation issue in the portfolio selection decision-making process.

The main objective of this paper is to describe an integrated decision analysis framework which has been developed to support portfolio selection in private banking. This research proposes a hybrid approach, which integrates fuzzy multi-attribute decision-making (MADM) and fuzzy MOO with a view to optimizing portfolios. On the one hand, the fuzzy MADM component enables measurements to be made of the adequacy for each kind of asset class available in a bank (fixed income, stocks, commodities, etc.) regarding some criteria (return, risk, liquidity, investment objectives, etc.) which are in keeping with investor's profiles (i.e. whether conservative, moderate, bold, and audacious). On the other hand, the MOO model obtains an optimal proportion for each asset in the investor portfolio. The proposed portfolio selection process consists of four steps. In the first step, the problem is structured, mainly in terms of regulation aspects, possible investments, evaluation criteria, and criteria weights. In the second step, the alternatives are classified by a fuzzy MADM, which evaluates the investment options based on the criteria established, and takes the risk classification of the alternatives, and the investor's profiles into account. In the third step, the results of the second step are integrated into a MOO model, considering a triple objective optimization problem, where maximizing both the expected return and the investor profile are taken into consideration, while minimizing risks. Finally, in the last step, a risk management out-sample analysis is conducted on the portfolios obtained and compared to other concurrent approaches with a view to validating the whole process. Numerical experiments, which emulate real world situations, were carried out.

In this paper, we make the following contributions: (i) We develop a hybrid decision-making framework to select a portfolio in private banking, while fully considering regulation issues by structuring an innovative Operational Research (OR) application; (ii) We propose an innovative metamodel that combines elements of fuzzy-MADM and fuzzy-MOO, by taking advantage of the best characteristics that each method has to offer; (iii) We apply a linguistic approach based on fuzzy sets that allowed us to incorporate vague classifications, such as investor profiles, into our modeling framework, and (iv) We develop a novel classification method called FTOPSIS-Class, which is able to measure the adequacy of an investment option for an investor's risk profile. Further, we focus on the transparency and ease with which the approach can be understood rather than on using very sophisticated mathematical tools. As the framework deals with a wide spectrum of factors characterized by vagueness and uncertainties, its successful application depends greatly on active cooperative teamwork between the investors, bank analysts, and optimization approach, which enables the investor to be actively engaged during the entire decision-making process. This is one of the major advantages of our framework, and one with which all other activities interact. In summary, the main contribution of our study is to introduce the development and application of a systemic and integrated decision making method for portfolio selection in private banking, combining fuzzy multi-attribute decision making and optimization-based approach, that considers simultaneously regulation issues, and investors' objectives and preferences.

The remainder of this paper is organized as follows. Section 2 characterizes the problem, by describing the regulatory context in which the decision-making process occurs. Section 3 introduces some of the background required to understand the framework developed. Section 4 describes in details the methodological aspects of each step of the developed framework. A numerical application of the framework using real financial investment data is described in Section 5. Finally, Section 6 presents some final remarks, draws some conclusions and suggests some further lines of future research.

2. Definition of the problem

As previously mentioned, investment in private banking is highly regulated. Before suggesting products, services, and operations, the bank is obliged to apply a questionnaire to each client so as to place the investor in a predefined category, which depends on matters such as risk, the objective of the investment, grace period, and the client's knowledge of the market. Although each bank has its own questionnaire, they are very similar. Based on the answers given, the bank classifies the client into one of the categories, defined as investor's profiles. In general, there are four profiles, as follows:

Conservative: Prioritize security as the key aspect in investments. In this profile, it is advisable to keep a higher percentage of investments in low-risk products, but a small portion can be allocated to products that offer higher levels of risk, in order to achieve greater long-term gains.

Moderate: Emphasizes security in the investments, but also opts for products that can deliver greater long-term gains. Diversifying the resources is the most advisable investment strategy in this profile.

Bold: Investors in this class look for possibilities of greater gains and, as consequence, take higher risks. However, even for bolder strategies, it is advisable to keep part of the resources in lower risk products, as a way of protecting the investor's assets.

Aggressive: Has a strong tolerance for risks, and regards these as opportunities for greater gains. The investor seeks a return on his/her investments in the long-term and, thus, adapts his/her portfolio to oscillations in the market verified in the short term. A representative part of his/her investments is allocated in the stock, option and derivative markets, special attention being paid to new sectors.

This classification is in general valid for some specified period, after which a revaluation is carried out. Banks are also obliged to classify their funds by categories mainly based on the market and credit risks. Market risks are associated with the oscillation in price of assets in their respective negotiation markets, while credit risks are associated with the possibility that the issuer of financial securities cannot fulfill the contractual terms, partially or completely. In general, funds are classified into the following categories:

Very low-risk funds: Funds that have low market risk, measured by the price variation of post-fixed bonds. They have low credit risk.

Low-risk funds: Funds that have low market risk, measured by the price variation of post-fixed and pre-fixed bonds. They have low credit risk.

Moderate-risk funds: Funds that have medium market risk, measured by the price variation of fixed-rate securities, prefixed and associated with inflation indices. They also bear this classification because they may contain medium credit risk.

High-risk funds: Funds that may present high market risk, measured by the price variation of fixed-rate, pre-fixed bonds, linked to inflation indices, foreign currencies, stock prices and derivative prices. They also bear this classification because they may contain high credit risk.

Very high-risk funds: Funds that may present very high market risk, measured by the price variation of fixed-rate, pre-fixed bonds, linked to inflation indices, foreign currencies, stock prices and derivative prices. This category includes External Mirror Funds and may contain high credit risk.

In general, a bank can only offer products, services, and operations in categories which are compatible with the profile of a specific client. However, a client can order an operation that it is not totally in accordance with his/her profile. This is only acceptable if the client enters into a formal agreement, stating that he/she is aware that this specific operation is based on either a non-updated profile or without the correct adjustment to his/her updated profile.

The main objective of this paper is to introduce a decision-making framework that supports private banking, in terms of selecting the best suitable portfolio for a specific client or a client's profile, in an explicit, dynamic, and flexible way, by addressing three key issues: (i) a procedure to capture clients' preferences concerning the products, services, and operations offered by a bank, considering several relevant attributes; (ii) a mathematical modeling tool to identify the best portfolio for each client based on multiple objectives, and a set of constraints defined by customers and a bank's financial analysts; and (iii) an evaluation tool to compare, considering several risk measures, the performance of the suggested portfolio with other portfolios which are defined by using similar or simplistic strategies.

3. Preliminaries

Before describing the approach developed, we introduce some notation. An MADM problem may be described by means of the following sets: $A = \{A_1, A_2, \ldots, A_m\}$ the set of m alternatives, and $C = \{C_1, C_2, \ldots, C_n\}$ the set of n criteria or attributes (these terms will be used as similar and interchangeable in the context of this study). Let R_{ij} be the matrix of the performance ratings to each alternative i with respect to each criterion j, and let w_j be the weight of criteria $j \in C$.

3.1. Fuzzy set theory

In this section, some definitions concerning fuzzy set theory related to the method developed are reviewed.

Definition 1 (Chen, Lin, & Huang (2006))**.** A fuzzy set $\tilde{A}$ in a universe of discourse X is characterized by a membership function $\mu_{\tilde{a}}(x)$ which associates with each element x in X a real number in the interval [0,1].

Definition 2 (Kaufmann & Gupta (1991))**.** A trapezoidal fuzzy number $\tilde{a}$ can be defined as $\tilde{a} = (a_1, a_2, a_3, a_4)$ according to the membership function $\mu_{\tilde{a}}(x)$ defined as follows:

$$\mu_{\tilde{a}}(x) = \begin{cases} f^L_{\tilde{a}}(x), & a_1 \le x \le a_2 \\ 1, & a_2 \le x \le a_3 \\ f^R_{\tilde{a}}(x), & a_3 \le x \le a_4 \\ 0, & \text{otherwise} \end{cases}$$

where $f^L_{\tilde{a}}(x) : [a_1, a_2] \to [0, 1]$ is a strictly increasing function and $f^R_{\tilde{a}}(x) : [a_3, a_4] \to [0, 1]$ is a strictly decreasing function.

Definition 3 (Chen et al. (2006))**.** The fuzzy sum $\oplus$ and fuzzy subtraction $\ominus$ of any two trapezoidal fuzzy numbers are also trapezoidal fuzzy numbers by the extension principle (Dubois & Prade, 1980). However, the multiplication of any two trapezoidal fuzzy numbers $\otimes$ is only an approximate trapezoidal fuzzy number. Given two positive trapezoidal fuzzy numbers, $\tilde{a} = (a_1, a_2, a_3, a_4)$ and $\tilde{b} = (b_1, b_2, b_3, b_4)$, and a non-fuzzy number $r \ge 0$, where $0 \le a_1 \le a_2 \le a_3 \le a_4$ and $0 \le b_1 \le b_2 \le b_3 \le b_4$, then the fuzzy operations of sum, subtraction, multiplication and multiplication by a scalar can be expressed by, respectively:

$$\tilde{a} \oplus \tilde{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$$
$$\tilde{a} \ominus \tilde{b} = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1)$$
$$\tilde{a} \otimes \tilde{b} \equiv (a_1 \times b_1, a_2 \times b_2, a_3 \times b_3, a_4 \times b_4)$$
$$\tilde{a} \otimes r \equiv (a_1 \times r, a_2 \times r, a_3 \times r, a_4 \times r)$$

Definition 4 (Chen (2000))**.** The vertex distance $\delta(\tilde{a}, \tilde{b})$ between two trapezoidal fuzzy numbers $\tilde{a} = (a_1, a_2, a_3, a_4)$ and $\tilde{b} = (b_1, b_2, b_3, b_4)$ is defined as follows:

$$\delta(\tilde{a}, \tilde{b}) = \sqrt{\frac{1}{4}[(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2 + (a_4 - b_4)^2]}$$

Definition 5 (Chen (2000))**.** A matrix $\tilde{M}$ is a fuzzy matrix if, at least, one of its elements is a fuzzy number.

Definition 6 (Zimmermann (1978))**.** A linguistic variable is a variable whose values are expressed in linguistic terms, i.e., words or sentences in a natural or artificial language, such as “high”, “average” and “low”, for example.

3.2. The Fuzzy-TOPSIS method

Chen (2000) extended the classic TOPSIS (Technique for Order of Preference by Similarity to Ideal Solution) method proposed by Hwang and Yoon (1981) towards solving decision-making problems under fuzzy environments, where linguistic variables are used to assess the rating of alternatives and criteria weights. As the seminal TOPSIS method, Fuzzy-TOPSIS is also based on choosing the best alternatives such as the one closest to a positive ideal solution and that farthest from a negative ideal solution. Fuzzy-TOPSIS has been widely used in the literature as have its adapted versions such as FETOPSIS (Caetani, Ferreira, & Borenstein, 2016; Santi, Ferreira, & Borenstein, 2015). Algorithm 1 summarizes the Fuzzy-TOPSIS engine mechanism, whose parameters given by the DMs are: (i) the fuzzy ratings of the alternative i with respect the criterion j, $\tilde{R} = [\tilde{r}_{ij}]_{m\times n}$; and, the fuzzy weights of the criteria, $\tilde{W} = [\tilde{w}_j]_n$.

Algorithm 1 Fuzzy-TOPSIS routine.

Step 1: Structure the decision problem, by identifying DMs, the set of criteria and alternatives;

Step 2: Choose the linguistic terms to assess the relative importance of the criteria and to evaluate the rating of the alternatives;

Step 3: Construct the normalized decision matrix $\tilde{R} = [\tilde{r}_{ij}]_{m\times n}$ as follows:

$$\tilde{r_{ij}} = \begin{cases} \left(\frac{a_{ij}}{d^*_j}, \frac{b_{ij}}{d^*_j}, \frac{c_{ij}}{d^*_j}, \frac{d_{ij}}{d^*_j}\right) & \text{if } j \in B, \text{ where set } B \text{ is associated with benefit criteria, and } d^*_j = \max_i d_{ij} \\ \left(\frac{a^-_j}{a_{ij}}, \frac{a^-_j}{b_{ij}}, \frac{a^-_j}{c_{ij}}, \frac{a^-_j}{d_{ij}}\right) & \text{if } j \in C, \text{ where set } C \text{ is associated with cost criteria, and } a^-_j = \min_i a_{ij} \end{cases}$$

Step 4: Construct the weighted normalized fuzzy decision matrix $\tilde{V} = [\tilde{v}_{ij}]_{m\times n}$ from $\tilde{R} = [\tilde{r}_{ij}]$ and $\tilde{W} = [\tilde{w}_j]$ as $\tilde{v}_{ij} = \tilde{r}_{ij} \otimes \tilde{w}_j$;

Step 5: Determine the positive ideal $(\tilde{A}^*)$ and the negative ideal solutions $(\tilde{A}^-)$ as $\tilde{A}^* = \{\tilde{v}^*_1, \tilde{v}^*_2, ..., \tilde{v}^*_n\}$, $\tilde{A}^- = \{\tilde{v}^-_1, \tilde{v}^-_2, ..., \tilde{v}^-_n\}$, where $\tilde{v}^*_j = \max_i \{v_{ij4}\}$, $\tilde{v}^-_j = \min_i \{v_{ij1}\}$, $i = 1, 2, ..., m$, and $j = 1, 2, ..., n$.

Step 6: Calculate the distances of each alternative i in relation to the ideal solutions as follows:

$$\tilde{d}^*_i = \sum_{j=1}^{n} \delta(\tilde{v}_{ij}, \tilde{v}^*_j), \; i = 1, 2, ..., m$$

$$\tilde{d}^-_i = \sum_{j=1}^{n} \delta(\tilde{v}_{ij}, \tilde{v}^-_j), \; i = 1, 2, ..., m$$

Step 7: Calculate the closeness coefficient of each alternative i as $CC_i = \frac{\tilde{d}^-_i}{\tilde{d}^*_i + \tilde{d}^-_i}, i = 1, 2, ..., m.$

Step 8: Sort the alternatives in descending order. The highest CC_i value indicates the best performance in relation to the evaluation criteria.

4. Proposed framework

The portfolio selection problem in private banking is broken down into two connected subproblems: (i) to measure the adequacy of each alternative of investment $i \in A$ for each investor profile $p \in P$; (ii) to allocate the available resources to banking products or services, taking into account several objectives, including the adequacy of the asset for the investor's profile, and restrictions. The first problem is formulated as an MADM one, in which the main objective is to define an estimation of the adequacy (A_{ip}) of investment $i \in I$ for each profile $p \in P$, considering simultaneously several different attributes, such as the investor's objectives, investment risk, and the investor's knowledge of the product or service. The second problem was modelled as a fuzzy MOO portfolio selection that consists of finding the proportions of various assets to be held in a portfolio that achieve a good compromise solution for the established objectives set, subject to real life features, which are represented as constraints.

Fig. 1 presents how the framework for this decision-making process was structured, and describes the methodological procedures used. The process was divided into four interconnected steps. The first step comprises the definition and scope of the personal portfolio selection, while the second step uses a novel MADM method, FTOPSIS-Class, to classify the investment alternatives according to the DMs' preferences and the bank's classification of risk. The third step uses a fuzzy multi-objective linear integer optimization model to allocate the clients' resources into a set of banking products. One of the objectives of this is to test the adequacy of each investment alternative for each investor's profile, which was computed in the second phase, and constraints defined by the in-

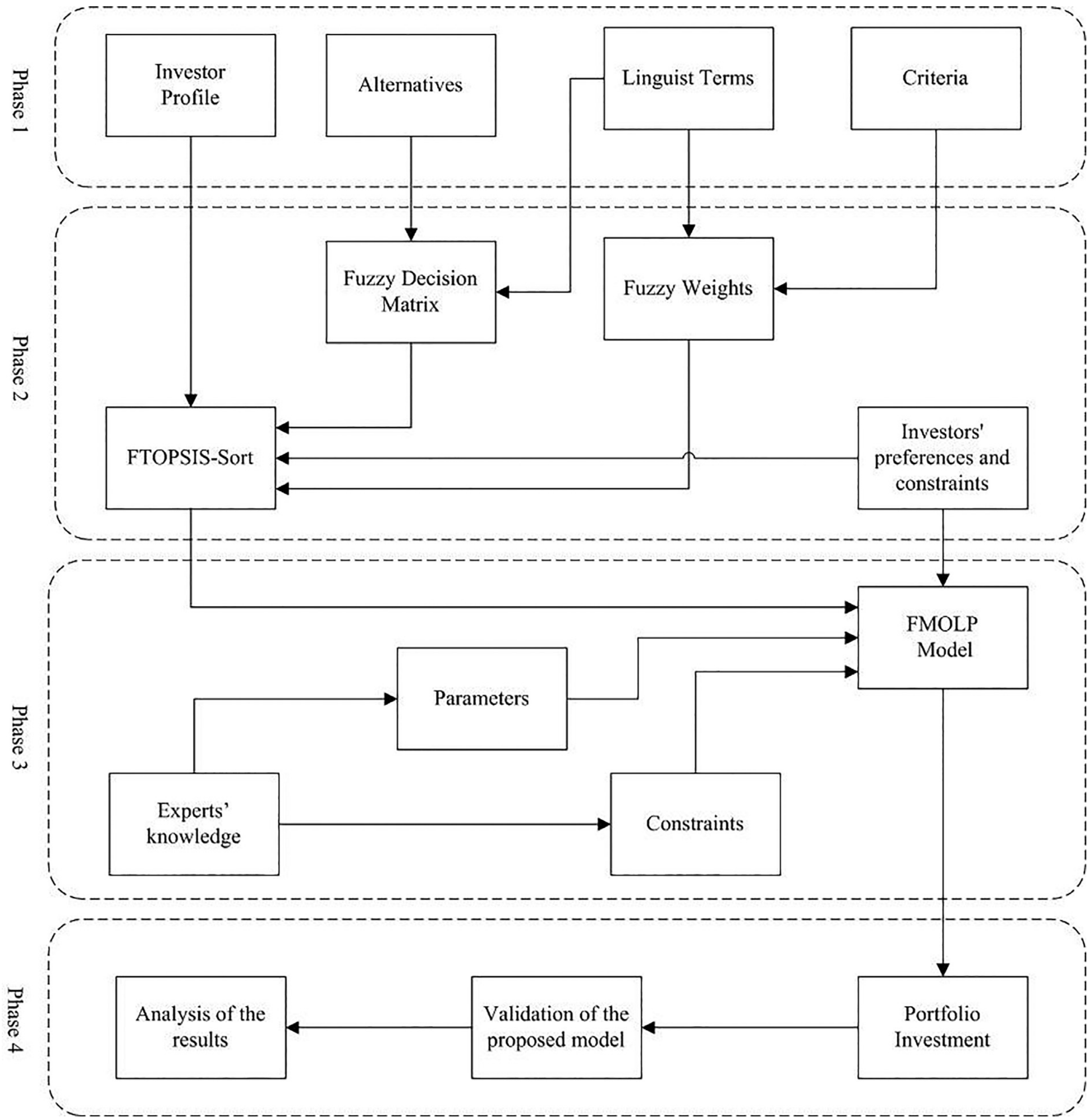

Fig. 1. Phases of the proposed framework.

vestor. Finally, the last step is used to validate and analyse the results by using measurements of risk and return on performance.

4.1. Phase 1—problem structuring

This first step deals with structuring the problem, in terms of identifying and making explicit investment objectives, alternative funds, and investor' preferences. It is suggested that this step is carried out by a bank analyst who should interacting with the investor. For the full application of the framework it is assumed that: (i) the bank has already defined the investor's profile; and (ii) the bank has also classified the investment options in line with the categories presented in Section 2.

The main objective of this step is to feed the models developed with the required parameters and variables. The regulation norms by governmental agencies can be used as a basis for developing a set of attributes to measure the adequacy of an investment option for the investor's profile. The set of criteria should incorporate the following regulation issues: (i) the banks can only offer products, services, and operations that are in accordance with the investor's profile and objectives; (ii) the investor should understands the risks related to a product, service, or operation; and (iii) the investor's financial situation is compatible with the product, service, or operation. This step should also elicit the real features that the investor wishes to consider when defining the MOO model.

4.2. Phase 2—classification procedure

The main objective of the second step is to define the adequacy of an investment $i \in I$ for a client profile $p \in P$, based on the attributes defined in Step 1. This is essentially a Multiple Criteria Nominal Classification (MCNC) problem (Chen, 2006), where the main objective is to assign the investment options into predefined homogeneous groups, specified by multiple characteristics (clients' profiles). Several MADM methods have been developed for classification problems, categorized as follows (Doumpos, Marinakis, Marinaki, & Zopounidis, 2009): (i) based on value functions (Greco, Mousseau, & Słowiński, 2010); (ii) symbolic methods based on decision rules (Greco, Matarazzo, & Slowinski, 2001); and (iii) outranking methods (Bouyssou & Marchant, 2015). However, these traditional classification methods focus on sorting alternatives into groups ordered by preference (Zopounidis & Doumpos, 2002). However, in sorting problems, an alternative belongs to only one group, while in MCNC problems some alternatives may be assigned to more than one group and some may not be assigned to any group. As a result, the classification methods based on sorting ones are of little relevance to the MCNC (Chen, 2006).

The literature in MCNC is still scant, being represented by only a few published research works, such as those by Perny (1998), Scarelli and Narula (2002), and Malakooti and Yang (2004). In these papers, outranking methods, based on ELECTRE-III, are developed to solve special types of MCNC problems, but without a systematic analysis Chen (2006). As ELECTRE and its variants, including the fuzzy extensions, have been criticized in the literature (Wang & Triantaphyllou, 2008; Zandi & Roghanian, 2013) for admitting sophisticated preference assumptions, Chen (2006) presents a Simple Multi-Attribute Rating Technique (SMART)-based optimization model to solve MCNC problems. Although widely applied to several business cases due to its simplicity and good results, SMART demands that decision-makers assess value functions for each of the lowest level attributes, a difficult task for criteria without an interval scale (Barfod & Leleur, 2014).

Based on this criticism, we decided to develop our own method. As the investor's profiles and classification of the investments were defined using linguistic terms, as presented in Section 2, we decided to implement a fuzzy based MADM sorting method. The proposed approach is suitable for situations when a compensatory rationale is acceptable and may be used for large-scale applications, especially when the use of linguistic statements are required which is the case in the private banking context. Considering the success of TOPSIS and its fuzzy implementations in the MADM community (Behzadian, Otaghsara, Yazdani, & Ignatius, 2012), we decided to develop a specialized classification variant of Fuzzy-TOPSIS, called FTOPSIS-Class. Although Sabokbar, Hosseini, Banaitis, and Banaitiene (2016) have introduced a TOPSIS classification/sorting approach, this method does not discriminate between alternatives and profiles towards using the traditional TOPSIS algorithm. Furthermore, this method does not cope with fuzzy environments. We slightly changed Fuzzy-TOPSIS following an intuitive idea. The score of an alternative $i \in I$ in a profile $p \in P$ is the closeness coefficient (CC_i^p) which is computed based on the distances of the alternative i to the positive ideal solution of profile p ($A_p^* = [\tilde{v}_{pi}^*]_m$), and to the negative ideal solution of profile p ($A_p^- = [\tilde{v}_{qi}^-]_m$). The positive and negative ideal solutions of profile p are computed using matrix $\tilde{Q} = [\tilde{q}_{pj}]_{|P|\times n}$, where $\tilde{q}_{pj}$ is the linguistic term associated with the main reference to classify the profile p in the evaluation criterion j. The values of matrix $\tilde{Q}$ should be defined by a finance expert.

Algorithm 2 outlines FTOPSIS-Class. Suppose that the decision problem involves $\tilde{R} = [\tilde{r}_{ij}]_{m\times n}$ and $\tilde{W} = [\tilde{w}_j]_m$, as in Algorithm 1.

4.3. Phase 3—asset allocation

The main objective of this phase is to define the weights of each investment option in the portfolio according to the investor's profile, goals and preferences. Following the state-of-the-art in portfolio optimization (Zopounidis, Galariotis, Doumpos, Sarri, & Andriosopoulos, 2015), the asset allocation strategy is represented by the following general multi-objective model as follows:

$$\max Z_1, Z_2, \ldots, Z_k \quad (1)$$

Algorithm 2 FTOPSIS-Class routine.

Step 1: Execute Steps 1–4 of the Fuzzy-TOPSIS method.

Step 2: For each profile $p = 1, 2, \ldots, |P|$, do:

Step 2.1: Set the positive ideal solution regarding the profile p as $\tilde{A}_p^* = \{\tilde{v}_{p1}^*, \tilde{v}_{p2}^*, ..., \tilde{v}_{pn}^*\}$, where $\tilde{v}_{pj}^* = \tilde{q}_{pj}$, since the goal of the model is to maximize the adequacy of the alternative i in relation to category p, thereby minimizing the distance between $\tilde{A}_q^*$ and the reference values of each category;

Step 2.2: Set the negative ideal solution regarding the category p as $\tilde{A}_p^- = \{\tilde{v}_{p1}^-, \tilde{v}_{q2}^-, ..., \tilde{v}_{pn}^-\}$, where $\tilde{v}_{p'j}^-$ are the values of the farthest profile p' from p, and the distance to be maximized.

Step 2.3: Calculate the distances of each alternative i in relation to category p as follows:

$$\tilde{d}_i^{p^*} = \sum_{j=1}^{n} \delta(\tilde{v}_{ij}, \tilde{v}_{pj}^*), \; i = 1, 2, ..., m$$

$$\tilde{d}_i^{p^-} = \sum_{j=1}^{n} \delta(\tilde{v}_{ij}, \tilde{v}_{pj}^-), \; i = 1, 2, ..., m$$

Step 2.4: Calculate the closeness coefficient of each alternative i regarding profile p as $CC_i^p = \frac{\tilde{d}_i^{p^-}}{\tilde{d}_i^{p^*} + \tilde{d}_i^{p^-}}, i = 1, 2, ..., m$.

Step 3 (Sorting): For each alternative i, we find class $p_i^* = \text{argmax}_{p \in P}\{CC_i^p\}$; that is, p_i^* is the category with the highest value of CC_i^p for alternative i.

$$\min Z_{k+1}, Z_{k+2}, \ldots, Z_l \quad (2)$$

st

$$x \in X \quad (3)$$

where $Z_1, Z_2, \ldots, Z_k$ are the positive objectives for maximization, such as return, $Z_{k+1}, Z_{k+2}, \ldots, Z_l$ are the negative objectives for minimization, such as risk, and X is the set of feasible solutions.

Several solution methods were developed to solve such an MOO optimization problem. These methods aim to single out a specific solution, which is regarded as an "optimal" compromise solution, from the set of all non-dominated or Pareto optimal solutions of the problem. Multi-objective optimization methods are classified into the following groups according to Hwang and Yoon (1981): no-preference, *a priori*, *a posteriori*, and interactive methods. The first group assumes that a neutral compromise solution is identified without preference information. The other classes involve the DM's preference information in different ways, generally described by the name of the method. To save space, we refer the reader to Hwang and Yoon (1981) for a complete description of these methods.

However, in real problems, such as portfolio selection in private banking, the DM might assume a fuzzy perspective, in which membership functions $\mu(Z_j(x))$ are defined for each objective, and the compromise solution is the one that achieves all objectives given a certain tolerance limit under the system constraints. In this case, the problem 1)–((3) consists of finding a vector $\mathbf{x}^T$ to satisfy the following formulation (Amid, Ghodsypour, & O'Brien, 2011):

$$\tilde{Z}_i \geq\sim Z_i^o \qquad i = 1, \ldots, k \tag{4}$$

$$\tilde{Z}_j \leq\sim Z_j^o \qquad j = k+1, \ldots, l \tag{5}$$

st

$$g_s(x) = \sum_{i=1}^{n} a_{si} x_i \leq b_s \qquad \forall s \tag{6}$$

$$x_i \geq 0 \qquad \forall i \tag{7}$$

where Z_k^o and Z_l^o are the aspiration levels that the DM wants to reach, a_{si} and b_s are crisp values, and symbol $\sim$ indicates the fuzzy environment.

To solve this problem, Lin (2004) expanded the max–min operator approach by Zimmermann (1978), by proposing a weighted max–min model, in which the DM provides relative weights (θ_k) for the fuzzy goals with corresponding membership functions. This model finds an optimal feasible solution such that the ratio of the levels achieved is as close to the ratio of the weights as possible. This model can be stated as follows (Amid et al., 2011):

$$\max \lambda \tag{8}$$

st

$$\theta_k \lambda \leq f_{\mu Z_k}(x) \qquad k = 1, 2, \ldots, l \tag{9}$$

$$g_s(x) \leq b_s \qquad \forall s \tag{10}$$

$$\lambda \in [0, 1] \tag{11}$$

$$\sum_{k=1}^{l} \theta_k = 1 \tag{12}$$

$$\theta_k \geq 0 \qquad k = 1, 2, \ldots, l \tag{13}$$

$$x_i \geq 0 \qquad i = 1, 2, \ldots, m \tag{14}$$

where the membership function for maximization objectives (Z_k), and for minimization ones (Z_l) are as follows, respectively:

$$\mu(Z_k(x)) = \begin{cases} 0, & Z_k \leq Z_k^- \\ 1, & Z_k \geq Z_k^+ \\ f_{\mu(Z_k)} = \frac{Z_k^+ - Z_k(x)}{Z_k^+ - Z_k^-}, & Z_k^- \leq Z_k(x) \leq Z_k^+ \end{cases} \tag{15}$$

$$\mu(Z_l(x)) = \begin{cases} 1, & Z_l \leq Z_l^- \\ 0, & Z_l \geq Z_l^+ \\ f_{\mu(Z_l)} = \frac{Z_l(x) - Z_l^-}{Z_l^+ - Z_l^-}, & Z_l^- \leq Z_l(x) \leq Z_l^+ \end{cases} \tag{16}$$

where Z_k^+ and Z_l^- are the optimal single objective functions (individual maximum and minimum solutions) of positive objective Z_k and negative objective Z_l, respectively, and Z_k^- and Z_l^+ are the minimum and maximum values (worst solutions) of objectives Z_k and Z_l, respectively (Lai & Hwang, 1993).

The weighted max-min models for the portfolio optimization problem are obtained by using Algorithm 3. Appendix A presents an example of a model generated by this algorithm, used in the case study described in the next section.

Algorithm 3 Fuzzy MOLP Generator Algorithm.

Step 1: Formulate the crisp portfolio selection model, using the objectives and constraints defined interactively by the bank analyst and investor.

Step 2: Solve the MOLP as a single objective problem for each objective $i, i = 1, \ldots, k$. As this is the best value for each objective, set Z_i^+ as the upper bound of the i-th objective.

Step 3: Solve the MOLP as a single objective, changing the optimization direction of each objective $j = k+1, \ldots, l$. As this is the worst value for each objective, set Z_j^- as the lower bound of the j-th objective.

Step 4: For each objective $i = 1, \ldots. k$ find the membership function by using (15).

Step 5: For each objective $j = k, \ldots, l$ find the membership function by using (16).

Step 6: Define the weights ($\theta_i, i = 1, \ldots. l$) of the objectives.

Step 7: Formulate the problem as a multi-objective weighted max-min model (8) – (14).

Step 8: Solve the model of Step 7 so as to find the optimal compromise solution of the problem formulated in Step 1.

4.4. Phase 4—validation of the results

We should expect that the natural trade-off between risk and return (riskier strategies lead to higher expected returns) would be present. This is the logic associated with using performance metrics based on the Sharpe ratio. Nonetheless, such behaviour might change for each investment profile i due to including adequacy as one of the objectives in the optimization model. In this sense, a good strategy X will exhibit a satisfactory balance between return and risk on out-sample performance, at least in relation to some benchmark.

For each investment profile $p \in P$, let $\mathbf{x} = \{x_i, i \in I\}$ be the vector of optimal weights obtained from our optimization problem, and $\mathbf{r} = \{r_i, i \in I\}$ the vectors of asset returns (typically daily log-returns) that comprise the portfolio. Then, $X = \mathbf{x}^T \mathbf{r}$ is the vector of returns that the portfolio obtains. While the return is easily and directly computed given some sample, there is no consensus about a definitive or superior risk measure to be used in comparisons of different approaches, as pointed out recently by Emmer, Kratz, and Tasche (2015). Hence, we considered the risk measures most widely used in both the literature and industry.

We considered the usual standard deviation (StD), as a measure of variability, in order to represent the classic Markowitz approach. More recently, the occurrence of critical events has focused attention on the tail risk measurement. Thus, our next choice is the canonical tail risk measure, the well-known and widely-used Value at Risk (VaR), which represents the α-quantile of interest. Nonetheless, VaR fails to be a coherent risk measure in the sense of Artzner, Delbaen, Eber, and Heath (1999), since it is not convex. Thus, we also consider the Expected Shortfall (ES), proposed by Acerbi and Tasche (2002), which represents the expected value of a loss, given that it is beyond the α-quantile of interest, and directly linked to the VaR concept. ES is a coherent risk measure. Finally, we consider a risk measure that has recently gained more attention, the Expectile Value at Risk (EVaR). This measure is linked to the concept of expectile. Ziegel (2016) and Bellini and Di Bernardino (2015) studied EVaR from both theoretical and em-

Table 1
Fund description.

Fund	Risk	D_i (days)	B_i ($)	I_i ($)	R_i ($)
Fixed Income ST 200 (F_1)	Very low	0	50.00	200.00	7.81
Fixed Income ST 50,000 (F_2)	Very low	0	50.00	50,000.00	9.63
Fixed Income interbank referenced ST 500 (F_3)	Low	0	50.00	500.00	8.71
Fixed Income Interbank referenced ST 50,000 (F_4)	Low	0	10.00	50,000.00	9.95
Fixed Income LT 10,000 (F_5)	Medium	0	50.00	10,000.00	10.26
Fixed Income LT 50,000 (F_6)	Medium	0	10.00	50,000.00	10.01
Fixed Income price index LT 5000 (F_7)	High	3	50.00	5000.00	12.70
Multi market LT 200 (F_8)	High	1	200.00	200.00	−2.11
Dollar exchange LT 200 (F_9)	Very high	0	200.00	1000.00	10.97
Equities LT 200 (F_{10})	Very high	4	200.00	200.00	2.18

pirical points of view, and verified that it is the only coherent risk measure, beyond the mean loss, that possesses the property of elicitability, which allows a function to have its forecasts evaluated.

More formally, we considered that the random result X is defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $E[X]$ is the expected value of X under $\mathbb{P}$. All equalities and inequalities are considered to be almost surely in $\mathbb{P}$. F_X is the probability function of X and its inverse is F_X^{-1}. Thus, for $0 \le \alpha \le 1$, we have the following formulations:

$$StD(X) = \sqrt{E[(X - E[X])^2]} \tag{17}$$

$$VaR^{\alpha}(X) = -F_X^{-1}(\alpha) \tag{18}$$

$$ES^{\alpha}(X) = -E[X | X \le F_X^{-1}(\alpha)] \tag{19}$$

$$EVaR^{\alpha}(X) = -\arg\min_{\gamma} E[(\alpha - \mathbf{1}_{X \le \gamma})(X - \gamma)^2] \tag{20}$$

5. Numerical application

In this section, we illustrate our proposed approach in a real numerical example, emulating a real personal banking experience. Our objective is to define the best portfolios, considering different investor's profiles, from a small number of funds offered by a Brazilian bank. Following Calvo et al. (2016), we used a reduced size problem to provide more information concerning the application. It should be noted that this approach can be used in large-scale problems. The presentation of the application follows the steps of the proposed framework which were presented in the previous section.

5.1. Phase 1—problem structuring results

Initially, the main parameters and variables used to classify the funds according to the profiles were identified and collected. Table 1 describes the 10 funds used in our experiments, all of which are managed by the largest Brazilian bank (Bank of Brazil), and includes the following information: risk, the investment liquidity (D_i), minimum balance (B_i), initial investment (I_i) and net return (R_i). Detailed information about these funds can be found at http://www.bb.com.br. We considered two investment options for each of the four risk classifications for funds which are set out in Section 2. These investments represent funds comprising distinct classes of asset, including Brazilian treasury bonds, equities, and exchange rate funds. We chose to collect daily data from 2014 and 2015. We consider the data from 2014 for the application of the proposed framework, while we use data from 2015 to out-sample performance analysis of our approach against two very well-known benchmarks in portfolio problems, as follows: (i) the Equally Weighted (EW) approach, which give equal participation (1/10 in our case) to each asset; and the (ii) Markowitz (MW) approach, which seeks to maximize the expected return on a portfolio and to minimize variance.

In the first phase of the framework, we defined a set of criteria and sub-criteria to measure the adequacy of each fund i with respect to each risk profile p, CC_i^p as computed in Algorithm 2 (see Table 2). The criteria were directly defined based on the regulatory instructions from governmental agencies. The criteria can be divided into two groups, the first comprises the criteria related to the performance of funds (C_1, C_2 and C_3), while the second is associated with an investor's knowledge and objectives (C_4, C_{51}, C_{52} and C_{53}).

The decision-making process requires the DMs to reflect explicitly on their preferences and values. Thus, having adequate and consistent criteria weighting is an essential condition to ensure an effective process. Although TOPSIS and its variants do not include any specific mechanism for supporting this task, several techniques may be applied in defining the relative importance of criteria, including analytical procedures, simulations or empirical approaches. Sensitivity analysis was used in our case, where the weights of the criteria were defined after a set of experiments with different levels of importance. Since there was not a qualitative distinction in results obtained *a posteriori*, we choose to keep just the initial specification due to lack of space. Nonetheless, all results are available upon request. A set of linguistics terms was used to assess the importance level of each criterion, as well as the ratings of the alternatives (see Table 3).

Tables 4 and 5 summarize the data required for the sorting procedure (see Algorithm 2). The former table comprises decision matrix $\tilde{R}$, e.g. the ratings of each fund regarding each criterion, while the latter provides the main reference matrix $\tilde{Q}$ for each profile, both using the linguistic terms presented in Table 3. This process was conducted by an experience fund manager, assuming the point of view of each investor's profile.

5.2. Phase 2—sorting procedure results

After all the parameters required for *Phase 2* were defined, the sorting algorithm FTOPSIS-Class was used to obtain the adequacy of each fund for each investor's profile, computed as the closeness coefficient CC_i^p. Table 6 presents the final results with criteria weight vector $\tilde{\mathbf{w}} = [H, H, H, H, H, H, H]$. The values of CC_i^p were computed following Algorithm 2. The higher the value of the closeness coefficient (CC_i^p) for a fund i regarding an investor's profile p, the more adequate this fund is for the profile p, for example, F_1 can be better classified as a Conservative fund. For validation purposes, we also present the class computed by our procedure, using Step 3. Notice that there was a complete convergence between the computed class and the risk classification of the bank for each fund (see Table 1), thus showing the effectiveness of our procedure.

Several different combinations of criteria weights were used, however without significant changes in the classification results.

Table 2
Criteria and sub-criteria definitions.

Criteria	Sub-criteria
1.Risk (C_1)	
2.Number of days to withdraw the money (C_2)	
3.Net return (C_3)	
4.Compatibility of investor's level of knowledge on financial markets (C_4)	
5.Investment objectives	Wealth preservation (C_{51})
	Wealth generation in the short term (C_{52})
	Wealth accumulation in the long term (C_{53})

Table 3
Linguistic variables.

Ratings	Fuzzy numbers	Weights	Fuzzy numbers
Very low (VL)	(0.0,0.0,0.1,0.2)	Unimportant (U)	(0.0,0.0,0.1,0.2)
Low (L)	(0.1,0.2,0.3,0.4)	Moderately important (MI)	(0.1,0.2,0.3,0.4)
Medium (M)	(0.3,0.4,0.5,0.6)	Important (I)	(0.3,0.4,0.5,0.6)
High (H)	(0.5,0.6,0.7,0.8)	Very important (VI)	(0.5,0.6,0.7,0.8)
Very high (VH)	(0.7,0.8,0.9,1.0)	Extremely important (EI)	(0.7,0.8,0.9,1.0)

Table 4
Fuzzy decision matrix ($\tilde{R}$) for each investment fund.

Fund	C_1	C_2	C_3	C_4	C_{51}	C_{52}	C_{53}
F_1	VL	VL	H	VL	VH	VL	M
F_2	VL	VL	VH	VL	VH	VL	M
F_3	L	VL	H	L	H	L	M
F_4	L	VL	VH	L	H	L	M
F_5	M	VL	VH	M	M	M	VH
F_6	M	VL	VH	M	M	M	VH
F_7	H	H	VH	H	L	H	VH
F_8	H	L	VL	H	L	H	H
F_9	VH	VL	VH	VH	VL	VH	H
F_{10}	VH	VH	L	VH	VL	VH	VH

Table 5
Main reference matrix ($\tilde{Q}$) for each profile.

Profile	C_1	C_2	C_3	C_4	C_{51}	C_{52}	C_{53}
Conservative	L	VL	VL	VL	VH	VL	H
Moderate	L	VL	L	L	H	L	H
Bold	H	VL	H	H	L	H	L
Aggressive	VH	VL	VH	VH	VL	VH	VL

Table 6
Results—sorting procedure. The bold values indicate the largest values of the closeness coefficient for each fund.

Fund	Closeness coefficient – CC_i^p			
	Conservative	Moderate	Bold	Aggressive
F_1	**0.79866**	0.70388	0.27493	0.20134
F_2	**0.7516**6	0.66237	0.30022	0.24833
F_3	0.68649	**0.78957**	0.44210	0.31350
F_4	0.63540	**0.73081**	0.44897	0.36460
F_5	0.50332	**0.63794**	0.59904	0.49668
F_6	0.50332	**0.63794**	0.59904	0.49668
F_7	0.37450	0.45127	**0.72643**	0.62549
F_8	0.51017	0.53095	**0.66981**	0.48982
F_9	0.13143	0.15093	0.71979	**0.86856**
F_{10}	0.34331	0.40799	0.61318	**0.65668**

The funds were always classified as in Table 6, there being slightly insignificant differences in the final values of CC_i^p. As a consequence, we defined the aforementioned vector weight as the most suitable. Although the experimentation was brief, the decision model seems robust to changes in criteria weights. However, additional experiments are required before this statement can be generalized.

5.3. Phase 3—fuzzy multiple objective model

Based on the general model (1)–(3), we formulated a linear integer multi-objective model for choosing a Pareto-frontier solution in terms of the asset weights in the portfolio, assuming a single investment period and m available assets. The model has the following parameters: R_i is the 2014 compound net return of investment i in a previously considered period, C is the total available capital for investment, I_i is the minimal initial investment in fund i, and CC_i^p is the closeness coefficient computed by Algorithm 2, measuring the adequacy of investment i for investor's profile p, as shown in Table 6. The values of R_i and I_i used in the experiments are presented in Table 1. We now introduce the decision variables. Let x_i be a decision variable representing the amount invested (or weight) in fund F_i, and z_i be a binary decision variable, in which $z_i = 1$ if an investment is made in fund i, otherwise $z_i = 0$. The crisp formulation for the personal banking portfolio selection problem is presented as follows:

$$\max Z_1 = \sum_{i\in I}\sum_{p\in P} CC_i^p x_i \tag{21}$$

$$\max Z_2 = \sum_{i\in I} R_i x_i \tag{22}$$

$$\min Z_3 = \frac{1}{2m}\sum_{i=1}^{m}\left|R_i - \left(\sum_{i=1}^{m} x_i R_i\right)\right| \tag{23}$$

st

$$x_i \geq \frac{I_i}{C} z_i \qquad \forall i \in I \tag{24}$$

$$\sum_{i\in I} x_i = 1 \tag{25}$$

$$x_i \leq z_i \qquad \forall i \in I \tag{26}$$

$$\sum_{i\in I} z_i \leq N \tag{27}$$

$$x_i \geq 0,\ z_i \in \{0,1\} \qquad \forall i \in I \tag{28}$$

Table 7
Upper and lower bounds.

Profile	Z_1^+	Z_1^-	Z_2^+	Z_2^-	Z_3^+	Z_3^-
Conservative	0.799	0.131	1.0	0.0	0.158	0.0928
Moderate	0.790	0.151	1.0	0.0	0.158	0.0928
Bold	0.726	0.275	1.0	0.0	0.158	0.0928
Aggressive	0.869	0.105	1.0	0.0	0.158	0.0928

Three objective functions, namely, the adequacy of the investment for the client's profile (21), the mean expected return (22), and the mean absolute deviation (MAD) model (23) were formulated to maximize the adequacy of the investments for the client's profile, to maximize the return on the portfolio, and to minimize the portfolio risk, respectively. The MAD model was selected to measure portfolio risk. MAD was introduced by Konno and Yamazaki (1991) in portfolio optimization, and it has been extensively tested on various stock markets (Mansini, Ogryczak, & Speranza, 2014). Constraints (24) ensure that the minimum amount demanded by each fund is respected. Constraint (25) assures that all the client's resources are used. Constraints (26) ensure that resources are only allocated to assets selected by the program. Constraint (27) restricts the number of assets in the portfolio to N assets. The domain for the decision variables x and z are defined in constraints (28).

It should be noted that this is a reference model that can be expanded, which introduces real features such as investment threshold constraints, or can be reduced, if the cardinality constraint (26) is not relevant, thereby eliminating the need for decision variable z_i, and constraint (26) and (27). We run the model with and without these constraints. However, since there are no significant changes in the results, we choose to keep the more flexible specification. Nonetheless, all results are available upon request.

Next, for each profile, we solved the multi-objective problem as three single objective problems and found the upper (Z_i^+) and lower (Z_i^-) limits for each objective, as presented in Table 7. The values were normalized to interval [0,1]. It should be noted that only objective Z_1 changes with the investor's profile, since it uses the adequacy of investment i for investor's profile p. The remaining ones do not depend on the profile.

Based on the weighted max-min models (8) – (14), the crisp single objective formulation is as follows:

$$\text{maximize } \lambda \tag{29}$$

subject to

$$\theta_1(Z_1^+ - Z_1^-)\lambda - Z_1 \leq -Z_1^- \tag{30}$$

$$\theta_2(Z_2^+ - Z_2^-)\lambda - Z_2 \leq -Z_2^- \tag{31}$$

$$\theta_3(Z_3^+ - Z_3^-)\lambda + Z_3 \leq Z_3^+ \tag{32}$$

$$(11) - (12), (24) - (28)$$

The mathematical packages AMPL and Knitro were used to solve this last formulation. Different combinations of weights for each objective were used. We identified 4 weight combinations that better characterized the different investors' attitudes concerning the importance of the three considered objectives, always offering different portfolios from the remaining combinations. Table 8 presents the results for these 4 combinations, totaling 16 investment portfolios, and for the two validation benchmarks. Appendix A illustrates the formulation for *Case 3* of the Conservative profile.

In general, the allocation of resources is coherent with the investor's profiles when the objective adequacy is the most important one. In cases 1 and 2 of all investor's profiles, in which adequacy presents much higher weights, there is a clear concentration on the fund(s) that were classified into the risk class most aligned with the investor's profile. For instance, for the profile moderate, the funds F_3 and F_4 received all resources, while in different proportions in both cases. Both funds were classified as moderate by our FTOPSIS-Class as shown in Table 6. As risk becomes a more relevant objective, presenting higher weights, our approach increased the diversity in the portfolio. This behaviour is highly convergent with the consensus among financial experts that diversification is the most important component for helping investors to reach their financial goals while minimizing their risk. In case 4 of all investor's profiles, resources were allocated to six or more funds. In our experiments, the objective return is used by our approach as a tie-breaking criterion when defining proportions in the same fund classification as can be seen in the results shown in Table 6. Overall, selected portfolios respected the importance of the objectives defined and regulatory issues, while they were also in line with the common-sense financial notions assumed by experts.

5.4. Phase 4—validation of the results

The approach developed was evaluated by comparing its results with the usual benchmarks, such as the classic MW mean-variance, and EW naive strategy. Based on risk and return performance criteria, we computed for the out sample period, which comprised daily log-returns of 2015, the average returns and the aforementioned risk measures for each portfolio in Table 8.

We carried out the MW approach in three frameworks: (i) minimization of variance with expected return as a constraint; (ii) maximization of the expected return with variance as a constraint; and (iii) maximization of the ration between the expected return and variance in a Sharpe ratio inspired framework. As expected, the results obtained were quite similar. Following DeMiguel, Garlappi, and Uppal (2009) and Righi and Borenstein (2017), we have conducted the analysis of performance for several control periods comprising 3, 3, 4, 6, and 12 months on our out-of-sample data set. Results have not exhibited qualitative distinctions, perhaps because our data set consists of funds which include a high percentage of bonds. Moreover, we used MAD in our proposed approach in order to keep the illustration linear. In the MW we choose to keep the original framework. The recent axiomatic theory of generalized deviations, initiated by Rockafellar, Uryasev, and Zabarankin (2006), shows that p-norm deviations, with MAD and Standard deviation (variance) as particular cases, are intrinsically linked. Moreover, the recent paper of Righi and Borenstein (2017) explored the role of distinct risk measures on portfolio problems with no relevant empirical distinction in the performance of measures based on deviations. We have replaced variance by MAD in preliminary results for the study but we have not found significant changes.

Risk measures were computed by historical simulation (HS), which is an empirical method that creates no assumptions about the data and is the most extensively used method (Pérignon & Smith, 2010). Although HS has been criticized (Pritsker, 2006), our focus is not to discuss estimation details or even compare different measurement models, but rather to use risk measures to evaluate performance. We considered $\alpha = 5\%$, a reasonable and typical quantile of interest in financial applications. Other values of α were used, without qualitatively changing the analysis.

Considering both the large number of results obtained in our experiments and the nonexistence of significant differences among the results from the different aforementioned experimental settings, we decided only to show the results for the following settings: one-year control period, MV model as the maximization of

Table 8
Resulting portfolios.

Profile	Case	θ_1	θ_2	θ_3	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
Conservative	1	0.8	0.1	0.1	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	2	0.4	0.2	0.4	0.289	0.711	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	3	0.2	0.2	0.6	0.206	0.000	0.113	0.000	0.194	0.000	0.363	0.029	0.061	0.033
	4	0.1	0.2	0.7	0.112	0.000	0.125	0.000	0.239	0.000	0.326	0.033	0.118	0.047
Moderate	1	0.8	0.1	0.1	0.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	2	0.4	0.2	0.4	0.000	0.000	0.500	0.500	0.000	0.000	0.000	0.000	0.000	0.000
	3	0.2	0.2	0.6	0.124	0.000	0.183	0.000	0.266	0.000	0.253	0.029	0.099	0.047
	4	0.1	0.2	0.7	0.092	0.000	0.110	0.000	0.212	0.000	0.362	0.047	0.121	0.055
Bold	1	0.8	0.1	0.1	0.000	0.000	0.000	0.000	0.000	0.000	0.823	0.000	0.177	0.000
	2	0.4	0.2	0.4	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.072	0.928	0.000
	3	0.2	0.2	0.6	0.086	0.000	0.113	0.000	0.242	0.000	0.280	0.037	0.179	0.064
	4	0.1	0.2	0.7	0.084	0.000	0.098	0.000	0.212	0.000	0.352	0.047	0.145	0.061
Aggressive	1	0.8	0.1	0.1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000	0.000
	2	0.4	0.2	0.4	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.899	0.101
	3	0.2	0.2	0.6	0.080	0.000	0.096	0.000	0.000	0.000	0.306	0.049	0.395	0.073
	4	0.1	0.2	0.7	0.122	0.000	0.146	0.000	0.244	0.000	0.210	0.043	0.205	0.031
EW	–	–	–	–	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100
MW	–	–	–	–	0.243	0.135	0.145	0.030	0.134	0.302	0.001	0.001	0.006	0.004

Table 9
Results of the validation experiments.

		Proposed vs. MW					Proposed vs. EW				
Profile	Case	Mean	StD	VaR	ES	EVaR	Mean	StD	VaR	ES	EVaR
Conservative	1	−0.11	−0.72	−0.22	−0.38	−0.02	0.19	−0.99	−1.07	−1.06	−1.12
	2	−0.01	−0.72	−0.38	−0.56	−0.14	0.32	−0.99	−1.08	−1.07	−1.13
	3	−0.15	20.61	12.33	21.5	7.55	0.14	−0.22	−0.31	−0.10	−0.23
	4	−0.03	19.9	11.7	19.65	6.97	0.30	−0.25	−0.35	−0.18	−0.30
Moderate	1	−0.04	−0.6	−0.33	−0.45	−0.11	0.29	−0.99	−1.08	−1.06	−1.13
	2	0.01	−0.65	−0.41	−0.55	−0.18	0.36	−0.99	−1.09	−1.07	−1.14
	3	−0.05	17.06	10.51	16.6	6.06	0.27	−0.35	−0.42	−0.32	−0.41
	4	−0.08	23.11	14.3	22.99	8.25	0.24	−0.13	−0.19	−0.04	−0.15
Bold	1	0.16	38.97	21.43	37.77	13.51	0.56	0.44	0.25	0.61	0.47
	2	2.48	101.95	57.41	84.48	31.21	3.68	2.70	2.45	2.65	2.54
	3	0.08	21.55	12.49	19.32	7.3	0.45	−0.19	−0.30	−0.20	−0.26
	4	−0.03	23.43	13.85	22.67	8.19	0.30	−0.12	−0.21	−0.05	−0.16
Aggressive	1	2.71	110.95	62.23	91.52	33.92	3.99	3.03	2.75	2.96	2.86
	2	1.99	89.53	50.09	74.97	28.09	3.02	2.26	2.00	2.23	2.18
	3	0.59	35.2	19.99	30.82	11.46	1.14	0.30	0.16	0.30	0.23
	4	0.32	18.06	10.5	16.05	5.99	0.77	−0.31	−0.42	−0.34	−0.42

the ration between the expected return and variance, MAD as the risk measure, and $\alpha = 5\%$. Table 9 presents the results relative to MW and EW as benchmarks for these settings. The omitted results are available on request. The values in the table are relative changes, the benchmark values being defined as references. Based on the obtained results, our approach has a better performance than EW for all cases. Moreover, in most cases, especially for the conservative and moderate profiles, our approach has produced greater returns with less risk. Of course, EW is a naive strategy. MW is a more robust strategy since it balances mean and variance. In this case, our approach does not exhibit a clear advantage. Nonetheless, it is competitive since for two compositions of the three objectives on two investment profiles there was a slight reduction on the returns but with an important decrease in risk. These cases are precisely those that give more importance to the adequacy of the investment profile, which draws attention to the relevance of our approach.

6. Conclusions

Portfolio optimization in private banking is a complex problem due to the several issues that need to be involved in the decision-making process. In this paper we introduce a fuzzy integrated approach, combining MADM and MOOM, to support DMs to reach their objectives and preferences, while taking into consideration the regulatory issues defined by governmental agencies. The portfolio selection problem was broken down into two interconnected problems. First, the adequacy of each investment option for each investor's profile is measured, using a fuzzy MADM sorting method. Trapezoidal fuzzy numbers were used to represent the investor's profiles and the classification of risk used by banks for their investment options. Next, a multi-objective optimization model is used to find the percentage of each option in the portfolio, which simultaneously maximizes net returns and the adequacy of the options for the profile considered (measured in the first step), and minimizes the financial risks of the portfolio. The model is transformed into a weighted max-min fuzzy programming model, such that the achievement level of the objective functions matches their weights, which the DM has defined. This transformation reduces the computational complexity, thereby facilitating the calculation of the solution that the model gives. In proposing a structured way to present the problem and to analyse alternative solutions and different investor's profiles, the proposed framework provided a basis on which DMs can understand and exploit the problem, considering simultaneously distinct and conflicting dimensions.

A numerical application based on real financial data was used to demonstrate the potential application of the approach, considering different investor's profiles and different combinations of weights of the objective functions in the multi-objective programming formulation. Further, the approach was validated, by comparing the results it produced with those of Markowitz's classic mean-variance method and the equally weighted naive strategy. For the comparison, the average returns and several risk measures were used, these being computed from historical data. Our method outperformed the equally weighted strategy for the portfolios obtained for the conservative, moderate, and bold profiles. For the aggressive profile, the developed approach improved the return, but obtained worse risks. Further, our method did not present a clear advantage in comparison with the Markowitz approach. These results were expected since we are taking into consideration the investor's profiles and the risk classification of the investment options, which the Makowitz approach neglects. As a consequence, we presented portfolios with slightly poorer returns, but with lower risks, since these are better suited for private banking.

Given the complexity of the problem, there is a natural gap between the existence of a solution algorithm and its application to real world cases. As a result, we are planning to develop a decision support system to be offered to private banking institutions. The development of a computer-based tool, which is capable of providing a user-friendly environment and emphasizing flexibility, efficiency and adaptability, becomes an important aspect in terms of the effective use of the developed approach, thereby allowing additional tests towards assessing the range of the optimization-based approach. Further, future research will seek to use other risk measures towards a better convergence of this measure and the investor's approach to financial risks. Also, we intend to expand the scope of the developed approach to other allocation optimization problems, such as portfolio selection in different financial contexts, and project portfolio analysis. These expansions will require new ways of modeling and solving the problems, since the requirements and dimensions involved differ significantly from private banking and from each other.

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Appendix A. Example of multi-objective programming model

This appendix presents the crisp single objective formulation for *case 3*, a conservative profile (see Table 8) as follows:

$$\max \lambda$$

st

$$0.2\lambda \leq \frac{f_1 - 0.275}{0.451}$$

$$0.2\lambda \leq f_2$$

$$0.6\lambda \leq 0.1589 - f_3$$

$$x_1 - \frac{200}{100000} z_1 \geq 0$$

$$x_2 - \frac{50000}{100000} z_2 \geq 0$$

$$x_3 - \frac{500}{100000} z_3 \geq 0$$

$$x_4 - \frac{50000}{100000} z_4 \geq 0$$

$$x_5 - \frac{10000}{100000} z_5 \geq 0$$

$$x_6 - \frac{50000}{100000} z_6 \geq 0$$

$$x_7 - \frac{5000}{100000} z_7 \geq 0$$

$$x_8 - \frac{200}{100000} z_8 \geq 0$$

$$x_9 - \frac{1000}{100000} z_9 \geq 0$$

$$x_{10} - \frac{200}{100000} z_{10} \geq 0$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} = 1$$

$$x_1 - z_1 \leq 0, x_2 - z_2 \leq 0, x_3 - z_3 \leq 0, x_4 - z_4 \leq 0, x_5 - z_5 \leq 0$$

$$x_6 - z_6 \leq 0, x_7 - z_7 \leq 0, x_8 - z_8 \leq 0, x_9 - z_9 \leq 0, x_{10} - z_{10} \leq 0$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \geq 1$$

where,

$$f_1 = 0.275x_1 + 0.3x_2 + 0.442x_3 + 0.449x_4 + 0.599x_5 + 0.599x_6 + 0.726x_7 + 0.67x_8 + 0.72x_9 + 0.613x_{10}$$

$$f_2 = 0.6753x_1 + 0.7897x_2 + 0.7278x_3 + 0.811x_4 + 0.8303x_5 + 0.815x_6 + x_7 + 0.8835x_9 + 0.2894x_{10}$$

$$f_3 = \frac{1}{20}(| 0.6753 - f_2 | + | 0.7897 - f_2 | + | 0.7278 - f_2 | + | 0.811 - f_2 | + | 0.8303 - f_2 | + | 0.815 - f_2 | + | 1 - f_2 | + | 0.8835 - f_2 | + | 0.2894 - f_2 |)$$

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